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DERIVATION OF THE EFFECTIVE NONLINEAR SCHRÖDINGER EQUATIONS FOR DARK AND POWER LAW SPATIAL PLASMON-POLARITON SOLITONS USING NANO SELF-FOCUSING

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Abstract—An effective Nonlinear Schrödinger Equation for propagation is derived for optical dark and power law spatial solitons at the subwavelength with a surface plasmonic interaction. Starting with Maxwell's Nonlinear Equations a model is proposed for TM polarized type spatial solitons on a metal dielectric interface. Two separate systems are considered in which one metal dielectric interface has a dielectric Kerr medium that has self-defocusing and another similar interface which the dielectric Kerr medium that has self-focusing depending on the modulus of the electric field to some power law variable p. The beam dynamics are analytically studied for these nanowaveguides.

1. INTRODUCTION

Optical spatial solitons have been studied for the past few decades due to their potential applications to optical pulse compression [4], all optical switching [5], logic devices [6] and others [1]. The advantage of these optical beams is that they are self trapped due to balance between diffraction and nonlinearity (self-focusing) [16–19]. With this, there may be no need of a cladding for a waveguide to keep the light beams from escaping from the core. These spatial solitons interact with one another while displaying properties that are normally

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associated with real particles. Bright spatial soliton amplitude profiles are displayed as a hyperbolic secant function and hold there profile at short distances on the order of centimeters. Dark spatial solitons are similar but have hyperbolic tangent type profiles. Dark spatial solitons were first observed by Jerominek in 1985 and Belanger and Mathieu in 1987 in an experiment of an optical branching effect in a Titanium and Lithium Niobate planar waveguide. Spatial solitons have been observed also in other experiments [1–3]. Gisin and Malomed [7] showed that bright and dark spatial solitons beam size cannot be smaller than half the carrier wavelength in the transverse magnetic (TM) case. In the transverse electric (TE) case, bright and dark spatial solitons are subject to strong instability [8]. Recently, intense study has gone into an area of nano-optics called plasmonics due to its potential application for development of smaller optical devices such as nano-waveguides circuits at the subwavelength [9–11]. Plasmons were first observed as quantized bulk oscillations by Pine and Bohm in 1952 as explanation for energy losses of fast electrons passing through metal foils. Surface plasmons-polaritons (SPPs) are a quantized excitation at the metal/dielectric (or metal/vacuum) interface with one material having a negative permittivity (usually metal) and free charge carriers and the other material having a positive permittivity (dielectric or Kerr medium). These collective oscillations of surface charge behave like a particle. SPPs support phenomena such as Surface Enhanced Raman Scattering (SERS), fluorescence, enhanced nanoscale transmission, and second harmonic generation in metals from subwavelength slits. These surface plasmons (SPs) are the key in the development of nano-antennas and platforms for sensing biological and chemical agents [12–14]. Nonlinearity plays important role in increasing efficiency of nanocircuit and waveguide applications. Past studies have been done on nonlinear guided waves in metallic/dielectric structures localized in the transverse dimension [22, 31]. Furthermore, losses in the visible frequency range have to be considered when studying soliton generation and propagation for plasmonic structures [1, 2]. Davoyan et al. have shown that the bright spatial soliton beam size limitation can be overcome theoretically by using surface plasmons [15]. They showed that a nonlinear dielectric depending on the modulus squared of the total electric field and metal dielectric interface can create what is called "spatial plasmon-polariton solitons". This was verified by them through full vector finite-difference time domain (FDTD) simulations with commercial software Rsoft. In this paper, nonlinear dielectrics and the behavior of SPs with self defocusing and SPs with power law nonlinearity is considered theoretically at the interface.

In Section 2, a nonlinear Schrödinger equation is derived for self-defocusing based on metal-dielectric boundary conditions. Next, the one soliton solution is considered from the effective nonlinear Schrödinger Equation. Expressions are developed for the integrals of motion. Spatial frequency, amplitude relation and spatial velocity are also derived. Briefly, the requirements for dark soliton creation and prediction are presented which apply at the nanoscale based on the effective nonlinear Schrödinger Equation.

In Section 3, an effective nonlinear Schrödinger equation is considered for self-focusing with a power law Kerr like medium based on metal-dielectric boundary conditions. With no loss, this equation is usually solved using the inverse transform method [1, 19]. In this case, a secant type soliton solution is proposed and conditions for existence are derived. The results include equations for spatial frequency with a nonlinear correction to plasmon frequency. This also includes equations for the amplitude and spatial velocity. In the case with no attenuation, the integrals of motion are conserved. Lastly, the case with attenuation is considered which the integrals of motion energy, momentum, and the Hamiltonian are dynamic. The integral coefficients in the proposed model nonlinear evolution equation contribute to derived adiabatic expressions with no losses and attenuation.

2. SELF DEFOCUSING WITH METALLIC DIELECTRIC INTERFACE

In order to create self defocusing in the dielectric nonlinear medium the linear dielectric constant has to counter the intensity of the dielectric medium. Consider an illuminated TM polarized light that is created to be narrowly focused to some beamwidth R_0 in close proximity to a microscopic planar waveguide with the metal on top and the nonlinear dielectric on the bottom (see Figure 1 below). Plasmons are excited by light scattered on the metal corner so plane waves will not be excited in the dielectric material. In addition, a thin metal screen would have to be place in front of the dielectric surface [15]. To derive the ENLSE (Effective Nonlinear Schrödinger Equation) we start Maxwell's equations for planar waveguide interface with a dielectric and a metal (see Figure 1 below). But first we consider the eigenmodes for the case which is linear (or no Kerr effect and complex dielectric). Recall that Maxwell's time dependent vector equations are:

$$\begin{aligned} \nabla \times \zeta + \frac{\partial B}{\partial t} &= 0, \quad B = \mu_0 h, \\ \nabla \times h - \frac{\partial D}{\partial t} &= 0, \quad D = \varepsilon_0 \varepsilon = \varepsilon_0 n^2 \zeta, \end{aligned} \tag{1}$$

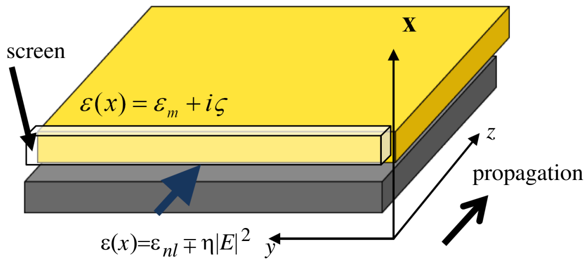

Figure 1. A diagram of the nonlinear dielectric/metal interface in order to avoid plane wave stimulation in the dielectric material [15].

where μ_0, n, and ε_0 is the permeability of free space, the index of refraction, and the permittivity of free space. Let us consider ζ and h as time dependent monochromatic fields defined as

$$h = \frac{1}{\mu_0 c} H(x, y, z) e^{-i\omega t} \quad \text{and} \quad \zeta = E(x, y, z) e^{-i\omega t}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) reduces to time independent equations

$$\nabla \times E - ikH = 0, \quad \text{and} \quad \nabla \times H + ikn^2 E = 0, \quad \text{with} \quad k = 2\pi/\lambda. \tag{3}$$

Taking the curl of the above Eq. (3) and substituting the curl of H into the first part of Eq. (3) and normalize by $1/k$ this leads to the Wave Equation:

$$\nabla \times \nabla \times E - \varepsilon(x) E = 0. \tag{4}$$

The free space wavelength is λ. The electric field E is the new field normalized to k. In this case assume the ε depends on x for a planar structure. Also, we assume that these waves are transverse magnetic (TM) waves so $E = (E_x, 0, E_z)$ or $E_y = H_x = H_z = 0$. Now, the metal-dielectric interface is designed as such the permittivity for $x > 0$, $\varepsilon(x) = \varepsilon_m + i\varsigma$ for metal and for $x < 0$, $\varepsilon(x) = \varepsilon_{lin} \pm \eta|E|^2$ for a Kerr nonlinear material that has self-defocusing (+ for self-focusing). ς is the imaginary part of the dielectric for metal and η is the nonlinear coefficient. Moreover, the dielectric tensor could depend on either the transverse or longitudinal field components which helps simplify the problem and has been verified through simulations [32–34]. If we separate the normalized wave equation into its vector components the resulting equations are

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_z}{\partial x \partial z} \mp \varepsilon(x) E_x = 0 \quad \text{for} \quad x \quad \text{and} \tag{5}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \frac{\partial^2 E_x}{\partial z \partial x} \mp \varepsilon(x) E_z = 0 \quad \text{for} \quad z. \tag{6}$$

Equations (5) and (6) cannot be solved analytically due to the dependences of field components to nonlinear propagation. We are looking for solutions for E that are invariant in the y directions so

$$E = GE_{x0}e^{-i\beta z}\hat{x} + iGE_{z0}e^{-i\beta z}\hat{z} \tag{7}$$

$$\beta = \left(\frac{\varepsilon_m \varepsilon_{nl}}{\varepsilon_m + \varepsilon_{nl}}\right)^{1/2}, \tag{8}$$

where G is the amplitude and β is the propagation constant or plasmon wavenumber normalize to k. x and z are unit vectors in the x and z directions. β is also the effective dielectric constant similar to effective index of refraction usually seen in classical guided mode theory [20]. E_{xo} and E_{zo} are evanescent type fields or surface plasmons at the metal dielectric interface with boundary conditions

$$\begin{aligned} E_{x0} &= \frac{-\beta}{\kappa_1}e^{\kappa_1 x} \text{ and } E_{z0} = e^{\kappa_1 x} \text{ with } \kappa_1^2 = \beta^2 - \varepsilon_m \text{ for } x < 0 \\ E_{x0} &= \frac{\beta}{\kappa_2}e^{-\kappa_2 x} \text{ and } E_{z0} = e^{-\kappa_2 x} \text{ with } \kappa_2^2 = \beta^2 - \varepsilon_{nl} \text{ for } x > 0. \end{aligned} \tag{9}$$

Referring to the above Eq. (7), assume that TM polarized field and E_y is neglected. The beam in the y direction is self-focusing in the paraxial approximation. These are considered paraxial plasmonic beams. We consider a solution,

$$E = C_1 E_{x0} e^{-i\beta z}\hat{x} + iC_2 E_{z0} e^{-i\beta z}\hat{z}, \tag{10}$$

with C_1 and C_2 being slow varying amplitudes and $E_0 = (E_{x0}, 0, E_{z0})$ is linear Plasmon profile. Also, the plasmon transverse structure does not change significantly (or $\partial_x C_1 \approx 0$ and $\partial_x C_2 \approx 0$) [15]. Substitute Eq. (10) into Eqs. (5) and (6) taking into consideration boundary conditions (Eq. (9)), simplifies to

$$\begin{aligned} & E_{x0}\left[\frac{\partial^2 C_1}{\partial y^2} + \frac{\partial^2 C_1}{\partial z^2} - 2i\beta\frac{\partial C_1}{\partial z}\right] - i\frac{\partial C_2}{\partial z}\frac{\partial E_{z0}}{\partial x} \\ & + i\varsigma C_1 E_{x0} \mp \eta\left[|C_1|^2 E_{x0}^2 + |C_2|^2 E_{z0}^2\right] C_1 E_{x0} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} & E_{z0}\left[\frac{\partial^2 C_2}{\partial y^2}\right] + i\frac{\partial C_1}{\partial z}\frac{\partial E_{x0}}{\partial x} \\ & + i\varsigma C_2 E_{z0} \mp \eta\left[|C_1|^2 E_{x0}^2 + |C_2|^2 E_{z0}^2\right] C_2 E_{z0} = 0. \end{aligned} \tag{12}$$

Multiply Eq. (11) by E_{x0} and Eq. (12) E_{z0}. Assume $C_1 \approx C_2$ with a slow vary approximation $|\partial^2 C_1/\partial z^2| \ll |2i\beta \partial C_1/\partial z|$ or the length scale of the waveguide (in microns) is much greater than the carrier wavelength. With neglecting $\partial^2 C_1/\partial z^2, C_1 \equiv C$ and integrating with respect to x, Eqs. (11) and (12) become

$$-i\beta \frac{\int E_{x0}^2 dx}{\int \left|\vec{E}_0\right|^2 dx} \frac{\partial C}{\partial z} + \frac{1}{2}\frac{\partial^2 C}{\partial y^2} \mp \frac{1}{2}\frac{\int \eta(x)\left|\vec{E}_0\right|^4 dx}{\int \left|\vec{E}_0\right|^2 dx} |C|^2 C$$

$$+i\frac{1}{2}\frac{\int \varsigma(x)\left|\vec{E}_0\right|^2 dx}{\int \left|\vec{E}_0\right|^2 dx} C = 0. \tag{13}$$

A more familiar form of Eq. (13) is

$$-i\beta W \frac{\partial C}{\partial z} + \frac{1}{2}\frac{\partial^2 C}{\partial y^2} \mp \xi |C|^2 C + i\Omega C = 0 \tag{14}$$

with

$$W = \frac{\int E_{x0}^2 dx}{\int \left|\vec{E}_0\right|^2 dx} \tag{15}$$

$$\xi = \frac{1}{2}\frac{\int \eta(x)\left|\vec{E}_0\right|^4 dx}{\int \left|\vec{E}_0\right|^2 dx} \tag{16}$$

$$\Omega = \frac{1}{2}\frac{\int \varsigma(x)\left|\vec{E}_0\right|^2 dx}{\int \left|\vec{E}_0\right|^2 dx}. \tag{17}$$

So Eq. (13) is a nonlinear Schrödinger Equation with integral coefficients that are a function of x for a dark or bright solitons. Again, $\eta(x)$ and $\varsigma(x)$ are the Kerr nonlinear term and the imaginary part of the dielectric constant for the metal. Ω is described as the attenuation of surface plasmon soliton [15, 21, 22]. Considering a dark soliton and $\Omega = 0$ or no attenuation we get

$$-i\beta W \frac{\partial C}{\partial z} + \frac{1}{2}\frac{\partial^2 C}{\partial y^2} - \xi |C|^2 C = 0. \tag{18}$$

This is the effective nonlinear Schrödinger Equation (ENLSE) for plasmonic dark spatial solitons with no losses with an integral coefficient contribution from W and ξ the Kerr coefficient ($x < 0$). With $\xi = 0$, the model (Eq. (18)) is considered a model of beam diffraction at the nanoscale only.

With the above NLSE equation, there exist a stationary fundamental one soliton solution, N solitons, and even algebraic solutions. For simplicity, we consider the fundamental mode (or lowest order solitary wave) and assume a stationary solution in which an Ansatz is described as

$$C(y,z) = Ag[D(y-\bar{y})]\exp[i(-K_s y + \omega_s z + \sigma_0)] \tag{19a}$$

$$\text{and} \quad v = \frac{d\bar{y}}{dz} \tag{19b}$$

which the function g is described as the shape of the soliton and. ν is the mean velocity of the soliton with mean position $\bar{y}(z)$. The constants A and D represent the soliton amplitude and width. K, ω_s, and σ_0 are the soliton frequency, the wave number, and the center of the phase. This frequency ω_s is also the nonlinear correction to the plasmon wave number. The phase is defined as

$$\Phi(y,z) = -\kappa y + \omega_s z + \sigma_0. \tag{20}$$

With $\tau_{shape} = D(y-\bar{y})$ we substitute a dark spatial soliton solution for g:

$$C(y,z) = A\tanh(\tau_{shape})\exp[i\Phi(y,z)] \tag{21}$$

Substituting Eq. (21) into Eq. (18) we get the following relations

$$\kappa = \beta W v \tag{22a}$$

or

$$\kappa = \beta v \frac{\int E_{x0}^2 dx}{\int \left|\vec{E}_0\right|^2 dx} \quad \text{or} \quad \omega_s = \frac{1}{2\beta W}\left(D - \frac{\kappa^2}{D}\right)\xi^{1/2} \tag{22b}$$

$$\omega_s = \frac{1}{2\beta W}\left(D - \frac{\kappa^2}{D}\right)\left(\frac{\int \eta(x)\left|\vec{E}_0\right|^4 dx}{\int \left|\vec{E}_0\right|^2 dx}\right)^{1/2} \tag{22c}$$

$$D = \xi^{1/2} A \tag{22d}$$

with

$$C(y,z) = A\tanh(A\sqrt{2\xi}(y-\bar{y}))\exp[i\Phi(y,z)]. \tag{22e}$$

Equation (22e) is a dark spatial plasmon polariton soliton amplitude which is the solution to the ENLSE Eq. (18) (see Figure 2(b)). Dark solitons have boundary conditions which $C(0,y)$ approaches C_0 the background amplitude with y approaching $+\infty$ while $C(0,y)$ approaches $C_0 e^{i\theta}$ as y approaching $-\infty$ with theta being the constant phase. These profiles exhibit an intensity dip near the center $y = 0$. The magnitude of the dip at the center is governed by the amplitude A. The phase

$$\phi = \cos^{-1}\left(\frac{\sqrt{\xi}A}{C_0}\right), \tag{23}$$

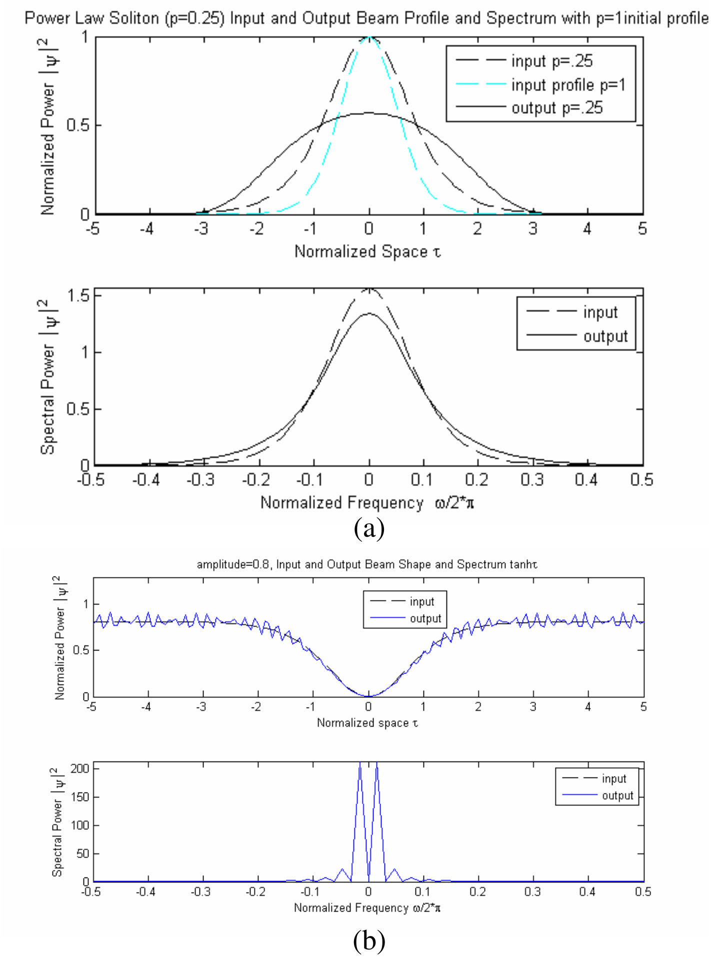

Figure 2. This is Split Step Fourier simulation of the normalized NLSE Eq. (14) with $\Omega = 0$ for a soliton calculating power and spectral power normalized amplitude ψ with initial amplitudes. (a) Power law bright soliton $p = 0.25$ with $p = 1$ initial profile for comparison. (b) 0.8 for a dark soliton without attenuation.

is a single parameter that characterizes the dark soliton solution [1, 3, 19]. The dark spatial plasmon polariton soliton can be created with an arbitrary small dip on a continuous wave background. The condition for existence of a dark soliton at the nanoscale would be that the input beam would be [1, 3]

$$C(0, y) = C_0 \exp(i\theta) + C_{01}(y) \quad (\text{with} \quad C_{01} \to 0 \quad \text{as} \quad |y| \to \infty) \quad (24)$$

and

$$\Lambda \equiv Re\left(\exp(-i\theta)\int_{-\infty}^{\infty} C_{01}(y)dy\right) < 0. \tag{25}$$

Under this condition there exist two discrete eigenvalues in the associated inverse scattering problem such that

$$\gamma_{1,2} = \pm C_{01}(1 - 1/2\Lambda^2) \tag{26}$$

This corresponds to a two dark solitons with equal amplitudes and opposite velocities [1, 3]. With the above conditions, the intensity dip always produces a pair of dark solitons.

The equations for the integrals of motion renormalized due to concern of divergence are

$$P_r = \int_{-\infty}^{\infty} (C_0^2 - |C|^2)dy \tag{27a}$$

$$M_r = \frac{i}{2}\int_{-\infty}^{\infty} (C_y^* C - C_y C^*)\left(1 - \frac{C_0^2}{|C|^2}\right)dy \tag{27b}$$

$$H_r = \int_{-\infty}^{\infty}\left(\frac{1}{2}\left|\frac{dC}{dy}\right|^2 + \int_{C_0^2}^{|C|^2}[F(C_0^2) - F(I)]dI\right)dy \tag{27c}$$

with $F(I) = -I$ and $F(C_0^2) = -C_0^2$ for the dark plasmon soliton. P_r, M_r, and H_r are renormalized soliton power, momentum, and the Hamiltonian quantities. So the equations for power and Hamiltonian for the dark plasmon soliton are

$$P_r = 2\sqrt{\xi}A \tag{28}$$

$$H_r = 4\frac{\sqrt{\xi}}{3}(C_0^2 - v^2)^{\frac{3}{2}}. \tag{29}$$

To include attenuation, Eq. (14) for self-defocusing is normalized to form the effective NLSE

$$i\frac{\partial\psi}{\partial\vartheta} + \frac{1}{2}\frac{\partial^2\psi}{\partial\tau^2} - |\psi|^2\psi + iP\psi = 0 \tag{30}$$

with

$$\begin{aligned} C &= \frac{1}{R_0}\left(\frac{-\beta W}{\xi}\right)^{1/3}\psi \quad y = \frac{R_0}{(-\beta W)^{1/2}}\left(\frac{-\beta W}{\xi}\right)^{1/6}\tau \\ z &= R_0^2\left(\frac{-\beta W}{\xi}\right)^{1/3}\vartheta \quad P = \frac{\Omega R_0^2}{2(-\beta W)^{2/3}}(\xi)^{-1/3} \end{aligned} \tag{31}$$

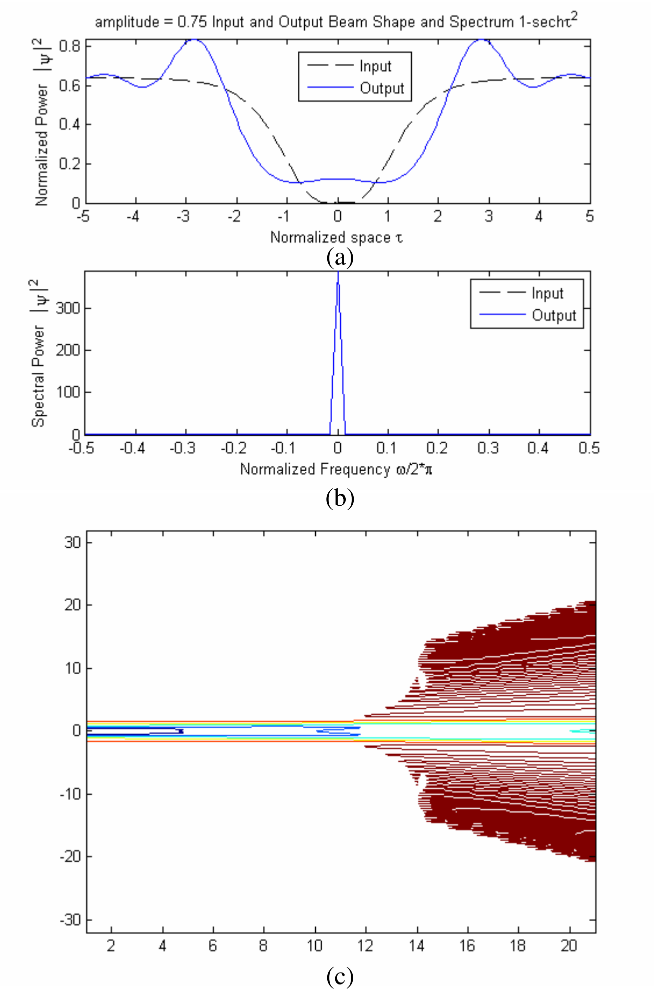

Figure 3. With initial profile $\psi(\tau, 0) = 1 - B_0 \text{sech}^2(\tau)$ simulation of the normalized NLSE overcome by losses with calculation of (a) power, (b) spectral power and (c) contour plot of the beam for a dark plasmon soliton with attenuation $\varepsilon_m = -25 + i1.74$, $\varepsilon_{linear} = 4.84$, $R_0 = 0.35\,\mu\text{m}$, $\eta = 3.0 \times 10^{-6}\,\text{V/m}^2$ and $B_0 = 0.65$.

where R_0 is the initial beam width. Again,with $P = 0$, Eq. (30) is the normalized effective NLSE which has solution of hyperbolic tangent function form (See Figure 3). Integrating W, Ω, and ξ which are indefinite and assuming η and ς are constants, the coefficients depend on dielectric constants for the metal and Kerr material. There is a discontinuity at the origin as far as the boundary conditions (see

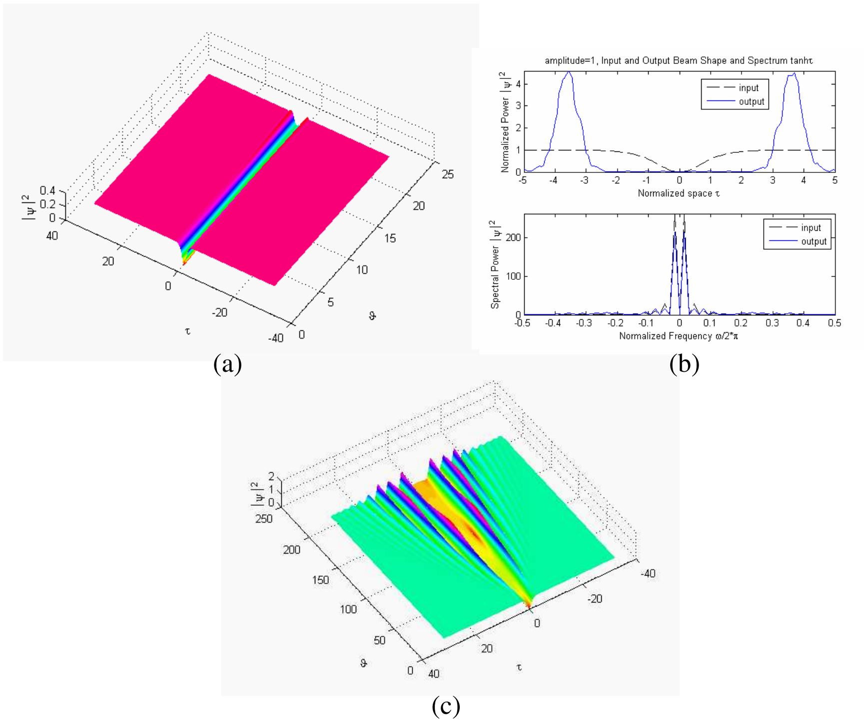

Figure 4. (a) Initial profile $\psi(\tau,0) = 1 - B_0\text{sech}^2(\tau)$ simulation plasmon soliton with $B_0 = 0.2$ without attenuation. An unstable dark spatial plasmon soliton simulation showing break up into Gaussian like spatial profiles. (b) $\psi(\tau,0) = B_0\tanh(\tau)$ with $B_0 = 1$. (c) $\psi(\tau,0) = 1 - B_0\text{sech}^2(\tau)$ with $B_0 = 0.8$.

Eq. (9)). In applying the Split Step Fourier Method Figures 3 and 4 below describes the propagation if $P \neq 0$ with the initial profile a hyperbolic tangent with $\varepsilon_m = -25+i1.74$, $\varepsilon_{linear} = 4.48$, $R_0 = 0.35\,\mu\text{m}$, $\eta = 3.0\times10^{-6}\,\text{V/m}^2$, and amplitude 0.8. The dark plasmon soliton breaks up into small Gaussian like spatial profiles moving into the $\pm\tau$ directions. With η and ς depending on x, the integral coefficients profiles would have to be determined for dielectric and nonlinear materials for certain frequencies which are not considered in this paper due complexity of the integral coefficients.

It has been shown that dark spatial solitons exist when attenuation is ignored. The soliton features at the nanoscale are similar to optical spatial solitons above the one half carrier wavelength limit with the appropriate assumptions and approximations. The attenuation has to be managed in order achieve self defocusing and generate dark solitons.

3. POWER LAW MEDIUM DIELECTRIC METALLIC INTERFACE

Spatial solitons have been known to be created in materials with power nonlinearity such as semiconductors and other materials [23, 24]. Considering dielectric $\varepsilon(x)$ is not dependent on the square of the modulus of electric field but the electric field to some number say p, the equation for the dielectric is:

$$\varepsilon(x) = \varepsilon_{linear} + \eta\,|E|^{2p} \tag{32}$$

The power law in NLSEs has been studied extensively for optical temporal solitons [18, 19, 25–28]. For $p = 1$, Davoyan et al., theoretically and numerically using FDTD (Finite Difference Time Domain), showed bright spatial plasmon polariton soliton propagation for a nanosized dielectric interface [15]. Following the same assumptions for Eqs. (1)–(10) similar in the previous section, but use Eq. (32) instead of a Kerr nonlinear material ($p = 1$) based on Maxwell's equations we get:

$$\begin{aligned}
&E_{x0}\left[\frac{\partial^2 C_1}{\partial y^2}+\frac{\partial^2 C_1}{\partial z^2}-2i\beta\frac{\partial C_1}{\partial z}\right]-i\frac{\partial C_2}{\partial z}\frac{\partial E_{z0}}{\partial x}+i\varsigma C_1E_{x0}\\
&+\eta\left[|C_1|^{2p}E_{x0}^{2p}+|C_2|^{2p}E_{z0}^{2p}\right]C_1E_{x0}=0,\\
&E_{z0}\left[\frac{\partial^2 C_2}{\partial y^2}\right]+i\frac{\partial C_1}{\partial z}\frac{\partial E_{x0}}{\partial x}+i\varsigma C_2E_{z0}\\
&+\eta\left[|C_1|^{2p}E_{x0}^{2p}+|C_2|^{2p}E_{z0}^{2p}\right]C_2E_{z0}=0,
\end{aligned} \tag{33}$$

Again, we assume Eq. (33) has the same boundary conditions as Eq. (11), with $E_0 = (E_{x0}, 0, E_{z0}), E = C_1E_{x0}e^{-i\beta z}x + iC_2E_{z0}e^{-i\beta z}z, \partial_x C_1 \approx 0$ and $\partial_x C_2 \approx 0, C_1 \approx C_2 \equiv C$, and the slow varying envelope approximation, $|\partial^2C_1/\partial z^2| \ll |2i\beta\partial C_1/\partial z|$ is applied. Multiplying the first equation by E_{x0} and the second by E_{z0} taking the sum and integrating over transverse component x, we get the power law effective nonlinear Schrödinger equation:

$$\begin{aligned}
&-i\beta\frac{\int E_{x0}^2dx}{\int\left|\vec{E}_0\right|^2dx}\frac{\partial C}{\partial z}+\frac{1}{2}\frac{\partial^2 C}{\partial y^2}+\frac{\int\eta(x)\left[E_{x0}^{2p}+E_{z0}^{2p}\right]\left[E_{x0}^2+E_{z0}^2\right]dx}{\int\left|\vec{E}_0\right|^2dx}|C|^{2p}C\\
&+i\frac{1}{2}\frac{\int\varsigma(x)\left|\vec{E}_0\right|^2dx}{\int\left|\vec{E}_0\right|^2dx}C=0.
\end{aligned} \tag{34}$$

or

$$-i\beta W\frac{\partial C}{\partial z}+\frac{1}{2}\frac{\partial^2 C}{\partial y^2}+\xi_{2p}\left|C\right|^{2p}C+i\Omega C=0 \quad \text{with} \quad 0<p<2. \quad (35)$$

and

$$W = \frac{\int E_{x0}^2 dx}{\int \left|\vec{E}_0\right|^2 dx} \quad (36a)$$

$$\xi_{2p} = \frac{\int \eta(x)\left[E_{x0}^{2p}+E_{z0}^{2p}\right]\left[E_{x0}^2+E_{z0}^2\right]dx}{\int \left|\vec{E}_0\right|^2 dx} \quad (36b)$$

$$\Omega = \frac{1}{2}\frac{\int \varsigma(x)\left|\vec{E}_0\right|^2 dx}{\int \left|\vec{E}_0\right|^2 dx}. \quad (36c)$$

p is limited to $0<p<2$ due to self focusing singularity. Taking into consideration a generalized NLSE such as

$$i\frac{\partial\psi_g}{\partial z}+\frac{1}{2}\left(\frac{\partial^2\psi_g}{\partial x_1^2}+\ldots+\frac{\partial^2\psi_g}{\partial x_{Dim}^2}\right)+\left|\psi_g\right|^{2p}\psi_g=0, \quad (36d)$$

with Dim being the transverse dimension, there are 3 different cases to consider: 1) In the critical case $p\cdot Dim=2$ solutions become singular in a finite z. 2) With $p\cdot Dim<2$, which is the subcritical case, the diffraction always dominates. 3) In the supercritical case $p\cdot Dim>2$, there is a large class of smooth initial amplitudes for which a focusing singularity forms in a finite distance z. With the case between the subcritical and supercritical self-focusing, near balance is achieved between focusing nonlinearity and diffraction. With $\xi_{2p}=0$, Eq. (35) the model is described as diffraction at the nanoscale. We can solve for a stationary soliton solution by making $\Omega=0$ for attenuation. We propose the following Ansatz [29, 30]:

$$C(y,z)=\frac{A}{\cosh^m[B(y-\bar{y})]}\exp(-i\kappa y+i\omega_s z+i\sigma) \quad (37)$$

with $v=\frac{d\bar{y}}{dz}$.

Here, B, A, ν, K, and ω_s are the width, amplitude of the soliton, mean velocity, spatial frequency and the wave number. σ and $\bar{y}$ is centre of the soliton phase and the centre of the soliton. Substituting Eq. (37) into Eq. (35) with $\Omega=0$ using the relation $m(2p+1)=m+1$ for powers of hyperbolic cosine function in the resulting relations are

$$\kappa = \beta W v \quad (38a)$$

$$\omega = -\frac{B^2-\kappa^2}{2p^2\beta W} \quad (38b)$$

$$B = \xi_{2p}^{\frac{1}{2}} A^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} \tag{38c}$$

$$\text{and} \quad C(y,z) = \frac{A}{\cosh^{\frac{1}{p}}[B(y-vz-\bar{y})]} \exp(-i\kappa y + i\omega z + i\sigma). \tag{38d}$$

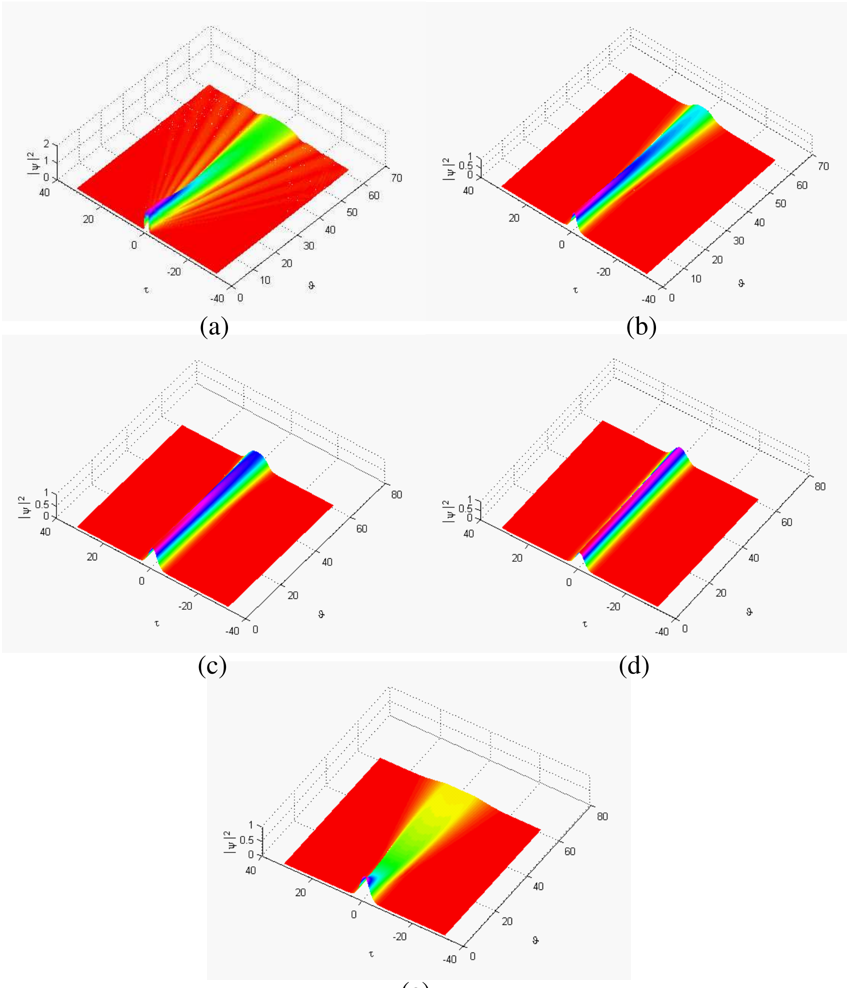

Figure 5. Simulation of power law plasmon soliton with $\psi = B_0 \mathrm{sech}^{1/p}(\tau)$, $B_0 = 0.8$, $p = 0.25$. (a) With no amplitude dependence in the phase. (b) With small attenuation $\Omega \neq 0$. (c) Phase depend on amplitude with $\Omega \neq 0$. (d) Stable with proper amplitude dependent phase (Eq. (38c)). (e) $\Omega \neq 0$ but complex.

The case $p = 1$ reduces to the Kerr law for the bright spatial plasmon soliton [15]. In this case our soliton parameter dynamics are:

$$\frac{dB}{dz} = 0 \quad \frac{dA}{dz} = 0 \quad \frac{d\bar{y}}{dz} = \frac{\kappa}{\beta W} \quad \frac{d\kappa}{dz} = 0 \tag{39}$$

The integral coefficients affect the power law soliton propagation, but do not change the usual properties (see Figure 5). The three conserved quantities energy (E), momentum (M), and the Hamiltonian (H) for the power law are

$$E = \int_{-\infty}^{\infty} |C|^2 \, dy = \left(\frac{\int \eta(x) \left[E_{x0}^{2p} + E_{z0}^{2p} \right] \left[E_{x0}^2 + E_{z0}^2 \right] dx}{\int \left| \vec{E}_0 \right|^2 dx} \right)^{\frac{1}{2} - \frac{1}{p}} B^{\frac{2-p}{p}} \left(\frac{1+p}{2p^2} \right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p} + \frac{1}{2}\right)} \tag{40}$$

$$\begin{aligned} M &= \frac{i}{2} \int_{-\infty}^{\infty} \left(C^* C_y - C C_y^* \right) dy \\ &= 2\kappa \left(\frac{\int \eta(x) \left[E_{x0}^{2p} + E_{z0}^{2p} \right] \left[E_{x0}^2 + E_{z0}^2 \right] dx}{\int \left| \vec{E}_0 \right|^2 dx} \right)^{\frac{1}{2} - \frac{1}{p}} \\ &\quad B^{\frac{2-p}{p}} \left(\frac{1+p}{2p^2} \right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p} + \frac{1}{2}\right)} \end{aligned} \tag{41}$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left(\frac{1}{2} |C_y|^2 - \frac{1}{1+p} |C|^{2p+2} \right) dy \\ &= \frac{\xi_{2p}^{\frac{1}{p}} A^2}{2p^2} \left[\left(\frac{\xi_{2p} A^{2p} \left(\frac{2p^2}{1+p} \right) + \kappa^2 p^2}{\xi_{2p}^{\frac{1}{2}} A^p \left(\frac{1+p}{2p^2} \right)^{\frac{1}{2}}} \right) \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{p} + \frac{1}{2}\right)} \right. \\ &\quad \left. - 2\xi_{2p}^{\frac{1}{2}} A^p \left(\frac{2p^2}{1+p} \right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p} + \frac{1}{2}\right)} \right] \end{aligned} \tag{42}$$

with

$$\xi_{2p} = \frac{\int \eta(x) \left[E_{x0}^{2p} + E_{z0}^{2p}\right] \left[E_{x0}^2 + E_{z0}^2\right] dx}{\int \left|\vec{E}_0\right|^2 dx} \tag{43}$$

where the Gamma function is defined as

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{44a}$$

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{2^m}\sqrt{\pi} \quad \text{for} \quad m=1,2,3\ldots. \tag{44b}$$

The conservative quantities depend on the initial beam profile and the integral coefficients. Referring back to Eq. (35), with $\Omega \neq 0$ the conservative quantities would change along with dynamic variables. To slowly add loss to the Eq. (35) we multiply $\alpha \ll 1$ to the loss term. The new equation with a perturbation is

$$-i\beta W \frac{\partial C}{\partial z} + \frac{1}{2}\frac{\partial^2 C}{\partial y^2} + \xi_{2p} \left|C\right|^{2p} C + i\alpha\Omega C = 0 \tag{45}$$

If we normalized Eq. (45), then resulting equation is

$$i\frac{\partial C'}{\partial z'} + \frac{1}{2}\frac{\partial^2 C'}{\partial y'^2} + \left|C'\right|^{2p} C' = -i\frac{\alpha\Omega}{\xi_{2p}}C' \tag{46a}$$

with transformations defined as

$$C\prime = C \tag{46b}$$

$$y' = \frac{1}{\sqrt{\xi_{2p}}}y \tag{46c}$$

$$\text{and} \quad z' = \frac{\xi_{2p}}{-\beta W}\, z. \tag{46d}$$

With the loss term include the adiabatic variation of the soliton parameters are

$$\frac{dA}{dz'} = \frac{\alpha}{2-p}A^{p-1}\left(\frac{2p^2}{1+p}\right)^{\frac{p-1}{2p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)}$$

$$\int_{-\infty}^{\infty} -\left(C^* \frac{\alpha\Omega}{\xi_{2p}}C + C\left(\frac{\alpha\Omega}{\xi_{2p}}\right)^* C^*\right) dy' \tag{47}$$

$$\frac{dB}{dz'} = \frac{\alpha p}{2-p} B^{\frac{p-2}{p}} \left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \int_{-\infty}^{\infty} -\left(C^* \frac{\alpha\Omega}{\xi_{2p}} C + C\left(\frac{\alpha\Omega}{\xi_{2p}}\right)^* C^*\right) dy' \tag{48}$$

$$\frac{d\kappa}{dz'} = \alpha B^{\frac{2p-2}{p}} \left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \left[i\int_{-\infty}^{\infty} -\left(C_y'^* \frac{\alpha\Omega}{\xi_{2p}} C - C_y'\left(\frac{\alpha\Omega}{\xi_{2p}}\right)^* C^*\right) dy' + \kappa \int_{-\infty}^{\infty} \left(C^* \frac{\alpha\Omega}{\xi_{2p}} C + C\left(\frac{\alpha\Omega}{\xi_{2p}}\right)^* C^*\right) dy' \right] \tag{49}$$

Transforming back from z' to z from the original NLSE and calculating integral on the right hand side of Eq. (47) through (49)

$$\frac{dA}{dz} = (\beta W)^{-1} \frac{\alpha}{2-p} A^{p-1} \left(\frac{2p^2}{1+p}\right)^{\frac{p-1}{2p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \alpha\Omega\sqrt{\xi_{2p}} 2E \tag{50}$$

$$\frac{dB}{dz} = (\beta W)^{-1} \frac{\alpha p}{2-p} B^{\frac{p-2}{p}} \left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \alpha\Omega\sqrt{\xi_{2p}} 2E \tag{51}$$

$$\frac{d\kappa}{dz} = \alpha (\beta W)^{-1} B^{\frac{2p-2}{p}} \left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{p}\right)} \kappa\alpha\Omega\sqrt{\xi_{2p}} 2E. \tag{52}$$

Equation (50) through (52) is ordinary (and possibly coupled) differential equations. The change in the soliton κ and amplitudes $A\&B$ depend on conserved energy, attenuation, and the integral coefficients of the initial input beam. Also, the change in energy, momentum, phase, and velocity as the evolution of the center of mass are:

$$\frac{dE}{dz} = (\beta W)^{-1} 2\sqrt{\xi_{2p}}\Omega\alpha E \tag{53}$$

$$\frac{dM}{dz} = 0 \tag{54}$$

$$\omega = \frac{d\phi}{dz} = \xi_{2p} (\beta W)^{-1} \left[\frac{B^2}{2} \frac{I_{0,0,2,0}}{I_{0,2,0,0}} - \frac{\kappa}{2} + \frac{1}{I_{0,2,0,0}} A^{2p} \int_{-\infty}^{\infty} \frac{1}{\cosh^{\frac{2}{p}+2}(s)} ds \right] \tag{55}$$

$$v = \frac{d\bar{y}}{dz} = \xi_{2p} \left(\beta W\right)^{-1} \left(\kappa + \frac{\alpha}{I_{0,2,0,0}} \frac{2B\alpha\Omega\sqrt{\xi_{2p}}}{A^2} \int_{-\infty}^{\infty} y \left|C\right|^2 dy \right) \quad (56)$$

with $I_{\gamma,\varsigma,\varphi,\varpi} = \int_{-\infty}^{\infty} \upsilon^{\gamma} g^{\varsigma}(\upsilon)(\frac{dg}{d\upsilon})^{\varphi}(\frac{d^2 g}{d\upsilon^2})^{\varpi} d\upsilon$ and $\upsilon = B(y - y(z))$.

The function $g(\upsilon)$ is the profile of the ansatz solution. With the loss term included in equation, one could also obtain a quasi-stationary solution but this will not be included here. Here, with some assumptions it has been shown that it is theoretically possible to have power law spatial soliton propagation at the nanoscale due to surface plasmons.

4. CONCLUSIONS

Although spatial solitons are diffraction limited in a Kerr medium alone, this limitation can be overcome by a nonlinear dielectric/metallic interface. It has been shown for this interface with a self defocusing medium dark spatial solitons exist theoretically called dark spatial plasmon polariton solitons. The proposed model for these dark plasmon solitons was a nonlinear evolution equation called the effective $(1 + 1)$ dimensional (first 1 is beam spread and the second 1 is propagation coordinate) Nonlinear Schrödinger Equation. Using soliton theory, adiabatic parameters were derived with the attenuation. Another type of interface was considered applying a power law material with a metal and then a power law type $(1+1)$ dimensional Nonlinear Schrödinger Equation was derived. Lastly, the attenuation was considered in the power law derived adiabatic equations for changes in the energy or power, momentum, spatial frequency, amplitude, velocity and phase.

One could also derive Algebraic and N soliton solutions for these dark and power law spatial plasmon solitons. One may also consider an approach with dielectric/metallic planar waveguides existence of vector solitons or derivation of a $(2 + 1)$ dimensional NLSE model. A more accurate approach to the above two systems would be to consider numerically the coupled Nonlinear Schrödinger Equation like equations which removes some approximations and assumptions. The dark and power law spatial plasmon solitons attenuate over short distance as they propagate along the dielectric metallic interface. This can possibly be minimized by considering solitons in tapered plasmonic waveguides, a subject of future investigation.

REFERENCES

1. Kivshar, Y. S. and G. P. Agrawal, *Optical Solitons from Fibers to Photonic Crystals*, Academic Press, 2003.
2. Stegeman, G. I., L. Jankovic, H. Kim, S. Polyakov, S. Carrasco, L. Torner, C. Bosshard, P. Gunter, M. Katz, and D. Eger, "Generation of and interactions between quadratic spatial solitons in non-critically-phase-matched crystals," *Journal of Nonlinear Optical Physics & Materials*, Vol. 12, No. 4, 447–466, 2003.
3. Hasegawa, A. and Y. Kodama, *Solitons in Optical Communications*, Oxford University Press, 1995.
4. Reitze, A. D., H. Weiner, and D. E. Leaird, "High power femtosecond optical pulse compression using spatial solitons," *Optics Letters*, Vol. 16, 1409–1411,1991.
5. Scheuer, J. and M. Orenstein, "Interactions and switching of spatial soliton pairs in the vicinity of a nonlinear interface," *Optics Letters*, Vol. 24, 1735–1737, 1999.
6. Pramono, Y. H., M. Geshiro, and T. Kitamura, "Optical Logic OR-AND-NOT and NOR Gates in waveguides consisting of nonlinear material," *IEICE Transactions on Electronics*, Vol. 11, 1755–1762, 2000.
7. Gisin, B. and B. A. Malomed, "One and two dimensional subwavelength solitons in saturable media," *Journal of Optical Society of America B*, Vol. 18, 1356–1361, 2001.
8. Gisin, B. and B. A. Malomed, "Subwavelength spatial solitons in optical media with non-Kerr nonlinearities," *Journal of Optics A: Pure and Applied Optics*, Vol. 3, 284–290, 2001.
9. Sarid, D. and W. Challener, *Modern Introduction to Surface Plasmons*, Cambridge Press, 2010.
10. Maier, S. A., *Plasmonics: Fundamentals and Applications*, Springer-Verlag, Berlin, 2007.
11. Bozhevolnyi, S. I., *Plasmonic Nanoguides and Circuits*, Pan Stanford Publ., Singapore, 2009.
12. Calzzani, F. A., R. Sileshi, A. Kassu, J. M. Taguenang, A. Chowdhury, A. Sharma, P. B. Ruffin, C. Brantley, and E. Edwards, "Detection of residual traces of explosives by surface enhance Raman scattering using gold coated substrates produced by nanospheres imprint technique," *Proceedings of SPIE*, Vol. 6949, 69451O, 2008.
13. Vincenti, M. A., et al., "Enhanced transmission and second harmonic generation from subwavelength slits on metal substrates,"

Proceedings of SPIE, Vol. 6987, 698700, 2008.
14. Scalora, M., G. D'Aguanno, N. Mattiucci, M. J. Bloemer, D. De Ceglia, M. Centini, A. Mandatori, C. Sibilia, N. Akozbek, M. G. Cappeddu, M. Fowler, and J. W. Haus, "Negative refraction and sub-wavelength focusing in the visible range using transparent metallo-dielectric stacks," *Optics Express*, Vol. 15, No. 508, 2007.
15. Davoyan, A. R., I. V. Shadrivov, and Y. S. Kivshar, "Self-focusing and spatial plasmon-polariton solitons," *Optics Express*, Vol. 17, No. 24, 21732–21737, 2009.
16. Edwards, M., S. H. Crutcher, M. D. Aggarwal, A. K. Batra, H. M. Jaenisch, A. J. Osei, and S. R. Bairavarasu, "Optical solitons in inhomogeneous media," *Proceedings of SPIE*, Vol. 5525, 2004.
17. Crutcher, S. H., M. Edwards, and A. Osei, "Optical spatial solitons, the power law, and the swing effect," *Proceedings of SPIE*, Vol. 7056, 70560Q-1, 2008.
18. Crutcher, S. H., M. E. Edwards, A. Biswas, and M. D. Aggarwal, "Oscillatory behavior of spatial soltions in two dimensional waveguides and stationary temporal power law solitons in optical fibers," *Journal of Electromagnetic Waves and Applications*, Vol. 20, No. 6, 761–772, 2006.
19. Trillo, S. and W. Torruellas, *Optical Spatial Solitons*, Springer Series in Optical Sciences, 2001.
20. Prade, B., J. Y. Vinet, and A. Mysyrowicz, "Guided optical waves in planar heterostructure with negative dielectric constant," *Physical Review B*, Vol. 44, No. 24, 556–572, 1991.
21. Bliokh, K. Y., Y. P. Bliokh, and A. Ferrando, "Resonant plasmon-soliton interaction," *Physical Review A*, Vol. 79, No. 24, 041803(R), 2009.
22. Feigenbaum, E. and M. Orenstein, "Plasmon-soliton," *Optics Letters*, Vol. 32, No. 6, 674–676, 2007.
23. Snyder, A. W. and D. J. Mitchell, "Spatial solitons of power law nonlinearity," *Optics Letters*, Vol. 18, No. 2, 101–103, 1993.
24. Mihalacke, D., M. Bertolotti, and C. Sibilia, *Progress in Optics XXVII*, E. Wolf (ed.), 228–309, Elsevier, Amsterdam, 1989.
25. Biswas, A., "Quasi-stationary optical solitons with power law nonlinearity," *Journal of Physics A: Mathematical and Theoretical*, Vol. 36, 4581–4589, 2003.
26. Biswas, A. and S. Konar, *Introduction to Non-Kerr Law Optical Solitons*, Chapman & Hall, 2007.
27. Sturdevant, B., D. A. Lott, and A. Biswas, "Topological solitons

in $1+2$ dimensions with time-dependent coefficients," *Progress In Electromagnetics Research Letters*, Vol. 10, 69–75, 2009.

28. Biswas, A., "1-soliton solution of (1 + 2)-dimensional nonlinear Schrödinger's equation in dual power law media," *Physics Letters A*, Vol. 372, 5941–5943, 2008.
29. Biswas, A., "Optical solitons in a parabolic law media with fourth order dispersion," *Applied Mathematics and Computation*, Vol. 208, 299–302, 2009.
30. Biswas, A., "1-soliton solution of 1 + 2 dimensional nonlinear Schrödinger's equation in power law media," *Communication in Nonlinear Science and Numerical Simulations*, Vol. 14, 1830–1833, 2009.
31. Boardman, A. D., G. S. Cooper, A. A. Maradudin, and T. P. Shen, "Surface — polariton solitons," *Physical Review B*, Vol. 34, 8273–8278, 1986.
32. Agranovich, V. M., V. S. Babichenko, and V. Ya Chernyak, "Nonlinear surface polaritons," *Soviet Physics. JETP Letters*, Vol. 32, 512–515, 1980.
33. Stegeman, G. H., C. T. Seaton, J. Ariyasu, R. F. Wallis, and A. A. Maradudin, "Nonlinear electromagnetic waves guided by a single interface," *Journal of Applied Physics*, Vol. 58, 2453–2459, 1985.
34. Boardman, A. D., G. H. Stegeman, A. A. Maradudin, T. Twardowski, and E. M. Wright, "Exact theory of nonlinear p-polarized optical waves," *Physical Review A*, Vol. 35, 1159–1164, 1987.