Progress In Electromagnetics Research Letters, Vol. 21, 147–157, 2011

ELECTROMAGNETIC CYLINDRICAL CONCENTRATORS IN INITIALLY HOMOGENEOUS OR INHOMOGENEOUS DIELECTRIC CORE MEDIA DESIGNED BY THE COORDINATE TRANSFORMATION METHOD

E. Cojocaru

Department of Theoretical Physics
Horia Hulubei National Institute of Physics and Nuclear Engineering
P. O. Box MG-6, Magurele, Bucharest 077125, Romania

Abstract—Reflectionless cylindrical concentrators have been designed up to now in an initially empty core space by applying form-invariant, spatial coordinate transformations of Maxwell's equations. We show that the reflectionless cylindrical concentrators in initially homogeneous and isotropic dielectric core media allow an enhanced concentration of the incident field. Designs of more complex reflectionless concentrators in initially inhomogeneous dielectric media represented by empty or dielectric-filled circular containers are also presented. The transformation equations are given explicitly, and the fields are expressed analytically. The functionality of these devices is illustrated by several examples.

1. INTRODUCTION

The form-invariant, spatial coordinate transformation method [1–3] provides a conceptually simple approach to the design of complex electromagnetic structures. Following this method, the invisibility cloaks have received much attention [4–12]. Apart from invisibility cloaks, other interesting metamaterial devices [13] such as rotators [14, 15], concentrators [16–19], illusion devices [20, 21], superscatterers [22, 23], equiscatterers [24], invisible tunnels [25], and beam shifters [26] were investigated.

The electromagnetic (EM) concentrators could be useful in the harnessing of light in solar cells or similar devices, where high-field

Received 10 February 2011, Accepted 3 March 2011, Scheduled 9 March 2011
Corresponding author: E. Cojocaru (ecojocaru@theory.nipne.ro).

intensities are required [16, 17]. A reflectionless circular cylindrical concentrator was first designed in [16] to focus the incident EM waves with wave vectors perpendicular to the cylinder axis, enhancing the EM energy density of incident waves in a given area. A similar, but more simple design, was then proposed in [17]. To our best knowledge, all researches on reflectionless EM concentrators reported so far have been focused on the case in which the light field concentration is accomplished in an initially empty space. However, the designs of concentrators in a homogeneous or inhomogeneous dielectric core medium have not been considered. Here, we consider first the simple case of a homogeneous dielectric as the initial core of the reflectionless EM concentrator. Then, we analyze the more complex case when the initial core of the reflectionless EM concentrator is an inhomogeneous dielectric represented by a circular dielectric container which is either empty or is filled with a dielectric, as for example with a liquid. The background medium of the concentrators is the empty space. The transformation relations are given explicitly, and the fields are expressed analytically. Examples that confirm the functionality of these devices are given. For simplicity, we consider in this paper the case of transverse-electric (TE) polarized incident plane waves with wave vectors perpendicular to the cylinder z-axis. Since only the z component of the electric field there exists, the concentrator is reduced to a bidimensional (2D) problem. Details concerning the electric field distribution for the TE plane wave incident on a circular cylindrical structure can be found for example in [6]. Note that the respective field distribution can be calculated and represented graphically by using an usual computational program like Mathematica or Matlab.

2. NOTATIONS AND GENERAL RELATIONS

Denote (r', θ', z') the cylindrical coordinates in the original, initial space. According to the coordinate transformation method, under a space transformation which radially maps points from a radius r' to a radius r and has the identity mapping for the axial transformation, we have

$$r' = f(r), \qquad \theta' = \theta, \qquad z' = z \tag{1}$$

where $f(r)$ is a monotonic differentiable function. The permittivity $\boldsymbol{\epsilon}$ and permeability $\boldsymbol{\mu}$ in the transformed space are given by [2]

$$\boldsymbol{\epsilon} = \mathbf{A}\epsilon'\mathbf{A}^T/\det\mathbf{A} \qquad \boldsymbol{\mu} = \mathbf{A}\mu'\mathbf{A}^T/\det\mathbf{A} \tag{2}$$

where ϵ' and μ' are the material constants in the original space, $\mathbf{A}$ is the Jacobian transformation matrix with components $A_{ij} = \partial x_i/\partial x'_j$. Throughout the paper we consider the material constants as relative, dimensionless quantities. For a homogeneous and isotropic medium in the original space, by the coordinate transformation $[f(r), \theta, z]$ one obtains the material constants

$$\frac{\boldsymbol{\epsilon}}{\epsilon'} = \frac{\boldsymbol{\mu}}{\mu'} = \mathrm{diag}\left[\frac{f}{rf'}, \frac{rf'}{f}, \frac{ff'}{r}\right] \tag{3}$$

where the argument of f is skipped and f' denotes the differentiation of f against r. With an $\exp(-i\omega t)$ time dependence, in the TE case, only three components, ϵ_z, μ_θ, and μ_r enter in the equation for determining the E_z field [6],

$$\frac{1}{\epsilon_z r}\frac{\partial}{\partial r}\left(\frac{r}{\mu_\theta}\frac{\partial E_z}{\partial r}\right) + \frac{1}{\epsilon_z r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_r}\frac{\partial E_z}{\partial \theta}\right) + k_0^2 E_z = 0 \tag{4}$$

where k_0 is the wave number in vacuum. Inserting the material constants given by (3), Eq. (4) can be solved by a separation of variables, $E_z = R(r)\exp(il\theta)$, where l is an integer number and $R(r)$ is a solution of equation

$$\frac{\partial^2 R}{\partial f^2} + \frac{1}{f}\frac{\partial R}{\partial f} + \left(k'^2 - \frac{l^2}{f^2}\right) R = 0 \tag{5}$$

with $k' = k_0\sqrt{\epsilon'\mu'}$. Therefore, there exists a simple set of solutions to E_z in the transformed medium of the form

$$F_l(k' f(r))\exp(il\theta) \tag{6}$$

where F_l is an lth order Bessel function.

3. EM CONCENTRATORS IN INITIALLY HOMOGENEOUS DIELECTRIC CORE MEDIA

Let a denote the radius of the concentration-field region I comprising initially a homogeneous dielectric core medium of material constants $(\epsilon_c, 1)$. Let b denote the external radius of region II delimiting the anisotropic and inhomogeneous shell covering the concentration-field medium in the transformed space. Following the design proposed in [17], one appropriate coordinate transformation function for the reflectionless circular cylindrical concentrator is given by

$$f(r) = \begin{cases} f_1(r) = Mr & \text{at} \quad 0 < r < a \\ f_2(r) = \frac{(b - Ma\sqrt{\epsilon_c})r + ab(M\sqrt{\epsilon_c} - 1)}{b-a} & \text{at} \quad a < r < b \end{cases} \tag{7}$$

where M is a constant, $f_2(a) = Ma\sqrt{\epsilon_c}$, and $f_2(b) = b$. The material constants in the transformed space are determined by (3),

$$\frac{\boldsymbol{\epsilon}}{\epsilon_c} = \boldsymbol{\mu} = \mathrm{diag}\left[1, 1, M^2\right] \tag{8}$$

in region I at $0 < r < a$, and

$$\boldsymbol{\epsilon} = \boldsymbol{\mu} = \mathrm{diag}\left[\frac{f_2}{r f_2'}, \frac{r f_2'}{f_2}, \frac{f_2 f_2'}{r}\right] \tag{9}$$

in region II at $a < r < b$. If $M < \frac{b}{a\sqrt{\epsilon_c}}$, both $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in region II which are determined by (9) have positive values; otherwise, if $M > \frac{b}{a\sqrt{\epsilon_c}}$, both $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ have negative values, the anisotropic and inhomogeneous medium in region II being a negative-index medium.

The electric field in each region is given by [6, 24]

$$\begin{aligned} r > b \qquad & E_z = \sum_l \left[\alpha_l^{(\mathrm{in})} J_l(k_0 r) + \beta_l^{(\mathrm{sc})} H_l(k_0 r)\right] \exp(il\theta), \\ a < r < b \qquad & E_z = \sum_l \left[\alpha_l^{(1)} J_l(k_0 f_2) + \beta_l^{(1)} H_l(k_0 f_2)\right] \exp(il\theta), \\ r < a \qquad & E_z = \sum_l \alpha_l^{(2)} J_l(k_c f_1) \exp(il\theta), \end{aligned} \tag{10}$$

where $k_c = k_0\sqrt{\epsilon_c}$; α_l and β_l are expansion coefficients determined by applying the boundary conditions at interfaces; J_l is the lth order Bessel function of the first kind; and H_l is the lth order Hankel function of the first kind. As a first example, we consider $a = 1\,\mathrm{m}$, $b = 3\,\mathrm{m}$, $M = 2$, $\epsilon_c = 1$, and the wavelength of the incident wave in vacuum $\lambda = 1\,\mathrm{m}$. In this case, with $\epsilon_c = 1$, $f(r)$ defined by (7) is like in [17]. Since $M < \frac{b}{a\sqrt{\epsilon_c}}$, the values of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in region II are positive.

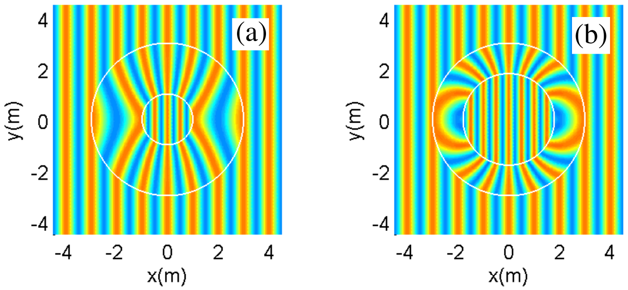

Figure 1. E_z field distributions for EM concentrators when the initial core medium is the empty space. (a) $M = 2 < \frac{b}{a}$, with $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ having positive values in the shell at $a < r < b$, which is delimited by the white solid lines. (b) $M = 2 > \frac{b}{a}$, the shell at $a < r < b$ comprising a negative-index medium.

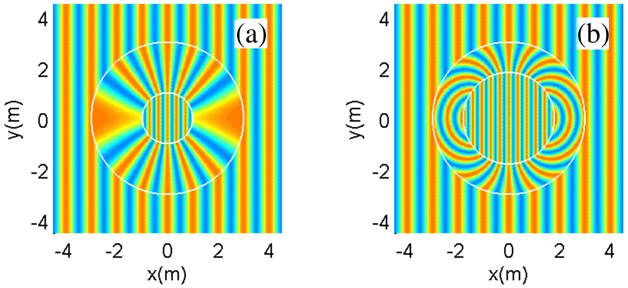

Figure 2. E_z field distributions for EM concentrators when the initial core medium is a dielectric of permittivity $\epsilon_c = 2$. (a) $M = 2 < \frac{b}{a\sqrt{\epsilon_c}}$, with $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ having positive values in the shell at $a < r < b$. (b) $M = 2 > \frac{b}{a\sqrt{\epsilon_c}}$, the shell at $a < r < b$ comprising a negative-index medium.

The electric field distribution which is determined by (10) is shown in Fig. 1(a). It is obvious that the concentrator is reflectionless and the energy density of the incident wave is enhanced by the factor M in the inner region I of radius a. If we consider $a = 1.8\,\text{m}$, the other parameters being the same, then $M > \frac{b}{a\sqrt{\epsilon_c}}$ and region II at $a < r < b$ comprises a negative-index medium. The respective electric field distribution is shown in Fig. 1(b). Fig. 2 shows the electric field distributions when $\epsilon_c = 2$, the other parameters being the same as those in Fig. 1. It is obvious that the concentrators in Figs. 2(a) and (b) are also reflectionless, but the energy density of the incident wave is enhanced by the factor $M\sqrt{2}$ in the inner region I of radius a.

4. EM CONCENTRATORS IN INITIALLY INHOMOGENEOUS DIELECTRIC CORE MEDIA

Consider that the initial concentration-field region of the concentrator is embedded in a dielectric circular container as shown in Fig. 3. The wall of the container, at $d < r < a$, is a dielectric of material constants $(\epsilon_w, 1)$. The concentration-field region I, at $r < d$, is initially a homogeneous dielectric medium of material constants $(\epsilon_c, 1)$. In the transformed space, a bilayer coating on the wall of the container is needed in order to get a reflectionless concentrator: the first layer, at $a < r < b$, representing region II, is a complementary negative-index shell that optically cancels the wall of the container, whereas the second layer, at $b < r < c$, representing region III, comprises a medium that restores the correct optical path in the canceled space [11, 20, 21]. One

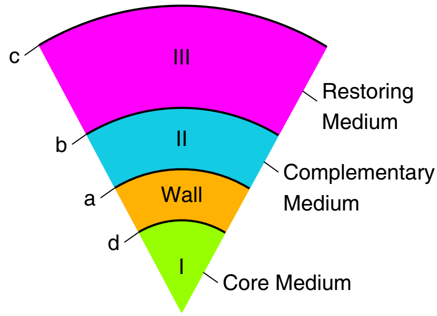

Figure 3. The component regions of an EM concentrator when the initial concentration-field medium is embedded in a dielectric wall of permittivity ϵ_w.

appropriate coordinate transformation function is given by

$$f(r) = \begin{cases} f_1(r) = Mr & \text{at} \quad 0 < r < d \\ r & \text{at} \quad d < r < a \\ f_2(r) = \frac{-(a-d)r+a(b-d)}{b-a} & \text{at} \quad a < r < b \\ f_3(r) = \frac{(c-Md\sqrt{\epsilon_c})r+c(Md\sqrt{\epsilon_c}-b)}{c-b} & \text{at} \quad b < r < c \end{cases} \tag{11}$$

where $f_2(a) = a$, $f_2(b) = d$, $f_3(c) = c$, and $f_3(b) = Md\sqrt{\epsilon_c}$. The material constants are $(\epsilon_w, 1)$ at $d < r < a$, and in the other regions, they are determined by (3),

$$\frac{\boldsymbol{\epsilon}}{\epsilon_c} = \boldsymbol{\mu} = \text{diag}\left[1, 1, M^2\right] \tag{12}$$

in region I, at $0 < r < d$,

$$\frac{\boldsymbol{\epsilon}}{\epsilon_w} = \boldsymbol{\mu} = \text{diag}\left[\frac{f_2}{rf_2'}, \frac{rf_2'}{f_2}, \frac{f_2 f_2'}{r}\right] \tag{13}$$

in the complementary region II, at $a < r < b$, and

$$\boldsymbol{\epsilon} = \boldsymbol{\mu} = \text{diag}\left[\frac{f_3}{rf_3'}, \frac{rf_3'}{f_3}, \frac{f_3 f_3'}{r}\right] \tag{14}$$

in the restoring region III, at $b < r < c$. From (11) and (14), we can see that, if $M < \frac{c}{d\sqrt{\epsilon_c}}$, both $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in the restoring region III have positive values; otherwise, if $M > \frac{c}{d\sqrt{\epsilon_c}}$, both the complementary and restoring regions II and III comprise negative-index media. The electric field in each region is given by

$$\begin{aligned}
r > c \qquad & E_z = \sum_l \left[\alpha_l^{(\mathrm{in})} J_l(k_0 r) + \beta_l^{(\mathrm{sc})} H_l(k_0 r)\right] \exp(il\theta), \\
b < r < c \qquad & E_z = \sum_l \left[\alpha_l^{(1)} J_l(k_0 f_3) + \beta_l^{(1)} H_l(k_0 f_3)\right] \exp(il\theta), \\
a < r < b \qquad & E_z = \sum_l \left[\alpha_l^{(2)} J_l(k_w f_2) + \beta_l^{(2)} H_l(k_w f_2)\right] \exp(il\theta), \\
d < r < a \qquad & E_z = \sum_l \left[\alpha_l^{(3)} J_l(k_w r) + \beta_l^{(3)} H_l(k_w r)\right] \exp(il\theta), \\
r < d \qquad & E_z = \sum_l \alpha_l^{(4)} J_l(k_c f_1) \exp(il\theta),
\end{aligned} \tag{15}$$

where $k_w = k_0\sqrt{\epsilon_w}$. The reflectionless condition and the continuity conditions of the tangential fields at interfaces impose on M a fixed value,

$$M = \sqrt{\epsilon_w}/\sqrt{\epsilon_c} \tag{16}$$

As an example, we consider the EM concentrator in an initially empty dielectric circular container, when $a = 1.2\,\mathrm{m}$, $b = 2\,\mathrm{m}$, $c = 3\,\mathrm{m}$, $d = 1\,\mathrm{m}$, $M = 2$, $\epsilon_w = 4$, $\epsilon_c = 1$, and $\lambda = 1\,\mathrm{m}$. The electric field distribution is shown in Fig. 4(a). Since for these values of parameters $f_3(r) = r$, the restoring medium in region III at $b < r < c$ is simply the

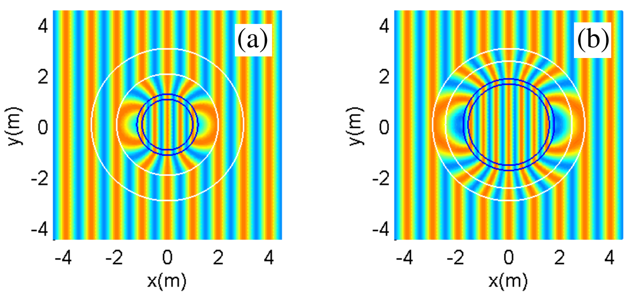

Figure 4. E_z field distributions for EM concentrators in empty circular dielectric containers with wall permittivity $\epsilon_w = 4$. (a) $M = \sqrt{\epsilon_w} < \frac{c}{d}$, with $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ having positive values ($\epsilon = \mu = 1$) in the restoring region III at $b < r < c$. (b) $M = \sqrt{\epsilon_w} > \frac{c}{d}$, both regions II and III comprising negative-index media. The white solid lines delimit the regions II and III, while the other circular solid lines delimit the dielectric wall.

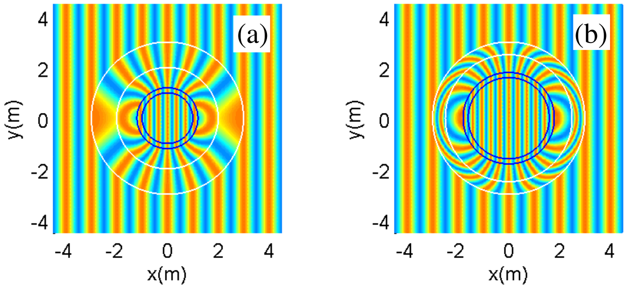

Figure 5. E_z field distributions for EM concentrators when the initial core medium is a dielectric of permittivity $\epsilon_c = 2$ embedded in the circular dielectric wall of permittivity $\epsilon_w = 8$. (a) $M = \frac{\sqrt{\epsilon_w}}{\sqrt{\epsilon_c}} < \frac{c}{d\sqrt{\epsilon_c}}$, with $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ having positive values in the restoring region III at $b < r < c$. (b) $M = \frac{\sqrt{\epsilon_w}}{\sqrt{\epsilon_c}} > \frac{c}{d\sqrt{\epsilon_c}}$, both regions II and III comprising negative-index media.

background medium. If we consider $a = 1.8\,\mathrm{m}$, $b = 2.5\,\mathrm{m}$, $d = 1.6\,\mathrm{m}$, the other parameters being the same, then $M = 2 > \frac{c}{d\sqrt{\epsilon_c}}$, and both layers coated on the container wall consist of negative-index media. The electric field distribution in this case is shown in Fig. 4(b). In both cases, reflectionless EM concentrators result. Fig. 5 shows the electric field distributions for EM concentrators like in Fig. 4, but when $\epsilon_w = 8$ and $\epsilon_c = 2$, the value of M being the same, $M = 2$. It is obvious that the energy density of the incident wave is enhanced in the core region I at $r < d$ by the factor M in Fig. 4, when the initial core medium is the empty space, and by the factor $M\sqrt{2}$ in Fig. 5, when the initial core medium is a dielectric of permittivity $\epsilon_c = 2$.

5. CONCLUSION

Designs of reflectionless EM circular cylindrical concentrators in initially homogeneous or inhomogeneous dielectric core media were presented. The simple approach of coordinate transformation method was applied. These designs are more complex than those of reflectionless EM circular concentrators in an initially empty core space. A numerical approach of such designs could be cumbersome comparing to the analytical approach. In the specific examples that were considered, we used only linear coordinate transformation functions in the coating region. Different transformation functions could be found not only for these examples, but also for other examples with different forms of the initial inhomogeneity in the core medium. For simplicity, we considered the empty space as the background medium of the concentrators. For another homogeneous or layered background medium [27], the designs could become more intricate. All examples referred to lossless media. However, when the dissipation is considered, the performances of the reflectionless EM circular concentrators can be drastically altered.

REFERENCES

1. Leonhardt, U. and T. G. Philbin, "General relativity in electrical engineering," *New J. Phys.*, Vol. 8, 247, 2006.
2. Schurig, D., J. B. Pendry, and D. R. Smith, "Calculation of material properties and ray tracing in transformation media," *Opt. Express*, Vol. 14, 9794–9804, 2006.
3. Chen, H. Y., C. T. Chan, and P. Sheng, "Transformation optics and metamaterials," *Nature Mater.*, Vol. 9, 387–396, 2010.
4. Pendry, J. B., D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," *Science*, Vol. 312, 1780–1782, 2006.
5. Schurig, D., J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, "Metamaterial electromagnetic cloak at microwave frequencies," *Science*, Vol. 314, 977–980, 2006.
6. Ruan, Z., M. Yan, C. W. Neff, and M. Qiu, "Ideal cylindrical cloak: Perfect but sensitive to tiny perturbations," *Phys. Rev. Lett.*, Vol. 99, 113903, 2007.
7. Li, C., K. Yao, and F. Li, "Two-dimensional electromagnetic cloaks with non-conformal inner and outer boundaries," *Opt. Express*, Vol. 16, 19366–19374, 2008.
8. Jiang, W. X., J. Y. Chin, Z. Li, Q. Cheng, R. Liu, and T. J. Cui, "Analytical design of conformally invisible cloaks for arbitrarily shaped objects," *Phys. Rev. E*, Vol. 77, 066607, 2008.
9. Yan, W., M. Yan, Z. Ruan, and M. Qiu, "Coordinate transformation make perfect invisibility cloaks with arbitrary shape," *New J. Phys.*, Vol. 10, 043040, 2008.
10. Nicolet, A., F. Zolla, and S. Guenneau, "Electromagnetic analysis of cylindrical cloaks of an arbitrary cross section," *Opt. Lett.*, Vol. 33, 1584–1586, 2008.
11. Lai, Y., H. Chen, Z.-Q. Zhang, and C. T. Chan, "Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell," *Phys. Rev. Lett.*, Vol. 102, 093901, 2009.
12. Liu, R., C. Ji, J. J. Mock, J. Y. Chin, T. J. Cui, and D. R. Smith,

"Broadband ground-plane cloak," *Science*, Vol. 323, 366–369, 2009.

13. Jiang, W. X., J. Y. Chin, and T. J. Cui, "Anisotropic metamaterial devices," *Mat. Today*, Vol. 12, 26–33, 2009.
14. Chen, H. and C. T. Chan, "Transformation media that rotate electromagnetic fields," *Appl. Phys. Lett.*, Vol. 90, 241105, 2007.
15. Silva-Macedo, J. A., M. A. Romero, and B.-H. V. Borges, "An extended FDTD method for the analysis of electromagnetic field rotations and cloaking devices," *Progress In Electromagnetics Research*, Vol. 87, 183–196, 2008.
16. Rahm, M., D. Schurig, D. A. Roberts, S. A. Cummer, D. R. Smith, and J. B. Pendry, "Design of electromagnetic cloaks and concentrators using form-invariant coordinate transformations of Maxwell equations," *Photon. Nanostruct.: Fundam. Applic.*, Vol. 6, 87–95, 2008.
17. Yaghjian, A. D. and S. Maci, "Alternative derivation of electromagnetic cloaks and concentrators," *New J. Phys.*, Vol. 10, 115022, 2008.
18. Jiang, W. X., T. J. Cui, Q. Cheng, J. Y. Chin, X. M. Yang, R. Liu, and D. R. Smith, "Design of arbitrarily shaped concentrators based on conformally optical transformation of nonuniform rational B-spline surfaces," *Appl. Phys. Lett.*, Vol. 92, 264101, 2008.
19. Yang, C., J. Yang, M. Huang, J. Peng, and W. Niu, "Electromagnetic concentrators with arbitrary geometries based on Laplace's equation," *J. Opt. Soc. Am. A*, Vol. 27, 1994–1998, 2010.
20. Lai, Y., J. Ng, H. Chen, D. Han, J. Xiao, Z.-Q. Zhang, and C. T. Chan, "Illusion optics: The optical transformation of an object into another object," *Phys. Rev. Lett.*, Vol. 102, 253902, 2009.
21. Cojocaru, E., "Illusion devices with internal or external circular objects designed by the coordinate transformation method," *Journal of Electromagnetic Waves and Applications*, Vol. 24, No. 16, 2309–2317, 2010.
22. Yang, T., H. Chen, X. Luo, and H. Ma, "Superscatterer: Enhancement of scattering with complementary media," *Opt. Express*, Vol. 16, 18545–18550, 2008.
23. Chen, H., X. Zhang, X. Luo, H. Ma, and C. T. Chan, "Reshaping the perfect electrical conductor cylinder arbitrarily," *New J. Phys.*, Vol. 10, 113016, 2008.

24. Cojocaru, E., "Equiscatterer with complementary coordinate-transformed cylindrical media," *J. Opt. Soc. Am. B*, Vol. 27, 895–898, 2010.
25. Zhang, J., Y. Luo, H. Chen, J. Huangfu, B.-I. Wu, L. Ran, and J. A. Kong, "Guiding waves through an invisible tunnel," *Opt. Express*, Vol. 17, 6203–6208, 2009.
26. Wang, M. Y., J. J. Zhang, H. Chen, Y. Luo, S. Xi, L.-X. Ran, and J. A. Kong, "Design and application of a beam shifter by transformation media," *Progress In Electromagnetics Research*, Vol. 83, 147–155, 2008.
27. Zhang, J., J. Huangfu, Y. Luo, H. Chen, J. A. Kong, and B.-I. Wu, "Cloak for multilayered and gradually changing media," *Phys. Rev. B*, Vol. 77, 035116, 2008.