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EQUIVALENT CIRCUIT MODEL FOR DESIGNING COUPLED RESONATORS PHOTONIC CRYSTAL FILTERS

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Abstract—A method for modeling and designing of coupled resonators photonic crystal (PC) filters for wavelength division multiplexing (WDM) systems is presented. This proposed method is based on coupling coefficients of intercoupled resonators and the external quality factors of the input and output resonators based on the circuit approach. A general formulation for extracting the two types of parameters from the physical structure of the PC filters is given. At last, we redesign a third-order Chebyshev filter which has a center frequency of 193.55 THz, a flat bandwidth of 50 GHz, and ripples of 0.1 dB in the pass-band. The filter's structure derived from the proposed method is more compact.

1. INTRODUCTION

Photonic crystal filters are the essential components of photonic integrated circuits and optical communication systems [1–5]. High-Q-factor optical resonant filters, utilizing a single-defect mode in PC, have been demonstrated experimentally. And the transmission spectra of such filters are Lorentzian [6, 7]. For optical resonant filters used in WDM optical communication systems, the transmission characteristics need to be improved, so as to have steep roll-off and flattened pass-band. This demands higher order filters. Higher order filters can be created by coupling multiple resonators. A third-order filter [8, 9] and an N-coupled-resonators filter [10] have been designed for improving the filtering performance. Presently, an approach based on the time domain coupled-mode theory (CMT) [11] was adopted for analyzing and designing many types of filters [12–15] including the coupled-resonators PC filters [9]. Based on the CMT method, the coupling

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between two resonators can be treated as if the resonators interact through a waveguide with a phase shift. However, the phase shift is determined by the center frequency of the filter, the effective index of the waveguide and the choice of reference planes. The exact computation of the phase shift would be difficult. The relatively simple model used for description of coupling structures, which was previously developed in solid state physics and known as tight-binding approximation [16], make them all more attractive for investigation and applications. The linear interaction of light with photonic cavities is analogous to interaction of electrons with quantum dots in solid state physics [17] once one ignores the polarization effects in the former and the multi-particle nature in the latter. If the cavities are relatively small and contain a few eigenstates, the light propagation through series of them is essentially one-dimensional. Therefore, a device built on these cavities is analogous to an electric circuit.

Recently, some approximate methods such as effective impedance model [18–20] and transmission-line model [21] which had been used in analyzing microwave phenomena are applied to analyze the PC and PC waveguides. In this paper, we propose a simple method to model photonic crystal filters by using the multiple cavities which based on the coupling matrix. The coupling matrix is important for representing a wide range of multi-coupled-resonator filters topologies [22, 23]. This idea is based on coupling coefficients of intercoupled resonators and the external quality factors of the input and output resonators. These parameters can be easily extracted by means of a numerical method, such as the finite-difference time-domain (FDTD) method. The frequency characteristic of the filter is developed directly from the circuit approach, which introduce the microwave filter theory to design the PC filters and avoid the calculations of the phase shift between the resonators. Although the derivations are based on circuit models, the outcomes are also valid for any other type of filter on a narrow-band basis.

2. THEORETICAL MODEL

The structure of the filter is shown in Fig. 1(a) with its schematic diagram shown in Fig. 1(b). It consists of n resonators and input/output waveguide. By lumped parameter approximation with lossless, the model of the filter is represented as the circuit which is illustrated in Fig. 1(c), where L, C and R denote the inductance, capacitance, and resistance, respectively; i represents the loop current; and e_s the voltage source. Generally, the coupling between two resonators can be magnetic or electric or even the combination of both.

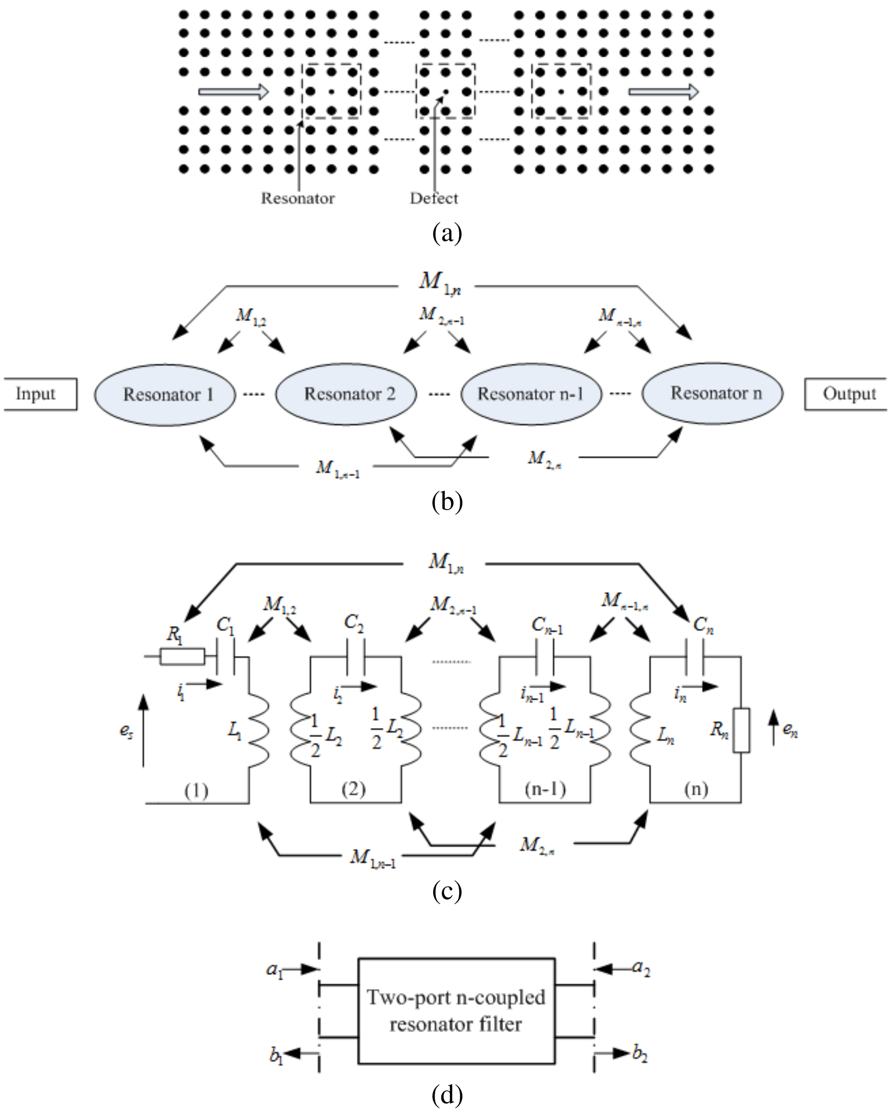

Figure 1. (a) Structure of the coupled resonator filter in a PC, which is composed of n resonators. (b) Schematic diagram of the filter. (c) Equivalent circuit of the structure in (a). (d) Its network representation.

Here we assume that the coupling is magnetic. By using the voltage law, which is one of Kirchhoff's two circuit laws and states that the algebraic sum of the voltage drops around any closed path in a network is zero, we can write down the loop equations for the circuit of Fig. 1(c):

$$\begin{aligned}
\left(R_1 + jwL_1 + \frac{1}{jwC_1}\right) i_1 - jwL_{12} i_2 \cdots - jwL_{1n} i_n &= e_s \\
-jwL_{21} i_1 + \left(jwL_2 + \frac{1}{jwC_2}\right) i_2 \cdots - jwL_{2n} i_n &= 0 \\
&\vdots \\
-jwL_{n1} i_1 - jwL_{n2} i_2 \cdots \left(R_n + jwL_n + \frac{1}{jwC_n}\right) i_n &= e_n
\end{aligned} \tag{1}$$

in which $L_{ij} = L_{ji}$ represents the mutual inductance between resonator i and j, and the all loop currents are supposed to have the same direction, so that the voltage drops due to the mutual inductance have a negative sign. This set of equations can be represented in matrix form

$$\begin{bmatrix} R_1 + jwL_1 + \frac{1}{jwC_1} & -jwL_{12} & \cdots & -jwL_{1n} \\ -jwL_{21} & jwL_2 + \frac{1}{jwC_2} & \cdots & -jwL_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -jwL_{n1} & -jwL_{n2} & \cdots & R_n + jwL_n + \frac{1}{jwC_n} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} = \begin{bmatrix} e_s \\ 0 \\ \vdots \\ e_n \end{bmatrix} \tag{2}$$

or

$$[Z] \cdot [i] = [e]$$

where $[Z]$ is an $n \times n$ impedance matrix.

For simplicity, let us first consider a synchronously tuned filter. In this case, the all resonators resonate at the same frequency, namely the mid-band frequency of filter $w_0 = 1/\sqrt{LC}$, where $L = L_1 = L_2 = \ldots = L_n$ and $C = C_1 = C_1 = \ldots = C_n$. The impedance matrix in Eq. (2) may be expressed by

$$[Z] = w_0 L \cdot [\bar{Z}] \tag{3}$$

where $[\bar{Z}]$ is the normalized impedance matrix, which in the case of synchronously tuned filter is given with assuming $w/w_0 \approx 1$ for a narrow-band approximation:

$$[\bar{Z}] = \begin{bmatrix} \frac{1}{Q_{e1}} + \Omega & -jM_{12} & \cdots & -jM_{1n} \\ -jM_{21} & \Omega & \cdots & -jM_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -jM_{n1} & -jM_{n2} & \cdots & \frac{1}{Q_{en}} + \Omega \end{bmatrix} \tag{4}$$

where $\Omega = j(w/w_0 - w_0/w)$ is the normalized complex low-pass frequency variable, $Q_{ei} = w_0 L/R_i$ (for $i = 1$ or n) is the external quality factor, and $M_{ij} = L_{ij}/L$ is the coupling coefficient.

A network representation of the circuit of Fig. 1(c) is shown in Fig. 1(d), a_1, a_2, b_1 and b_2 are the wave variables. Referring to the relationships between the wave variables and the voltage and current variables, we can get

$$a_1 = \frac{e_s}{2\sqrt{R_1}},\ b_1 = \frac{e_s - 2i_1 R_1}{2\sqrt{R_1}},\ a_2 = 0,\ b_2 = i_n\sqrt{R_n} \tag{5}$$

and hence the transmission coefficient and the reflection coefficient can be expressed as

$$S_{21} = \frac{b_2}{a_1}\ |_{a_2=0} = \frac{2}{\sqrt{Q_{e1}\cdot Q_{en}}}\left[\bar{Z}\right]_{n1}^{-1} \tag{6}$$

$$S_{11} = \frac{b_1}{a_1}|_{a_1=0} = 1 - \frac{2}{\sqrt{Q_{e1}}}\left[\bar{Z}\right]_{11}^{-1} \tag{7}$$

where $[\bar{Z}]_{ij}^{-1}$ denotes the ith row and jth column element of $[\bar{Z}]^{-1}$.

If the coupling is electric, the formulation of normalized admittance matrix is identical to that of normalized impedance matrix. It implies that we could have a unified formulation for an n-coupled resonator filter regardless of whether the couplings are magnetic or electric or even the combination of both. Accordingly, the transmission coefficient and the reflection coefficient may be incorporated into a general one:

$$S_{21} = 2\frac{1}{\sqrt{Q_{e1}\cdot Q_{en}}}[A]_{n1}^{-1} \tag{8}$$

$$S_{11} = \pm\left(1 - \frac{2}{\sqrt{Q_{e1}}}[A]_{11}^{-1}\right) \tag{9}$$

with $[A] = [Q] + \Omega[U] - j[M]$, where $[Q]$ is an $n \times n$ matrix with all entries zero, except for $Q_{11} = Q_{e1}$ and $Q_{nn} = Q_{en}$, $[U]$ is the $n\times n$ unit or identity matrix, and $[M]$ is the so-called general coupling matrix, which is an $n \times n$ reciprocal matrix (i.e., $M_{ij} = M_{ji}$). In this paper, we call the method the coupling resonator method (CRM).

For filters based on resonators structures, the external quality factor and the coupling coefficient can be extracted as [23]:

$$Q_e = \frac{2w_0}{\Delta w_{3\mathrm{dB}}} \tag{10}$$

$$M_{ij} = \pm\left(\frac{w_{0j}}{w_{0i}} + \frac{w_{0i}}{w_{0j}}\right)\sqrt{\left(\frac{w_j^2 - w_i^2}{w_j^2 + w_i^2}\right)^2 - \left(\frac{w_{0j}^2 - w_{0i}^2}{w_{0j}^2 + w_{0i}^2}\right)^2} \tag{11}$$

where w_0 and $\Delta w_{3\mathrm{dB}}$ represent the resonant frequency and bandwidth for a doubly loaded resonator respectively; $w_{0i} = (L_i C_i)^{-1/2}$ and $w_{0j} = (L_j C_j)^{-1/2}$ are the two resonant frequencies of uncoupled resonators, w_i and w_j are the characteristic frequencies of two coupled resonators corresponding to odd and even modes which are shown in Fig. 2.

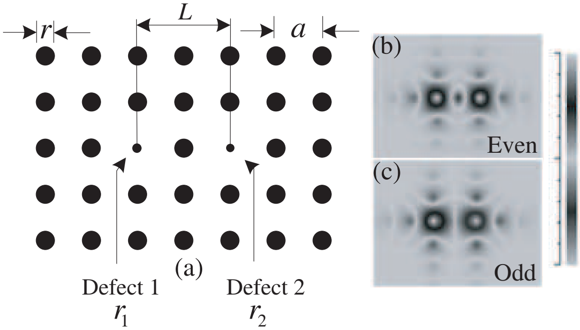

Figure 2. (a) Structure of two coupled resonators formed by point defects. And electric field profiles of two states of two coupled resonators: (b) even mode, (c) odd mode.

It is remarked that the interaction of the coupled resonators is mathematically described by the dot operation of their space vector fields, which allows the coupling to have either positive or negative sign. A positive sign would imply that the coupling enhances the stored energy of uncoupled resonators, whereas a negative sign would indicate a reduction. Therefore, the electric and magnetic couplings could either have the same effect if they have the same sign, or have the opposite effect if their signs are opposite. In this paper, we specify that the positive sign is taken when the frequency of odd mode is greater than that of even mode; conversely, the negative sign is taken.

If the coupling coefficients of the resonators are obtained, the coupling matrix is formed. Then the frequency response of the filter can be calculated by the Eq. (8) and Eq. (9). It can be seen that the normalized transmission spectrum is determined by two parameters: the quality factor Q_e, and the coupling coefficient M_{ij}.

At first the effect of the quality factor is considered in case of two identical resonators. The transmission spectra of the filters with the different quality factors are shown in Fig. 3(a). In this situation, we assume that the coupling between the two cavities is unchanged. It can be seen that the degree of resonance peaks' deviator factor decreases with decreasing quality factor but not with its bandwidth,

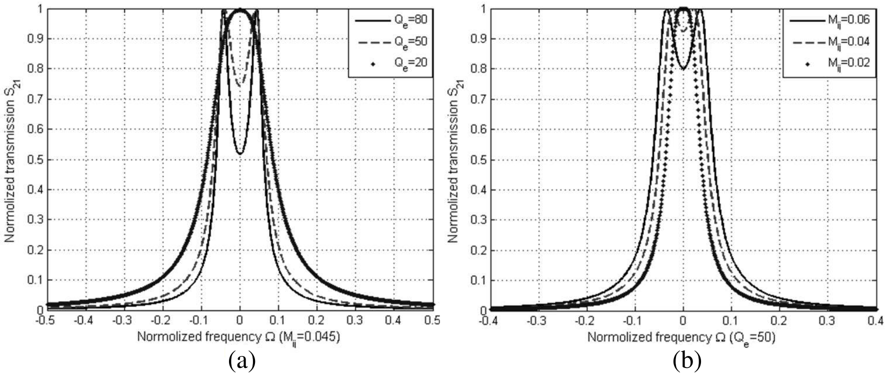

Figure 3. Comparison of the transmission spectra of the coupled cavities filter with different parameters. (a) Different external quality factors: $Q_e = 20$ (dot line), $Q_e = 50$ (dash line), and $Q_e = 80$ (solid line). (b) Different coupling coefficients: $M_{ij} = 0.02$ (dot line), $M_{ij} = 0.04$ (dash line), $M_{ij} = 0.06$ (solid line).

so the full-width at half maximum (FWHM) Δw remains virtually unchanged. And the larger of deviator factors the sharper roll-off of the transfer response. But the amplitude of the in-band ripple grows with increasing of the external quality factor. Subsequently, Fig. 3(b) shows the transmission spectra of the filters which are composed of the same two coupled cavities with different coupling coefficients and fixed quality factor. It shows that the bandwidth and the in-band ripple increase with the increase of coupling intensity, but the roll-off of transfer response is almost unchanged. So designing a filter with desired response requires a rational selection of external quality factors and coupling coefficients.

3. EXAMPLE

To demonstrate the robustness of the proposed method for modeling photonic crystal filters, we present an example in this section. In order to design the higher order PC filter, the resonators are separately designed in a 2-D PC waveguide to have the determined center frequency and the proper Q_e factors. Those parameters may be tuned by changing the radius of the defects and the radius of the rods near the defects, then, the coupling coefficient can be extracted by treating every two resonators as a whole. By adjusting the radius of the rods between the defects, the coupling coefficient can also be tuned to approach that we desired. With the software 'Rsoft' [24], the values of the above parameters can be obtained.

At last, we redesign the coupled-resonators band-pass photonic crystal filter by the method presented in the paper. The specifications for the filter under consideration are [9]:

- Center frequency 193.55 THz
- Flat bandwidth 50 GHz
- Pass-band ripple 0.1 dB

First, an $n = 3$ Chebyshev low-pass prototype is required, and, the external quality factors and the coupling coefficients can also be obtained by Synthesis methods [22, 23], which are shown followed:

- $Q_{e1} = Q_{e3} = 4800$
- $M_{1,2} = M_{2,3} = 0.093,\ M_{1,3} = 0.0084$

Second, we choose a PC topology structure which two parameters are closed to that obtained above. The rods in air type PC based on square lattice is adopted with the radius and the dielectric constant of the rods in the background 2-D PC are set to $0.2a$ (here $a = 580$ nm) and 11.56 respectively. In this paper, all the dimensions are specified in unit of the lattice constant so that future studies can make use of the scalability of PC. By plane-wave expansion technique, the normalized frequency of the TE band-gap of the PC is 0.28547 to 0.41987. To form a waveguide, a row of rods are removed in the complete PC and the dispersion diagram of the guide mode is shown in Fig. 4(a). Then, it is found that the resonance frequency of 193.55 THz is attained when the three defect rods $r_1 = r_2 = r_3 = 0.098a$. The dependence of the resonant frequency of the coupled cavity on the coupled defect radii is shown in Fig. 4(b).

It is known that the coupling coefficients are affected by the position of the resonators, so we first extract the coupling coefficients with different position and the dimension of the defects, which are shown in Table 1 and Table 2.

Substantial work is to set the three coupled cavities at proper position so that the coupling coefficients of every two resonators are

Table 1. Coupling coefficients of different length between the two resonators.

$L(a)$	3	4	5	6	7
M_{ij}	0.116	0.0502	0.0125	0.0032	0.00083

*The table shows the relation between the coupling coefficients and the different length of the two defects. The radius of the two defect rods are both $0.1a$.

Table 2. Coupling coefficients of two resonators with different defect radius.

r_i	0.1	0.09	0.08	0.07
(resonant frequency)	(0.32585)	(0.335044)	(0.343505)	(0.351339)
r_j	0.1	0.11	0.12	0.13
(resonant frequency)	(0.32585)	(0.316988)	(0.308489)	(0.30083)
M_{ij}	0.0502	0.2412	0.3463	0.4273

*The table shows the relation between the coupling coefficients and the radius of the two individual defects with its normalized resonator frequencies in the brackets. The length of the two defects L is $4a$.

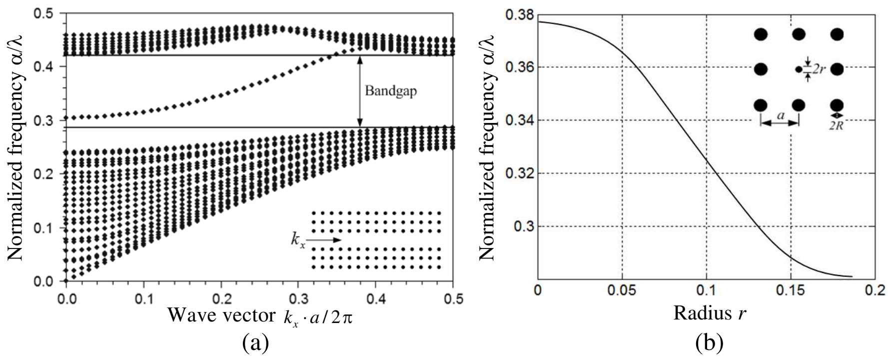

Figure 4. (a) Dispersion diagram of the guided mode inside the PBG (photonic band gap) in the Γ-X direction. (b) Dependence of the resonant frequency of the coupled cavity on the coupled defect radii.

close to the calculated values. The coupling coefficients are considered at first. We first put the three resonators in a straight line with $4a$ spacing of which the topology structure is shown in Fig. 5(a), then, extract the coupling coefficients of them, the results are: $M_{1,2} = M_{2,3} = 0.051$, $M_{1,3} = 0.000092$. It appears very different from the desired ones. After extensive calculations, it has been found that $M_{1,2}$ and $M_{2,3}$ and $M_{1,3}$ factors of 0.072, 0.072 and 0.0049 are attained when the second resonator is placed below the waveguide shown in Fig. 5(b).

Then, considering that the bandwidth and the ripple of the pass-band are both affected by coupling intensity, we first tune the second defect rod r_2 to $0.96a$ so as to make the band-width of the device approach the given specification, next, the radius of the rods r_a placed between defect and waveguide are fine tuned to $0.018a$ to make the pass-band ripple reduced to less than 0.1 dB. The theoretical result

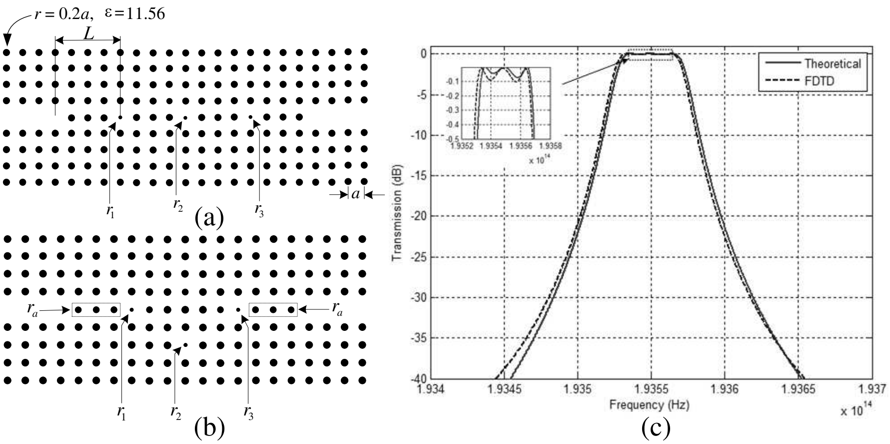

Figure 5. (a), (b) Schematic diagram of the designed filter, which is composed of 3 resonators. (c) Transmission spectra of the desired filter by the synthesized method (real line) and the simulation result by the FDTD method (dotted line).

which uses Eq. (8) is plotted as real line and compared to the result obtained by 2-D FDTD method with dotted line in Fig. 5(c). It indicates that the method presented in this paper is valid for designing coupled-resonators band-pass photonic crystal filters.

In order to obtain the desired filter based on photonic crystal, in general, the structure parameters need to be adjusted, that consumes substantial computation. However, based on the external quality factors and the coupling coefficients deduced from circuit approach model, it would be purposeful to change the structure when the relationship between the frequency characteristics and the structure parameters of the filter is known.

4. CONCLUSION

We presented a circuit-based approach for modeling and designing coupled-resonators band-pass photonic crystal filters. It was shown that a chain of serially coupled-resonators can be represented by an equivalent baseband LC ladder network in the narrowband approximation. By introducing the external quality factor and the coupling coefficient, the circuit model allows the standard analog-filter-realization techniques to be directly applied to design the coupled-resonators filters based on photonic crystal. Compared with the previous methods, the proposed method is simple and efficient for designing the band-pass PCs filters, which avoids the calculation of the phase shift between the resonators. And the structure derived from the method is more compact. Examples were provided to illustrate the application of the technique for designing the standard Chebyshev filters, and the characteristics were in good agreement with the design specifications, so the designed filter is suitable for the use in WDM optical communication systems.

REFERENCES

1. Joannopoulos, J. D., S. G. Johnson, J. N. Winn, and R. D. Meade, *Photonic Crystal: Molding the Flow of Light*, Princeton Univ. Press, Princeton, 1995.
2. Yanik, M. F., H. Altug, J. Vuckovic, and S. Fan, "Submicrometer all-optical digital memory and integration of nanoscale photonic devices without isolator," *IEEE J. Lightw. Techno.*, Vol. 22, 2316–2322, 2004.
3. Koshiba, M., "Wavelength division multiplexing and demultiplexing with photonic crystal waveguide coupler," *IEEE J. Lightw. Techno.*, Vol. 19, 1970–1975, 2001.
4. Mekis, M. Meier, A. Dodabalapur, R. E. Slusher, and J. D. Joannopoulos, "Lasing mechanism in two dimensional photonic crystal lasers," *Appl. Phys. A*, Vol. 69, 111–114, 1999.
5. Yanik, M. F., S. Fan, M. Soljacic, and J. D. Joannopoulos, "All-optical transistor action with bistable switching in a photonic crystal cross-waveguide geometry," *Optics Letters*, Vol. 28, 2506–2508, 2003.
6. Chen, J. C., H. A. Haus, S. Fan, P. R. Villeneuve, and J. D. Joannopoulos, "Optical filters from photonic band gap air bridges," *IEEE J. Lightw. Techno.*, Vol. 14, 2575–2580, 1996.
7. Imada, M., S. Noda, A. Chutinan, M. Mochizuki, and T. Tanaka, "Channel drop filter using a single defect in a 2-D photonic crystal slab waveguide," *IEEE J. Lightw. Techno.*, Vol. 20, 873–878, 2002.
8. Costa, R., A. Melloni, and M. Martinelli, "Bandpass resonant filters in photonic-crystal waveguides," *IEEE Photon. Techno. Letters*, Vol. 15, 401–403, 2003.
9. Park, D., S. Kim, I. Park, and H. Lim, "Higher order optical resonant filters based on coupled defect resonators in photonic crystals," *IEEE J. Lightw. Techno.*, Vol. 23, 1923–1928, 2005.
10. Li, X. C., J. Xu, K. Xu, A. Q. Liu, and J. T. Lin, "A side-coupled photonic crystal filter with sidelobe suppression," *Appl. Phys. A*, Vol. 89, 327–332, 2007.

11. Haus, H. A., *Wave and Fields in Optoelectronics*, Prentice-Hall, Englewood Cliffs, NJ, 1984.
12. Fan, S., P. Villeneuve, and J. Joannopoulos, "Channel drop filters in photonic crystals," *Opt. Express*, Vol. 3, 4–11, 1998.
13. Chen, C., X. Li, H. Li, K. Xu, J. Wu, and J. Lin, "Bandpass filters based on phase-shifted photonic crystal waveguide gratings," *Opt. Express*, Vol. 15, 11278–11284, 2007.
14. Fasihi, K. and S. Mohammadnejad, "Highly efficient channel-drop filter with a coupled cavity-based wavelength-selective reflection feedback," *Opt. Express*, Vol. 17, 8983–8997, 2009.
15. Akahane, Y., T. Asano, H. Takano, B. S. Song, Y. Takana, and S. Noda, "Two-dimensional photonic-crystal-slab channel-drop filter with flat-top response," *Opt. Express*, Vol. 13, 2512–2530, 2005.
16. Ashcroft, N. W. and N. D. Mermin, *Solid State Physics*, Saunders College, Philadelphia, 1976.
17. Kouwenhoven, L. P., C. M. Marcus, P. L. McEuen, S. Tarucha, R. M. Westervelt, and N. S. Wingreen, "Electron transport in quantum dots," *Proceedings of the NATO Advanced Study Institute on Mesoscopic Electron Transport*, Curaçao, Netherlands, Antilles, 1996.
18. Mohtashami, A., J. Zarbakhsh, and K. Hingerl, "Advances impedance matching in photonic crystal waveguides," *Opt. Quant. E*, Vol. 39, 387–394, 2007.
19. Biswas, R., Z. Y. Li, and K. M. Ho, "Impedance of photonic crystals and photonic crystal waveguides," *App. Physics Letters*, Vol. 84, 1254–1256, 2004.
20. Momeni, B., A. A. Eftekhar, and A. Adibi, "Effective impedance model for analysis of reflection at the interfaces of photonic crystals," *Optics Letters*, Vol. 32, 778–780, 2007.
21. Miri, M., A. Khavasi, K. Mehrany, and B. Rashidian, "Transimission-line model to design matching stage for light coupling into two-dimensional photonic crystals," *Optics Letters*, Vol. 35, 115–117, 2010.
22. Cameron, R. J., "General coupling matrix synthesis methods for chebyshev filtering functions," *IEEE Trans. on MTT*, Vol. 47, 433–442, 1999.
23. Hong, J. S. and M. J. Lancaster, *Microstrip Filters for RF/Microwave Applications*, John Wiley and Sons, Inc., 2001.
24. http://www.rsoftdesign.com/.