Progress In Electromagnetics Research B, Vol. 30, 27–46, 2011

DYADIC GREEN'S FUNCTIONS FOR UNBOUNDED AND TWO-LAYERED GENERAL ANISOTROPIC MEDIA

Y. Huang and J. K. Lee

Department of Electrical Engineering and Computer Science
Syracuse University, Syracuse, NY, USA

Abstract—The dyadic Green's functions (DGFs) for unbounded and layered general anisotropic media are considered in this paper. First, the DGF for unbounded problem is derived using the eigen-decomposition method. Two different approaches are proposed to obtain the DGF for layered problem when the source is located inside the anisotropic region. The first approach is to apply the modified symmetrical property of DGF to obtain the DGF for the field in the isotropic region when the source is located inside the anisotropic region, from the DGF for the field in anisotropic region when the source is in the isotropic region. This modified symmetrical property can be applied for the layered geometry with bounded anisotropic region being either reciprocal or non-reciprocal medium. However, this method can not give the DGF for the field inside the anisotropic region. Thus, the second approach is presented to obtain the complete set of DGFs for all the regions including the anisotropic region, by applying the direct construction method through eigen-decomposition together with matrix method.

1. INTRODUCTION

Radiation from the source embedded inside a layered anisotropic structure has been of considerable interest among the researchers for a long time [1–5]. One of the well-established tools for the analysis of electromagnetic radiation problems is the method of Green's function. Several different methods have been proposed so far to obtain the Green's function of a layered planar geometry. They include

Received 13 March 2011, Accepted 18 March 2011, Scheduled 28 April 2011
Corresponding author: Jay Kyoon Lee (leejk@syr.edu).

the Fourier transform method [6–10], the transition matrix method proposed by Krowne [11], the equivalent boundary method by Mesa et al. [12] and the cylindrical vector wave function method by Li and his group [13, 14]. The transmission line method is proposed in [15], in which an isotropic medium is analyzed based upon the decomposition of fields into TE and TM modes.

However, the DGFs of a layered structure obtained so far are mostly for the case of the source located inside the isotropic region or for the case of the tangential source parallel to the interface. In this paper, the DGF of a layered structure with an arbitrarily directed source embedded inside the general anisotropic region is considered. The region where the source is located is assumed to have electric anisotropy, which is characterized by a 3 by 3 permittivity tensor with no constraint imposed on the property of the medium.

If the region where the source is located is a reciprocal medium such as uniaxial medium, a common method to obtain the DGF is to apply symmetrical property of DGF [16] to the available DGF with the source inside the isotropic region. However, this symmetrical property has two limitations. First, it cannot be used to obtain the DGF for the layered geometry if the bounded region is non-reciprocal medium. Secondly, the symmetrical property can not give the DGF for the field in the region where the source is located. Thus, two corresponding methods are proposed in this paper to overcome the limitations of the existing symmetrical property.

This paper is organized as follows. First the DGF for the general unbounded anisotropic medium with no limitations on the permittivity and permeability tensor is presented using the eigen-decomposition method in Section 2. The complete DGFs for all the regions of a layered planar geometry with the source located in the isotropic medium are then presented in Section 3. The modified symmetrical property is proposed in Section 4 to obtain the DGF for the field in the isotropic region with the source located inside the general anisotropic region for a two layered geometry. Finally, the complete DGFs for all the regions of a two layered geometry with the source located inside the anisotropic slab are obtained via the direct construction method in Section 5, followed by discussion in Section 6 and conclusions in Section 7.

2. DYADIC GREEN'S FUNCTION FOR UNBOUNDED GENERAL ANISOTROPIC MEDIUM

DGF for a two layered geometry filled with uniaxial medium with arbitrarily oriented optic axis is presented in [6]. The similar method is then used to obtain the DGF of unbounded and layered biaxial anisotropic medium by Mudaliar and Lee [7] and unbounded gyroelectric medium by Eroglu and Lee [8]. The form of DGF shown above is dependent on the type of the medium and can not apply to the general anisotropic medium. In this section, a general formula of DGF obtained using the eigen-decomposition method is proposed for the unbounded general anisotropic medium characterized by $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ with all nine non-zero elements.

A medium is called anisotropic when its electrical and/or magnetic properties depend upon the directions of the field vectors. The relationship between the fields is given by $\overline{D} = \varepsilon_0\overline{\overline{\varepsilon}}\cdot\overline{E}$ and $\overline{B} = \mu_0\overline{\overline{\mu}}\cdot\overline{H}$, where $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ are the relative permittivity and permeability tensors, respectively. For a reciprocal medium such as uniaxial or biaxial medium, $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ are symmetric matrices. For a non-reciprocal medium such as gyrotropic medium, the permittivity and/or permeability matrices are not symmetric. Even in the principal coordinate (i.e., the coordinate axis aligned along the direction of the biasing magnetic field), the off-diagonal elements of the permittivity and permeability matrix are non-zero and satisfy $\overline{\overline{\varepsilon}} = \overline{\overline{\varepsilon}}^{\dagger}$ and $\overline{\overline{\mu}} = \overline{\overline{\mu}}^{\dagger}$ for loss-free medium [17]. It needs to be noted here that a coordinate-free approach can also be applied to obtain the spectral domain DGF for unbounded general anisotropic medium [18]. However, to obtain the DGF of the layered geometry, it is more convenient to use the DGF of the unbounded medium proposed in this section.

The Maxwell's equations (with a time variation of $e^{-i\omega t}$) for an unbounded medium are given as

$$\begin{aligned} \nabla\times\overline{E} &= i\omega\mu_0\overline{\overline{\mu}}\overline{H} \\ \nabla\times\overline{H} &= -i\omega\varepsilon_0\overline{\overline{\varepsilon}}\overline{E} + \overline{J} \end{aligned} \tag{1}$$

The electric field due to a current source in the unbounded medium can be written in terms of the dyadic Green's function $\overline{\overline{G}}_{ee}(\overline{r},\overline{r}')$ as

$$\overline{E} = \int_{v'} \overline{\overline{G}}_{ee}(\overline{r},\overline{r}')\cdot\overline{J}(r')d^3\overline{r}', \tag{2}$$

Substituting (2) into (1) and eliminating the magnetic field, a second order differential equation for the DGF is obtained as follows.

$$\left(\nabla\times\overline{\overline{\mu}}^{-1}\nabla\times\overline{\overline{I}} - k_0^2\overline{\overline{\varepsilon}}\right)\overline{\overline{G}}_{ee}(\overline{r},\overline{r}') = i\omega\mu_0\delta(\overline{r}-\overline{r}')\overline{\overline{I}} \tag{3}$$

Applying the Fourier transform to (3), we obtain the following in the spectral domain:

$$-(\overline{\overline{k}}\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}})\overline{\overline{G}}_{ee}(\overline{k},\overline{r}') = i\omega\mu_0 e^{-i\overline{k}\cdot\overline{r}'}, \tag{4}$$

where

$$\overline{\overline{k}} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{5}$$

The wave vector is given as $\overline{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$ and we used the fact that $\overline{k} \times \overline{\overline{G}} = \overline{\overline{k}} \cdot \overline{\overline{G}}$. The spatial domain DGF $\overline{\overline{G}}_{ee}(\overline{r}, \overline{r}')$ and spectral domain DGF $\overline{\overline{G}}_{ee}(\overline{k}, \overline{r}')$ are related by

$$\overline{\overline{G}}_{ee}(\overline{k}, \overline{r}') = \int_{-\infty}^{\infty} \overline{\overline{G}}_{ee}(\overline{r}, \overline{r}') e^{-i\overline{k}\cdot\overline{r}} d^3\overline{r} \tag{6a}$$

$$\overline{\overline{G}}_{ee}(\overline{r}, \overline{r}') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \overline{\overline{G}}_{ee}(\overline{k}, \overline{r}') e^{i\overline{k}\cdot\overline{r}} d^3\overline{k} \tag{6b}$$

The spectral domain DGF is given as solution to (4) in terms of the *electric wave matrix* $\overline{\overline{W_E}}$ as

$$\overline{\overline{G}}_{ee}(\overline{k}, \overline{r}') = -i\omega\mu_0 \overline{\overline{W_E}}^{-1} e^{-i\overline{k}\cdot\overline{r}'} \tag{7}$$

where

$$\overline{\overline{W_E}} = \overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}$$

Using the inverse Fourier transform (6b), we obtain the spatial domain DGF as

$$\begin{aligned}\overline{\overline{G}}_{ee}(\overline{r}, \overline{r}') &= \frac{-i\omega\mu_0}{(2\pi)^3} \int_{-\infty}^{\infty} \overline{\overline{W_E}}^{-1} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3\overline{k} \\ &= \frac{-i\omega\mu_0}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{adj\overline{\overline{W_E}}}{\left|\overline{\overline{W_E}}\right|} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3\overline{k} \end{aligned} \tag{8}$$

It can be seen that $|\overline{\overline{W_E}}|$ is a fourth order polynomial of k_z, with 4 roots corresponding to $|\overline{\overline{W_E}}| = 0$, so it can be written as

$$\left|\overline{\overline{W_E}}\right| = a_4(k_z - k_{zI}^u)(k_z - k_{zI}^d)(k_z - k_{zII}^u)(k_z - k_{zII}^d) \tag{9}$$

The subscripts *I*, *II* refer to two types of waves that exist in anisotropic medium. We will call them *type I* and *type II* wave. Out of the 4 roots, two roots k_{zI}^u and k_{zII}^u correspond to upward wave and the other two k_{zI}^d and k_{zII}^d correspond to downward wave. Substituting (9) into (8) and applying the Cauchy residue theorem to (8) along with the radiation boundary conditions, the 3D integration in the spectral domain shown in (8) can reduce to a 2D integration as follows.

$$\text{For}\quad z>z',\quad \overline{\overline{G}}_{ee}(\overline{r},\overline{r}')$$
$$=\frac{\omega\mu_0}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{adj\overline{\overline{W_E}}(k_{zI}^u)}{a_4(k_{zI}^u-k_{zI}^d)(k_{zI}^u-k_{zII}^u)(k_{zI}^u-k_{zII}^d)}e^{i\overline{k}_I\cdot(\overline{r}-\overline{r}')}\right.$$
$$\left.+\frac{adj\overline{\overline{W_E}}(k_{zII}^u)}{a_4(k_{zII}^u-k_{zI}^d)(k_{zII}^u-k_{zI}^u)(k_{zII}^u-k_{zII}^d)}e^{i\overline{k}_{II}\cdot(\overline{r}-\overline{r}')}\right)dk_xdk_y \tag{10a}$$

$$\text{For}\quad z<z',\quad \overline{\overline{G}}_{ee}(\overline{r},\overline{r}')$$
$$=\frac{\omega\mu_0}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{adj\overline{\overline{W_E}}(k_{zI}^d)}{a_4(k_{zI}^d-k_{zI}^u)(k_{zI}^d-k_{zII}^u)(k_{zI}^d-k_{zII}^d)}e^{i\overline{\kappa}_I\cdot(\overline{r}-\overline{r}')}\right.$$
$$\left.+\frac{adj\overline{\overline{W_E}}(k_{zII}^d)}{a_4(k_{zII}^d-k_{zI}^d)(k_{zII}^d-k_{zI}^u)(k_{zII}^d-k_{zII}^d)}e^{i\overline{\kappa}_{II}\cdot(\overline{r}-\overline{r}')}\right)dk_xdk_y \tag{10b}$$

where

$$\begin{aligned}\overline{k}_I&=\hat{x}k_x+\hat{y}k_y+\hat{z}k_{zI}^u,\quad \overline{k}_{II}=\hat{x}k_x+\hat{y}k_y+\hat{z}k_{zII}^u,\\ \overline{\kappa}_I&=\hat{x}k_x+\hat{y}k_y+\hat{z}k_{zI}^d,\quad \overline{\kappa}_{II}=\hat{x}k_x+\hat{y}k_y+\hat{z}k_{zII}^d\end{aligned} \tag{10c}$$

The adjoint matrix of electric wave matrix which is obtained from the second-order differential equation of electric field in the spectral domain shown above is closely related with the specific type of characteristic polarizations that can exist in the anisotropic medium. Applying the eigen-decomposition [17, 19], $adj\overline{\overline{W_E}}$ can be written as

$$adj\overline{\overline{W_E}}(k_z)=X\Lambda X^{-1} \tag{11}$$

where

$$X=\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix},\quad \Lambda=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix},\quad X^{-1}=\begin{bmatrix}v_1^{\mathrm{T}}\\ v_2^{\mathrm{T}}\\ v_3^{\mathrm{T}}\end{bmatrix}$$

If $k_z=k_{zp}^q$, $(p=I,II,q=u,d)$, which is the root to $|\overline{\overline{W_E}}|=0$ as in (9), two eigenvalues of the matrix $adj\overline{\overline{W_E}}(k_{zp}^q)$ will reduce to zero.

Denoting the non-zero eigenvalue of the matrix $adj\,\overline{\overline{W_E}}(k_{zp}^q)$ as λ_p^q, we obtain

$$adj\overline{\overline{W_E}}(k_{zp}^q)=\lambda_p^q\widehat{e_p^q}\left(\hat{v}_p^q\right)^{\mathrm{T}}\quad\text{for}\quad p=I,II;\quad q=d,u \tag{12}$$

where $\widehat{e_p^q}$ is the eigen-vector which corresponds to the eigenvalue λ_p^q.

It should be noted that $adj\overline{\overline{W_E}}$ is a symmetric matrix for all values of $\overline{k}$, if the medium is reciprocal. Thus we have

$$\hat{v}_p^q = \widehat{e_p^q} \quad \text{and} \quad \left(e_{px}^q\right)^2 + \left(e_{py}^q\right)^2 + \left(e_{pz}^q\right)^2 = 1 \tag{13a}$$

However, for a non-reciprocal medium, if k_{zp}^q is a real number, then $adj\overline{\overline{W_E}}(k_{zp}^q)$ is a hermitian matrix. We then have

$$\hat{v}_p^q = \widehat{e_p^q}^* \quad \text{and} \quad \left(\widehat{e_p^q}\right)^* \widehat{e_p^q} = 1 \tag{13b}$$

For a non-reciprocal medium with k_{zp}^q being a complex number or imaginary number, we have

$$\hat{v}_p^q \neq \widehat{e_p^q} \quad \text{and} \quad \left(\hat{v}_p^q\right)^{\mathrm{T}} \widehat{e_p^q} = 1 \tag{13c}$$

Using (12) in (10), the final dyadic form of DGF is given as

$z > z'$

$$\overline{\overline{G}}_{ee}(r, r') = \frac{i}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{0z}} \left(c_I^u(k_{zI}^u) \hat{e}_I^u \hat{v}_I^u e^{i\overline{k}_I \cdot (\overline{r} - \overline{r}')} + c_{II}^u(k_{zII}^u) \hat{e}_{II}^u \hat{v}_{II}^u e^{i\overline{k}_{II} \cdot (\overline{r} - \overline{r}')} \right) \tag{14a}$$

$z < z'$

$$\overline{\overline{G}}_{ee}(r, r') = \frac{i}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{0z}} \left(c_I^d(k_{zI}^d) \hat{e}_I^d \hat{v}_I^d e^{i\overline{\kappa}_I \cdot (\overline{r} - \overline{r}')} + c_{II}^d(k_{zII}^d) \hat{e}_{II}^d \hat{v}_{II}^d e^{i\overline{\kappa}_{II} \cdot (\overline{r} - \overline{r}')} \right) \tag{14b}$$

where

$$\begin{aligned}
c_I^u(k_{zI}^u) &= -\frac{2k_{0z}\lambda_I^u}{a_4(k_{zI}^u - k_{zII}^u)(k_{zI}^u - k_{zI}^d)(k_{zI}^u - k_{zII}^d)}, \\
c_{II}^u(k_{zII}^u) &= -\frac{2k_{0z}\lambda_{II}^u}{a_4(k_{zII}^u - k_{zI}^u)(k_{zII}^u - k_{zI}^d)(k_{zII}^u - k_{zII}^d)} \\
c_I^d(k_{zI}^d) &= \frac{2k_{0z}\lambda_I^d}{a_4(k_{zI}^d - k_{zII}^d)(k_{zI}^d - k_{zI}^u)(k_{zI}^d - k_{zII}^u)}, \\
c_{II}^d(k_{zII}^d) &= \frac{2k_{0z}\lambda_{II}^d}{a_4(k_{zII}^d - k_{zI}^d)(k_{zII}^d - k_{zI}^u)(k_{zII}^d - k_{zII}^u)}
\end{aligned}$$

The vectors $(\hat{e}_I^u, \hat{e}_{II}^u)$ and $(\hat{e}_I^d, \hat{e}_{II}^d)$ correspond to the *type-I* and *type-II* waves in a general anisotropic medium, with different characteristic polarizations propagating (or decaying) in the upward and downward directions, respectively. It can be shown that the above general expression for the wave vectors will reduce to the horizontally polarized h-wave ($\hat{e}_I = \hat{h}$) and the vertically polarized v-wave ($\hat{e}_{II} = \hat{v}$) for an isotropic medium, the ordinary wave ($\hat{e}_I = \hat{o}$) and the extraordinary wave ($\hat{e}_{II} = \hat{e}$) [6] for a uniaxial medium, and the a-wave ($\hat{e}_I = \hat{a}$) and b-wave ($\hat{e}_{II} = \hat{b}$) [7] for a biaxial medium, respectively. Thus, the DGF given in (14) can be easily calculated for many different types of anisotropic medium.

One interesting thing noted here is that the dyadic forms in the DGF of the unbounded anisotropic medium are not always composed of two same vectors. Dyad composed of two same vectors holds true only for the reciprocal medium such as uniaxial or biaxial medium. For non-reciprocal medium such as gyrotropic medium, the dyad is composed of two different vectors as shown in (13b) and (13c).

3. DYADIC GREEN'S FUNCTION FOR TWO LAYER GEOMETRY WITH SOURCE INSIDE THE ISOTROPIC REGION

In this section, the DGF of two-layer geometry with source located inside isotropic region is considered. The geometry is shown in Fig. 1. Region 0 is free space and Region 2 is denoted as isotropic medium with relative permittivity ε_2 and permeability of μ_2. Region 1 is an anisotropic medium of thickness d, with relative permittivity tensor $\overline{\overline{\varepsilon_1}}$ and permeability tensor $\overline{\overline{I}}\mu_1$. The current source is located at $z = z'$ in Region 0.

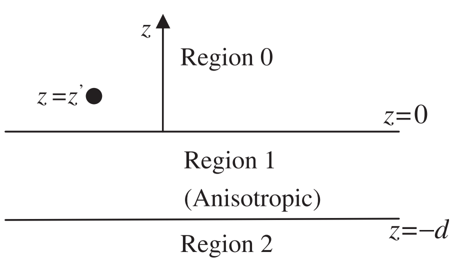

Figure 1. Geometry of the two layer problem.

Applying similar approach in [6, 7], the DGF of the layered geometry with Region 1 filled with general anisotropic medium are presented in this section. Denote $\overline{\overline{G}}_{ee}^{(0,0)}(\overline{r}, \overline{r}')$, $\overline{\overline{G}}_{ee}^{(1,0)}(\overline{r}, \overline{r}')$ and

$\overline{\overline{G}}_{ee}^{(2,0)}(\bar{r}, \bar{r}')$ as dyadic Green's functions for Region 0, Region 1 and Region 2, respectively, when the source is located at $z = z'$ in Region 0 for the two layer geometry problem and their expressions are given as follows.

$$\text{for} \quad 0 < z < z'$$

$$\overline{\overline{G}}_{ee}^{(0,0)}(\bar{r}, \bar{r}') = \frac{i}{8\pi^2}\int_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_0 z}\Bigg\{\Bigg[\hat{h}_0(-k_{0z})e^{i\bar{\kappa}_0\cdot\bar{r}} + R_{hh}\hat{h}_0(k_{0z})e^{i\bar{k}_0\cdot\bar{r}} + R_{hv}\hat{v}_0(k_{oz})e^{i\bar{k}_0\cdot\bar{r}}\Bigg]\hat{h}_0(-k_{0z}) + \Bigg[\hat{v}_0(-k_{0z})e^{i\bar{\kappa}_0\cdot\bar{r}} + R_{vh}\hat{h}_0(k_{0z})e^{i\bar{k}_0\cdot\bar{r}} + R_{vv}\hat{v}_0(k_{oz})e^{i\bar{k}_0\cdot\bar{r}}\Bigg]\hat{v}_0(-k_{0z})\Bigg\}e^{-i\bar{\kappa}_0\cdot\bar{r}'}, \tag{15a}$$

$$\text{for} \quad -d < z < 0$$

$$\overline{\overline{G}}_{ee}^{(1,0)}(\bar{r}, \bar{r}') = \frac{i}{8\pi^2}\int_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{0z}}\Bigg\{\Bigg[A_{heI}\hat{e}_I(k_{zI}^d)e^{i\bar{\kappa}_I\cdot\bar{r}} + B_{heI}\hat{e}_I(k_{zI}^u)e^{i\bar{k}_I\cdot\bar{r}} + A_{heII}\hat{e}_{II}(k_{zII}^d)e^{i\bar{\kappa}_{II}\cdot\bar{r}} + B_{heII}\hat{e}_{II}(k_{zII}^u)e^{i\bar{k}_{II}\cdot\bar{r}}\Bigg]\hat{h}_0(-k_{0z}) + \Bigg[A_{veI}\hat{e}_I(k_{zI}^d)e^{i\bar{\kappa}_I\cdot\bar{r}} + B_{veI}\hat{e}_I(k_{zI}^u)e^{i\bar{k}_I\cdot\bar{r}} + A_{veII}\hat{e}_{II}(k_{zII}^d)e^{i\bar{\kappa}_{II}\cdot\bar{r}} + B_{veII}\hat{e}_{II}(k_{zII}^u)e^{i\bar{k}_{II}\cdot\bar{r}}\Bigg]\hat{v}_0(-k_{0z})\Bigg\}e^{-i\bar{\kappa}_0\cdot\bar{r}'} \tag{15b}$$

$$\text{for} \quad z < -d$$

$$\overline{\overline{G}}_{ee}^{(2,0)}(\bar{r}, \bar{r}') = \frac{i}{8\pi^2}\int_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{0z}}\Bigg\{\Bigg[X_{hh}\hat{h}_2(-k_{2z})e^{i\bar{\kappa}_2\cdot\bar{r}} + X_{hv}\hat{v}_2(-k_{2z})e^{i\bar{\kappa}_2\cdot\bar{r}}\Bigg]\hat{h}_0(-k_{0z}) + \Bigg[X_{vh}\hat{h}_2(-k_{2z})e^{i\bar{\kappa}_2\cdot\bar{r}} + X_{vv}\hat{v}_2(-k_{2z})e^{i\bar{\kappa}_2\cdot\bar{r}}\Bigg]\hat{v}_0(-k_{0z})\Bigg\}e^{-i\bar{\kappa}_0\cdot\bar{r}'} \tag{15c}$$

In the above expression, $\bar{k}_n$ and $\bar{\kappa}_n$ denote the wave vectors of upward propagating (or decaying) wave and downward propagating (or decaying) waves along z-direction, respectively, in Region n ($n = 0$ or 2).

$$\begin{aligned} \bar{k}_0 &= \hat{x}k_x + \hat{y}k_y + \hat{z}k_{0z}, \quad \bar{\kappa}_0 = \hat{x}k_x + \hat{y}k_y - \hat{z}k_{0z} \\ \bar{k}_2 &= \hat{x}k_x + \hat{y}k_y + \hat{z}k_{2z}, \quad \bar{\kappa}_2 = \hat{x}k_x + \hat{y}k_y - \hat{z}k_{2z} \end{aligned} \tag{16}$$

$\hat{h_n}(\pm k_{nz})$ and $\hat{v_n}(\pm k_{nz})$ correspond to the horizontal and vertical polarizations of the electric field for upward and downward propagating waves in Region n ($n = 0$ or 2), respectively.

$$\begin{aligned} \hat{h_n}(+k_{nz}) &= \hat{h_n}(-k_{nz}) = \frac{\hat{z} \times \overline{k}_n}{k_\rho}, \quad k_\rho = \sqrt{k_x^2 + k_y^2} \\ \hat{v}_n(+k_{nz}) &= \frac{\hat{h_n}(+k_{nz}) \times \overline{k}_n}{k_n}, \quad \hat{v}_n(-k_{nz}) = \frac{\hat{h_n}(-k_{nz}) \times \overline{\kappa}_n}{k_n} \end{aligned} \tag{17}$$

As shown above, horizontal polarization corresponds to the wave polarized perpendicular to the plane of incidence while vertical polarization corresponds to the wave polarized parallel to the plane of incidence. The wave vectors and corresponding polarizations of characteristic waves of Region 1 (anisotropic region) are defined in Section 2 as $\overline{k}_I$, $\overline{k}_{II}, \overline{\kappa}_I, \overline{\kappa}_{II}$ and $\hat{e}_I^u(k_{zI}^u)$, $\hat{e}_{II}^u(k_{zII}^u)$, $\hat{e}_I^d(k_{zI}^d)$, $\hat{e}_{II}^d(k_{zII}^d)$.

The coefficients of the dyad in the DGF are obtained in terms of half-space reflection and transmission matrix using the matrix method of [6] as shown below.

$$\overline{\overline{R}} = \begin{bmatrix} R_{hh} & R_{vh} \\ R_{hv} & R_{vv} \end{bmatrix} = \overline{\overline{R}}^{01} + \overline{\overline{X}}^{10} \overline{\overline{R}}^{12} (I - \overline{\overline{R}}^{10} \overline{\overline{R}}^{12})^{-1} \overline{\overline{X}}^{01} \tag{18a}$$

$$\overline{\overline{D}} = \begin{bmatrix} A_{heI} & A_{veI} \\ A_{heII} & A_{veII} \end{bmatrix} = (I - \overline{\overline{R}}^{10} \overline{\overline{R}}^{12})^{-1} \overline{\overline{X}}^{01} \tag{18b}$$

$$\overline{\overline{U}} = \begin{bmatrix} B_{heI} & B_{veI} \\ B_{heII} & B_{veII} \end{bmatrix} = \overline{\overline{R}}^{12} (I - \overline{\overline{R}}^{10} \overline{\overline{R}}^{12})^{-1} \overline{\overline{X}}^{01} \tag{18c}$$

$$\overline{\overline{X}} = \begin{bmatrix} X_{hh} & X_{vh} \\ X_{hv} & X_{vv} \end{bmatrix} = \overline{\overline{X}}^{12} (I - \overline{\overline{R}}^{10} \overline{\overline{R}}^{12})^{-1} \overline{\overline{X}}^{01} \tag{18d}$$

The half-space reflection and transmission matrices $\overline{\overline{R}}^{10}$, $\overline{\overline{R}}^{12}$ and $\overline{\overline{X}}^{01}$, $\overline{\overline{X}}^{12}$ can be obtained from the boundary conditions at $z = 0$, and $z = -d$ [6]. It is noted here that in the above equation $\overline{\overline{R}}^{12}$ is reflection coefficient with reference plane at $z = 0$, the phase shift at $z = -d$ must be taken into account by the multiplication of the exponential terms. With the complete expression of the DGF for each region with source located inside Region 0, symmetrical property can be utilized to obtain the DGF with source inside the anisotropic region shown in the next section.

4. DYADIC GREEN'S FUNCTION FOR TWO LAYER GEOMETRY WITH SOURCE INSIDE THE ANISOTROPIC LAYER

If the source is located in Region 1 (anisotropic region), i.e., $-d < z' < 0$ instead of Region 0, the symmetrical property of DGF [16, 20] can be applied to obtain $\overline{\overline{G}}_{ee}^{(0,1)}$ from $\overline{\overline{G}}_{ee}^{(1,0)}$ using

$$\overline{\overline{G}}_{ee}^{(0,1)}(\overline{r}, \overline{r}') = \frac{\mu_1}{\mu_0} \left[\overline{\overline{G}}_{ee}^{(1,0)}(\overline{r}', \overline{r}) \right]^{\mathrm{T}} \tag{19}$$

provided that Region 1 is a reciprocal medium. Here, $\overline{\overline{G}}_{ee}^{(1,0)}$ is the DGF for the electric field in Region 1 with source located in Region 0, and $\overline{\overline{G}}_{ee}^{(0,1)}$ is the DGF for the field in Region 0 with source located in Region 1. The superscript T stands for the transpose. If Region 1 is non-reciprocal, i.e., the permittivity/permeability tensor is non-symmetric, and then the above relation (19) needs to be modified [21]. The modified symmetrical property of DGF for the non-reciprocal medium is given by

$$\overline{\overline{G}}_{ee}^{(0,1)}(\overline{r}, \overline{r}')\bigg|_{\overline{\overline{\varepsilon}}(\text{Region 1})=\overline{\overline{\varepsilon}}_1} = \frac{\mu_1}{\mu_0} \left[\overline{\overline{G}}_{ee}^{(1,0)}(\overline{r}', \overline{r})\bigg|_{\overline{\overline{\varepsilon}}(\text{Region 1})=\overline{\overline{\varepsilon}}_1^{\mathrm{T}}} \right]^{\mathrm{T}} \tag{20}$$

Thus for a non-reciprocal medium, in order to interchange the source and the field points in different regions, the medium property of the anisotropic region needs to be transposed. For a gyrotropic medium it implies that the direction of the biasing magnetic field needs to be reversed. It can be observed that (19) is a special case of (20) as $\overline{\overline{\varepsilon}}_1^{\mathrm{T}} = \overline{\overline{\varepsilon}}_1$ for a reciprocal medium. Applying the modified symmetrical property to (15b) in Section 3, we thus have for a non-magnetic medium

$$\begin{aligned}
\overline{\overline{G}}_{ee}^{(0,1)}(\overline{r}, \overline{r}') &= \left[\overline{\overline{G}}_{ee}^{(1,0)}(\overline{r}', \overline{r}) \right]^{\mathrm{T}} = \frac{i}{8\pi^2} \int_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{0z}} \\
&\Big\{ \hat{h}_0(-k_{0z}) e^{-i\overline{\kappa}_0 \cdot \overline{r}} \Big[A_{heI} \hat{e}_I(k_{zI}^d) e^{i\overline{\kappa}_I \cdot \overline{r}'} + B_{heI} \hat{e}_I(k_{zI}^u) e^{i\overline{k}_I \cdot \overline{r}'} \\
&+ A_{heII} \hat{e}_{II}(k_{zII}^d) e^{i\overline{\kappa}_{II} \cdot \overline{r}'} + B_{heII} \hat{e}_{II}(k_{zII}^u) e^{i\overline{k}_{II} \cdot \overline{r}'} \Big] \\
&+ \hat{v}_0(-k_{0z}) e^{-i\overline{\kappa}_0 \cdot \overline{r}} \Big[A_{veI} \hat{e}_I(k_{zI}^d) e^{i\overline{\kappa}_I \cdot \overline{r}'} + B_{veI} \hat{e}_I(k_{zI}^u) e^{i\overline{k}_I \cdot \overline{r}'} \\
&+ A_{veII} \hat{e}_{II}(k_{zII}^d) e^{i\overline{\kappa}_{II} \cdot \overline{r}'} + B_{veII} \hat{e}_{II}(k_{zII}^u) e^{i\overline{k}_{II} \cdot \overline{r}'} \Big] \Big\}
\end{aligned} \tag{21}$$

It needs to be noted here that $\overline{\overline{G}}_{ee}^{(1,0)}(\overline{r}', \overline{r})$ is obtained for the source inside the isotropic region with medium permittivity matrix of the anisotropic region as the transpose of the initial medium. Since the source and the field points are interchanged, the directions of the fields inside Region 0 need to be upward wave. Hence, the following transformations are applied:

$$\begin{aligned} & k_x \to -k_x, k_y \to -k_y \\ & \overline{\kappa}_0(-k_x, -k_y, -k_{0z}) = -\overline{k}_0(k_x, k_y, k_{0z}) \\ & \hat{h}_0(-k_x, -k_y, -k_{0z}) = -\hat{h}_0(k_x, k_y, k_{0z}) \\ & \hat{v}_0(-k_x, -k_y, -k_{0z}) = \hat{v}_0(k_x, k_y, k_{0z}) \end{aligned} \tag{22}$$

Using (22), (21) reduces to:

$$\begin{aligned} & z > 0 \quad \overline{\overline{G}}_{ee}^{(0,1)}(\overline{r}, \overline{r}') \\ = & \frac{i}{8\pi^2} \int_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{0z}} \Big\{ -\hat{h}_0(k_{0z}) e^{i\overline{k}_0 \cdot \overline{r}} \\ & \Big[A_{heI}(-k_x, -k_y) \hat{e}_I^d(-k_x, -k_y) e^{i\overline{\kappa}_I(-k_x, -k_y) \cdot \overline{r}'} \\ & + B_{heI}(-k_x, -k_y) \hat{e}_I^u(-k_x, -k_y) e^{i\overline{k}_I(-k_x, -k_y) \cdot \overline{r}'} \\ & + A_{heII}(-k_x, -k_y) \hat{e}_{II}^d(-k_x, -k_y) e^{i\overline{\kappa}_{II}(-k_x, -k_y) \cdot \overline{r}'} \\ & + B_{heII}(-k_x, -k_y) \hat{e}_{II}^u(-k_x, -k_y) e^{i\overline{k}_{II}(-k_x, -k_y) \cdot \overline{r}'} \Big] \\ & + \hat{v}_0(k_{0z}) e^{i\overline{k}_0 \cdot \overline{r}} \Big[A_{veI}(-k_x, -k_y) \hat{e}_I^d(-k_x, -k_y) e^{i\overline{\kappa}_I(-k_x, -k_y) \cdot \overline{r}'} \\ & + B_{veI}(-k_x, -k_y) \hat{e}_I^u(-k_x, -k_y) e^{i\overline{k}_I(-k_x, -k_y) \cdot \overline{r}'} \\ & + A_{veII}(-k_x, -k_y) \hat{e}_{II}^d(-k_x, -k_y) e^{i\overline{\kappa}_{II}(-k_x, -k_y) \cdot \overline{r}'} \\ & + B_{veII}(-k_x, -k_y) \hat{e}_{II}^u(-k_x, -k_y) e^{i\overline{k}_{II}(-k_x, -k_y) \cdot \overline{r}'} \Big] \Big\} \end{aligned} \tag{23}$$

The above equation is valid for the two-layer problem filled with either reciprocal or non-reciprocal medium. The wave vectors $\overline{k}_p(-k_x, -k_y)$ and $\overline{\kappa}_p(-k_x, -k_y)\,(p = I, II)$ are defined as in (10) with $\overline{k}_{zp}^u$ and $\overline{k}_{zp}^d$ that are obtained for the tangential wave number of $(-k_x, -k_y)$.

For a reciprocal medium, (21) can be further simplified using the relations of the *type-I* and *type-II* wave vectors and field vectors in

Region 1 as shown below.

$$\begin{aligned} \overline{\kappa}_I(-k_x, -k_y, k^d_{zI}) &= -\overline{k}_I(k_x, k_y, k^u_{zI}); \\ \overline{\kappa}_{II}(-k_x, -k_y, k^d_{zII}) &= -\overline{k}_{II}(k_x, k_y, k^u_{zII}) \\ \hat{e}^d_I(-k_x, -k_y, k^d_{zI}) &= -\hat{e}^u_I(k_x, k_y, k^u_{zI}); \\ \hat{e}^d_{II}(-k_x, -k_y, k^d_{zII}) &= \hat{e}^u_{II}(k_x, k_y, k^u_{zII}) \end{aligned} \tag{24}$$

Such substitution leads to the DGF for the two-layer uniaxial and biaxial medium shown in [6] and [10]. However, applying the above symmetrical property cannot provide the DGF for the field in the anisotropic region where the source is located. A direct construction method is provided in the following section to obtain the complete set of DGFs for all the regions of interest.

5. DYADIC GREEN'S FUNCTION FOR TWO LAYER GEOMETRY WITH SOURCE INSIDE THE ANISOTROPIC REGION USING DIRECT CONSTRUCTION METHOD

In the previous section, the DGF $\overline{\overline{G}}^{(0,1)}_{ee}(\overline{r}, \overline{r}')$ was obtained using the symmetrical property from $\overline{\overline{G}}^{(1,0)}_{ee}(\overline{r}, \overline{r}')$ with the source inside Region 0. The direct wave considered there was only the downward incident wave in the isotropic medium. Different from the problem with the source located inside the isotropic region, the method considered in this section consists of the direct upward and downward waves from the source as shown in Fig. 2.

In Fig. 2, 'a' wave and 'b' wave stand for the amplitude coefficient matrix of the unit vectors corresponding to the direct upward and downward wave generated by the source located inside the anisotropic region (Region 1) bounded by $z = 0$ and $z = -d$. 'A' and 'B' represent the amplitude coefficient matrix for unit vectors of the total upward and downward waves existing in Region 1 due to the multiple reflections of the 'a' and the 'b' waves at both the boundaries $z = 0$ and $z = -d$. 'C' and 'D' represent the amplitude coefficient matrix for unit vectors of the transmitted waves in Region 0 and Region 2, respectively. As seen from Fig. 2, all the waves in each region include two different polarizations. Inside the anisotropic region, the waves 'A' and 'B' include *type-I* and *type- II* polarizations, while the transmitted waves 'C' and 'D' inside the isotropic regions include the h and v-polarizations as described in Section 3. The above two-layer problem can be decomposed into two half-space problems with one corresponding to the reflection and transmission at the boundary $z = 0$

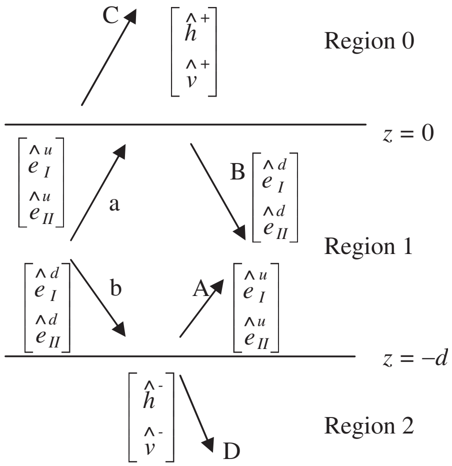

Figure 2. Representation of the waves and their corresponding polarizations existing in each region of the two layer problem with source inside the anisotropic region.

separating Region 0 and Region 1 and the other one corresponding to the reflection and transmission at the boundary $z = -d$ separating Region 1 and Region 2. The waves in each region can then be related through the half-space reflection and transmission coefficient matrices as follows.

$$\begin{aligned} A &= \overline{\overline{R}}^{12}(b+B), \quad B = \overline{\overline{R}}^{10}(a+A) \\ C &= \overline{\overline{X}}^{10}(a+A), \quad D = \overline{\overline{X}}^{12}(b+B) \end{aligned} \tag{25}$$

where $\overline{\overline{R}}^{ij}$ and $\overline{\overline{X}}^{ij}$ are the half-space reflection and transmission coefficient matrices with wave incident from Region i to Region j. Rewriting (23) such that A, B, C, and D are expressed in terms of the direct waves generated by the source, we obtain

$$\begin{aligned} A &= \overline{\overline{A}}^{a} a + \overline{\overline{A}}^{b} b, \quad B = \overline{\overline{B}}^{a} a + \overline{\overline{B}}^{b} b \\ C &= \overline{\overline{X}}^{a} a + \overline{\overline{X}}^{b} b, \quad D = \overline{\overline{T}}^{a} a + \overline{\overline{T}}^{b} b \end{aligned} \tag{26}$$

where

$$\begin{aligned} \overline{\overline{A}}^{a} &= \begin{bmatrix} R_{e_I^u e_I^u} & R_{e_{II}^u e_I^u} \\ R_{e_I^u e_{II}^u} & R_{e_{II}^u e_{II}^u} \end{bmatrix} = (I - \overline{\overline{R}}^{12}\overline{\overline{R}}^{10})^{-1}\overline{\overline{R}}^{12}\overline{\overline{R}}^{10} \\ \overline{\overline{A}}^{b} &= \begin{bmatrix} R_{e_I^d e_I^u} & R_{e_{II}^d e_I^u} \\ R_{e_I^d e_{II}^u} & R_{e_{II}^d e_{II}^u} \end{bmatrix} = (I - \overline{\overline{R}}^{12}\overline{\overline{R}}^{10})^{-1}\overline{\overline{R}}^{12} \end{aligned} \tag{27a}$$

$$\overline{\overline{B}}^a = \begin{bmatrix} R_{e_I^u e_I^d} & R_{e_{II}^u e_I^d} \\ R_{e_I^u e_{II}^d} & R_{e_{II}^u e_{II}^d} \end{bmatrix} = (\overline{\overline{I}} - \overline{\overline{R}}^{10}\overline{\overline{R}}^{12})^{-1}\overline{\overline{R}}^{10}$$
$$\overline{\overline{B}}^b = \begin{bmatrix} R_{e_I^d e_I^d} & R_{e_{II}^d e_I^d} \\ R_{e_I^d e_{II}^d} & R_{e_{II}^d e_{II}^d} \end{bmatrix} = (\overline{\overline{I}} - \overline{\overline{R}}^{10}\overline{\overline{R}}^{12})^{-1}\overline{\overline{R}}^{10}\overline{\overline{R}}^{12} \tag{27b}$$

$$\overline{\overline{X}}^a = \begin{bmatrix} X_{e_I^u h} & X_{e_{II}^u h} \\ X_{e_I^u v} & X_{e_{II}^u v} \end{bmatrix} = \overline{\overline{X}}^{10}(\overline{\overline{I}} - \overline{\overline{R}}^{12}\overline{\overline{R}}^{10})^{-1}$$
$$\overline{\overline{X}}^b = \begin{bmatrix} X_{e_I^d h} & X_{e_{II}^d h} \\ X_{e_I^d v} & X_{e_{II}^d v} \end{bmatrix} = \overline{\overline{X}}^{10}(\overline{\overline{I}} - \overline{\overline{R}}^{12}\overline{\overline{R}}^{10})^{-1}\overline{\overline{R}}^{12} \tag{27c}$$

$$\overline{\overline{T}}^a = \begin{bmatrix} T_{e_I^u h} & T_{e_{II}^u h} \\ T_{e_I^u v} & T_{e_{II}^u v} \end{bmatrix} = \overline{\overline{X}}^{12}(I - \overline{\overline{R}}^{10}\overline{\overline{R}}^{12})^{-1}\overline{\overline{R}}^{10}$$
$$\overline{\overline{T}}^b = \begin{bmatrix} T_{e_I^d h} & T_{e_{II}^d h} \\ T_{e_I^d v} & T_{e_{II}^d v} \end{bmatrix} = \overline{\overline{X}}^{12}(I - \overline{\overline{R}}^{10}\overline{\overline{R}}^{12})^{-1} \tag{27d}$$

The construction of the DGF in Region 1 is first considered and it needs special attention. The anisotropic medium is separated into two regions with one corresponding to the space above the source point (z') and the other corresponding to the space below the source point. For region above the source point, the direct wave includes the upward wave only, and for the region below source point, the direct wave includes downward wave only. With the DGF for the unbounded anisotropic region (Eq. (14)) and coefficients obtained from (27a) and (27b), we have for $z' < z < 0$,

$$\begin{aligned}
&\overline{\overline{G}}_{ee}^{(1,1)}(\overline{r},\overline{r}') \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \frac{i}{8\pi^2 k_{0z}}\bigg(c_I^u(k_{zI}^u)\hat{e}_I^u\hat{v}_I^u e^{i\overline{k}_I\cdot(\overline{r}-\overline{r}')} \\
&+c_{II}^u(k_{zII}^u)\hat{e}_{II}^u\hat{v}_{II}^u e^{i\overline{k}_{II}\cdot(\overline{r}-\overline{r}')} \\
&+\hat{e}_I^d e^{i\overline{\kappa}_I\cdot\overline{r}}\bigg(R_{e_I^u e_I^d} c_I^u\hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_I^d} c_{II}^u\hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'} \\
&+R_{e_I^d e_I^d} c_I^d\hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_I^d} c_{II}^d\hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\bigg) \\
&+\hat{e}_{II}^d e^{i\overline{\kappa}_{II}\cdot\overline{r}}\bigg(R_{e_I^u e_{II}^d} c_I^u\hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_{II}^d} c_{II}^u\hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'} \\
&+R_{e_I^d e_{II}^d} c_I^d\hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_{II}^d} c_{II}^d\hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\bigg) \\
&+\hat{e}_I^u e^{i\overline{k}_I\cdot\overline{r}}\bigg(R_{e_I^u e_I^u} c_I^u\hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_I^u} c_{II}^u\hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}
\end{aligned}$$

$$+R_{e_I^d e_I^u} c_I^d \hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_I^u} c_{II}^d \hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\Big)$$

$$+\hat{e}_{II}^u e^{i\overline{k}_{II}\cdot\overline{r}}\Big(R_{e_I^u e_{II}^u} c_I^u \hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_{II}^u} c_{II}^u \hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}$$

$$+R_{e_I^d e_{II}^u} c_I^d \hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_{II}^u} c_{II}^d \hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\Big)\Big), \tag{28a}$$

Similarly, for the region below the source point, we have for $-d < z < z'$,

$$\overline{\overline{G}}_{ee}^{(1,1)}(\overline{r},\overline{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \frac{i}{8\pi^2 k_{0z}}\Big(c_I^d(k_{zI}^d)\hat{e}_I^d\hat{v}_I^d e^{i\overline{\kappa}_I\cdot(\overline{r}-\overline{r}')}$$

$$+c_{II}^d(k_{zII}^d)\hat{e}_{II}^d\hat{v}_{II}^d e^{i\overline{\kappa}_{II}\cdot(\overline{r}-\overline{r}')}$$

$$+\hat{e}_I^d e^{i\overline{\kappa}_I\cdot\overline{r}}\Big(R_{e_I^u e_I^d} c_I^u \hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_I^d} c_{II}^u \hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}$$

$$+R_{e_I^d e_I^d} c_I^d \hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_I^d} c_{II}^d \hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\Big)$$

$$+\hat{e}_{II}^d e^{i\overline{\kappa}_{II}\cdot\overline{r}}\Big(R_{e_I^u e_{II}^d} c_I^u \hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_{II}^d} c_{II}^u \hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}$$

$$+R_{e_I^d e_{II}^d} c_I^d \hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_{II}^d} c_{II}^d \hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\Big)$$

$$+\hat{e}_I^u e^{i\overline{k}_I\cdot\overline{r}}\Big(R_{e_I^u e_I^u} c_I^u \hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_I^u} c_{II}^u \hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}$$

$$+R_{e_I^d e_I^u} c_I^d \hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_I^u} c_{II}^d \hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\Big)$$

$$+\hat{e}_{II}^u e^{i\overline{k}_{II}\cdot\overline{r}}\Big(R_{e_I^u e_{II}^u} c_I^u \hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'} + R_{e_{II}^u e_{II}^u} c_{II}^u \hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}$$

$$+R_{e_I^d e_{II}^u} c_I^d \hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'} + R_{e_{II}^d e_{II}^u} c_{II}^d \hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\Big)\Big) \tag{28b}$$

It is seen from (28a) and (28b), that the first two terms inside the integral represent the direct wave due to the sources, which are obtained from the DGF of the unbounded anisotropic region as shown in Section 2. All the other terms represent the upward ('A') and downward ('B') propagating waves reflected at the two boundaries.

Since the tangential electric field and magnetic field have to satisfy boundary condition at $z = 0$, the Green's function for Region 0 ($z > 0$) when source is located in the anisotropic region can be expressed in terms of coefficients from (27c) which are obtained for the transmitted wave 'C' as

$$\overline{\overline{G}}_{ee}^{(0,1)}(\overline{r},\overline{r}')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \frac{i}{8\pi^2 k_{0z}}$$
$$\left(\left[X_{e_I^u h}\hat{h}_0^+ e^{i\overline{k}_0\cdot\overline{r}} + X_{e_I^u v}\hat{v}_0^+ e^{i\overline{k}_0\cdot\overline{r}}\right] c_I^u(k_{zI}^u)\hat{v}_I^u e^{-i\overline{k}_I\cdot\overline{r}'}\right.$$
$$+\left[X_{e_{II}^u h}\hat{h}_0^+ e^{i\overline{k}_0\cdot\overline{r}} + X_{e_{II}^u v}\hat{v}_0^+ e^{i\overline{k}_0\cdot\overline{r}}\right] c_{II}^u(k_{zII}^u)\hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}$$
$$+\left[X_{e_I^d h}\hat{h}_0^+ e^{i\overline{k}_0\cdot\overline{r}} + X_{e_I^d v}\hat{v}_0^+ e^{i\overline{k}_0\cdot\overline{r}}\right] c_I^d(k_{zI}^d)\hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'}$$
$$\left.+\left[X_{e_{II}^d h}\hat{h}_0^+ e^{i\overline{k}_0\cdot\overline{r}} + X_{e_{II}^d v}\hat{v}_0^+ e^{i\overline{k}_0\cdot\overline{r}}\right] c_{II}^d(k_{zII}^d)\hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\overline{r}'}\right) \quad (28c)$$

Similarly, the DGF for the Region 2 (isotropic region) below the anisotropic slab can be derived from the coefficients (27d) corresponding to the transmitted wave 'D'. Thus we have for $z < -d$,

$$\overline{\overline{G}}_{ee}^{(2,1)}(\overline{r},\overline{r}')$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \frac{i}{8\pi^2 k_{0z}}\left(\left[T_{e_I^u h}\hat{h}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}} + T_{e_I^u v}\hat{v}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}}\right] c_I^u(k_{zI}^u)\hat{v}_I^u e^{-i\overline{k}_I\overline{r}'}\right.$$
$$+\left[T_{e_{II}^u h}\hat{h}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}} + T_{e_{II}^u v}\hat{v}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}}\right] c_{II}^u(k_{zII}^u)\hat{v}_{II}^u e^{-i\overline{k}_{II}\cdot\overline{r}'}$$
$$+\left[T_{e_I^d h}\hat{h}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}} + T_{e_I^d v}\hat{v}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}}\right] c_I^d(k_{zI}^d)\hat{v}_I^d e^{-i\overline{\kappa}_I\cdot\overline{r}'}$$
$$\left.+\left[T_{e_{II}^d h}\hat{h}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}} + T_{e_{II}^d v}\hat{v}_2^- e^{i\overline{\kappa}_2\cdot\overline{r}}\right] c_{II}^d(k_{zII}^d)\hat{v}_{II}^d e^{-i\overline{\kappa}_{II}\cdot\overline{r}'}\right) \quad (28d)$$

Note that $\hat{v}_I$, $\hat{v}_{II}$, the latter part of the dyad in (28a)–(28d) is taken from the result (Eq. (14)) of the DGF for the unbounded anisotropic medium in Section 2.

6. DISCUSSION

For a reciprocal medium, the coefficients of the dyad in the DGF for Region 0, with source located inside the anisotropic region, computed via the symmetrical property as in (23) and using the direct construction method as in (28c), agree with each other. The coefficients of the dyad in (23) are obtained from (18) with the substituted tangential wave vector $(-k_x, -k_y)$, while the coefficients of the dyad in (28) are obtained directly from (27) with the tangential wave vector of (k_x, k_y). Their expressions are repeated here for convenience.

$$\begin{aligned}\overline{\overline{D}} &= \begin{bmatrix} A_{heI} & A_{veI} \\ A_{heII} & A_{veII} \end{bmatrix} = (\overline{\overline{I}} - \overline{\overline{R}}^{10}\overline{\overline{R}}^{12})^{-1}\overline{\overline{X}}^{01} \\ \overline{\overline{U}} &= \begin{bmatrix} B_{heI} & B_{veI} \\ B_{heII} & B_{veII} \end{bmatrix} = \overline{\overline{R}}^{12}(\overline{\overline{I}} - \overline{\overline{R}}^{10}\overline{\overline{R}}^{12})^{-1}\overline{\overline{X}}^{01} \end{aligned} \tag{29a}$$

$$\begin{aligned}\overline{\overline{X}}^{a} &= \begin{bmatrix} X_{e_I^u h} & X_{e_{II}^u h} \\ X_{e_I^u v} & X_{e_{II}^u v} \end{bmatrix} = \overline{\overline{X}}^{10}(\overline{\overline{I}} - \overline{\overline{R}}^{12}\overline{\overline{R}}^{10})^{-1} \\ \overline{\overline{X}}^{b} &= \begin{bmatrix} X_{e_I^d h} & X_{e_{II}^d h} \\ X_{e_I^d v} & X_{e_{II}^d v} \end{bmatrix} = \overline{\overline{X}}^{10}(\overline{\overline{I}} - \overline{\overline{R}}^{12}\overline{\overline{R}}^{10})^{-1}\overline{\overline{R}}^{12} \end{aligned} \tag{29b}$$

It is verified numerically that the following relationship holds for (29a) and (29b),

$$\begin{aligned}&\begin{bmatrix} A_{heI}(-k_x,-k_y) & A_{heII}(-k_x,-k_y) \\ A_{veI}(-k_x,-k_y) & A_{veII}(-k_x,-k_y) \end{bmatrix} \\ &= \begin{bmatrix} c_I^u(k_x,k_y)X_{e_I^u h}(k_x,k_y) & -c_{II}^u(k_x,k_y)X_{e_{II}^u h}(k_x,k_y) \\ -c_I^u(k_x,k_y)X_{e_I^u v}(k_x,k_y) & c_{II}^u(k_x,k_y)X_{e_{II}^u v}(k_x,k_y) \end{bmatrix} \end{aligned} \tag{30a}$$

$$\begin{aligned}&\begin{bmatrix} B_{heI}(-k_x,-k_y) & B_{heII}(-k_x,-k_y) \\ B_{veI}(-k_x,-k_y) & B_{veII}(-k_x,-k_y) \end{bmatrix} \\ &= \begin{bmatrix} c_I^d(k_x,k_y)X_{e_I^d h}(k_x,k_y) & -c_{II}^d(k_x,k_y)X_{e_{II}^d h}(k_x,k_y) \\ -c_I^d(k_x,k_y)X_{e_I^d v}(k_x,k_y) & c_{II}^d(k_x,k_y)X_{e_{II}^d v}(k_x,k_y) \end{bmatrix} \end{aligned} \tag{30b}$$

The coefficients of the DGF obtained using the symmetrical property as in (29a) are composed of the half-space transmission matrix $\overline{\overline{X}}^{01}$ for the waves incident from Region 0 to Region 1. However, it does not represent the actual physical scenario of the problem of interest since the source is located inside Region 1.

On the other hand, the complete coefficients of the dyad obtained via the direct construction method, given by the RHS of (30a) and (30b), provide more physical insight to the problem. Each term in (30a) and (30b) is a product of the coefficients 'c' and 'X'. The coefficients 'c' (c_I^u, c_{II}^u and c_I^d, c_{II}^d) represent the amplitudes for the direct *type I* and *type II*, upward and downward waves due to the source in the unbounded anisotropic medium. The coefficients 'X' of (30a) and (30b) correspond to the two-layer transmission coefficients in (29b) for waves incident from Region 1 (where the source is) to Region 0. Each term of (29b) has its own physical interpretation. Since the source is embedded inside the bounded anisotropic slab, the direct wave generated by the source will experience multiple reflections at both boundaries. The total upward wave from the accumulation of all the reflections is indicated by $(\overline{\overline{I}} - \overline{\overline{R}}^{12}\overline{\overline{R}}^{10})^{-1}$. The transmission of the total upward wave from Region 1 to Region 0 is characterized by the half-space transmission matrix $\overline{\overline{X}}^{10}$.

7. CONCLUSIONS

Dyadic Green's functions are derived using eigen-decomposition and matrix method for the unbounded general anisotropic medium and layered anisotropic medium in this paper. It is shown that for unbounded reciprocal medium, the dyad in the DGF is composed of two same vectors which are the eigen-vectors of the adjoint of the wave matrix, corresponding to the characteristic polarizations of the waves in reciprocal medium. However, this relation does not hold for non-reciprocal medium and the different second vector has been derived.

Applying the concept of the eigen-decomposition for the general unbounded anisotropic medium and the matrix method to obtain the coefficients for the layered geometry, the DGFs for the layered problem with general anisotropic medium when the source is located inside the isotropic region are obtained. If the source is located inside the anisotropic region and the anisotropic region is a reciprocal medium, DGF of Region 0 above the source point can be obtained by applying the symmetrical property of DGF. If Region 1 is non-reciprocal medium, then the conventional symmetrical property needs to be modified. It is found that for a non-reciprocal medium such as a gyrotropic medium, an interchange of the source and the observation points in the two regions necessitates a reversal of the dc biasing magnetic field to calculate the corresponding DGF.

Modified symmetrical property of DGF simplifies the process to obtain the DGF. However, applying the symmetrical property can not provide the complete set of DGFs for all the regions when the source is located inside the anisotropic slab. Also, the available symmetrical property doesn't apply to the medium with magnetic anisotropy. A new method to construct the DGFs of layered medium with electric anisotropy directly from the characteristic waves in each region using the eigen-decomposition and matrix method is presented in this paper. This method can easily be extended to calculate the DGFs for a multilayered geometry filled with general anisotropic (electric or magnetic) medium with source located in any region.

Furthermore, the DGF obtained via direct construction method is compared with the DGF obtained using symmetrical property for the reciprocal medium case. Interesting relationship for the coefficients of the dyad in the DGFs obtained through two different methods is observed and discussed. Thus a straightforward physical insight to the DGFs is revealed when the source is located inside the anisotropic region as compared to the results obtained using the symmetrical property. The DGFs obtained in this paper have wide applications in the scattering and radiation from arbitrarily shaped 3D objects located inside the anisotropic slab.

REFERENCES

1. Bunkin, F. V., "On radiation in anisotropic media," *JETP Lett.*, Vol. 5, No. 2, 338–346, Sep. 957.
2. Arbel, E. and L. B. Felsen, "Theory of radiation from sources in anisotropic media. Part I: General sources in stratified media. Part II: Point source in infinite homogeneous medium," *Electromagnetic Waves*, 391–459, E. C. Jordan (ed.), Pergamon Press, New York, 1963.
3. Wu, C. P., "Radiation from dipoles in a magneto-ionic medium," *IEEE Trans. Antennas Propagation*, Vol. 11, 681–689, 1963.
4. Tsalamengas, J. L., "Electromagnetic fields of elementary dipole antennas embedded in stratified general anisotropic media," *IEEE Trans. Antennas Propagation*, Vol. 37, No. 3, 399–403, 1989.
5. Hsia, I. Y. and N. G. Alexopoulos, "Radiation characteristics of Hertzian dipole antennas in a nonreciprocal superstrate-substrate structure," *IEEE Trans. Antennas Propagation*, Vol. 40, No. 7, 782–790, Jul. 1992.
6. Lee, J. K. and J. A. Kong, "Dyadic Green's functions for layered anisotropic medium," *Electromagnetics*, Vol. 3, 111–130, 1983.
7. Mudaliar, S. and J. K. Lee, "Dyadic Green's function for two-layered biaxially anisotropic medium," *Journal of Electromagnetic Waves and Applications*, Vol. 10, No. 7, 909–923, 1996.
8. Eroglu, A. and J. K. Lee, "Dyadic Green's functions for an electrically gyrotropic medium," *Progress In Electromagnetics Research*, Vol. 58, 223–241, 2006.
9. Tsang, L., E. Njoku, and J. A. Kong, "Microwave thermal emission from a stratified medium with nonuniforn temperature distribution," *Journal of Applied Physics*, Vol. 46, No. 12, Dec. 1975.
10. Pettis, G. and J. K. Lee, "Radiation of Hertzian dipoles embedded in planarly layered biaxial media," *The Second*

IASTED International Conference on Antennas, Radar, and Wave Propagation, Banff, AB, Canada, Jul. 19–21, 2005.
11. Krowne, C. M., "Green's function in the spectral domain for biaxial and uniaxial anisotropic planar dielectric structures," *IEEE Trans. Antennas and Propagation*, Vol. 32, No. 12, Dec. 1984.
12. Mesa, F. L., R. Marques, and M. Horno, "A general algorithm for computing the bidimensional spectral Green's dyad in multilayered complex bianisotropic media: The equivalent boundary method," *IEEE Trans. Microwave Theory and Techniques*, Vol. 39, No. 9, Sep. 1991.
13. Li, L. W., P. S. Kooi, M. S. Leong, and T. S. Yeo, "On the eigenfunction expansion of dyadic Green's function in planarly stratified media," *Journal of Electromagnetic Waves and Applications*, Vol. 8, No. 6, 663–678, 1994.
14. Li, L. W., N. H. Lim, W.-Y. Yin, and J.-A. Kong, "Eigen functional expansion of dyadic Green's functions in gyrotropic media using cylindrical vector wave functions," *Progress In Electromagnetics Research*, Vol. 43, 101–121, 2003.
15. Vegni, L., R. Cicchetti, and P. Capece, "Spectral dyadic Green's function formulation for planar integrated structures," *IEEE Trans. Antennas and Propagation*, Vol. 36, No. 8, 1057–1065, Aug. 1988.
16. Lee, J. K. and J. A. Kong, "Active microwave remote sensing of an anisotropic random medium layer," *IEEE. Trans. Geoscience and Remote Sensing*, Vol. 23, No. 6, 910–923, Nov. 1985.
17. Chen, H., *Theory of Electromagnetic Wave: A Coordinate Free Approach*, McGraw-Hill, NY, 1983.
18. Zhuck, N. P. and A. S. Omar, "Radiation and low-frequency scattering of EM waves in a general anisotropic homogeneous medium," *IEEE Trans. Antennas and Propagation*, Vol. 47, No. 8, 1364–1373, Aug. 1999.
19. Trefethen, L. N. and D. Bau, *Numerical Linear Algebra*, SIAM, Society for Industrial and Applied Mathematics, 1997.
20. Tai, C. T., *Dyadic Green's Functions in Electromagnetic Theory*, 2nd Edition, IEEE Press, NY, 1994.
21. Huang, Y. and J. K. Lee, "The modified symmetrical property of the dyadic Green's functions for the non-reciprocal medium," EECS Dept. Technical Report, Syracuse University, Jun. 2010.