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# TRANSIENT SOLUTION FOR LOSSY TRANSMISSION LINE BY MEANS OF ORTHOGONAL PROJECTION METHOD

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**Abstract**—A novel electromagnetic transient analysis technique by means of the orthogonal projection method for lossy transmission line is proposed. By employing the proposed method, the traveling waves propagating from one terminal to another can be quickly obtained with less amount of computation at considerably large steps. First of all, the differential function to variable time can be approximated to be the convolution with a fixed vector relates to a certain set of orthogonal basis, e.g., Daubechies' basis. The partial differential telegraph equations related to both variable time $t$ and distance $x$ are then transformed to be differential equations only related to $x$. The solution of such equations can be obtained accordingly. The discrete coefficients of propagation function for lossy line are obtained as well, by which the propagating traveling waves can be calculated precisely at considerably large sampling periods with less amount of computation.

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## 1. INTRODUCTION

The transient analysis for transmission line plays an important role in the area of transient protective relaying. For example, traveling wave based differential protection, compared with the current differential protection, has advantages of high reliability and security, because the traveling wave based protection does not suffer the effect of the capacitive current generated by the distributed shunt capacitance [1–3], however, requires to calculate the traveling waves propagating from the other terminal in real time.

Most of transient calculation techniques are based on "characteristic method", in which the simultaneous telegraph equations are firstly transformed to frequency domain by Laplace transform, and then the solution is obtained in frequency domain. The time domain solution is given by taking inverse Laplace transform [4, 5].

Taking an example of a single phase transmission line MN, the line length is $L$, with the distributed parameters respectively series resistance $R_0$, inductance $L_0$, shunt conductance $G_0$ and capacitance $C_0$. The relation of traveling waves in frequency domain between two terminals (Terminal M and N) is shown in the following [8, 9]:

$$\begin{cases} F_N(s) = A_L(s)F_M(s) \\ B_M(s) = A_L(s)B_N(s) \end{cases} \tag{1}$$

where $F = U + Z_cI$ and $B = U - Z_cI$ are respectively the forward and backward voltage traveling waves; $A_L = \exp(-\gamma L)$ is the propagation function; $Z_c$ is the surge impedance; $\gamma$ is the propagation coefficient. The traveling waves in time domain should be taken the inverse Laplace transform as follows:

$$\begin{cases} f_N(t) = a_L(t) * f_M(t) \\ b_M(t) = a_L(t) * b_N(t) \end{cases} \tag{2}$$

where $a_L(t) = L^{-1}[A_L(s)]$ is the propagation function in time domain, which is the inverse Laplace transforms of $A_L(s)$. Symbol "*" means convolution operator.

For lossy transmission line, the propagation function $a_L(t)$ cannot be directly obtained. $A_L(s)$ is approximated to be rational functions with poles and zeros for taking inverse Laplace transform to convert it to time domain [6–8]. Moreover, the convolution integration is inconvenient for numerical computation.

Recently, several methods have been proposed to deal with these drawbacks as well as to avoid convolution integration. The Finite Element Method requires the line to be subdivided into a finite number of regions. This allows the telegraph equations to be converted into one-dimensional differential vector equations only related to variable

$t$, and the recursive formulas of each element can be obtained [9–11]. This method does not need inverse Laplace transform but needs large amount of recursive convolution computation. Waveform relaxation techniques avoid time domain convolution by solving the transmission line equations in the frequency domain and using the faster Fourier transform (FFT) to transform the results back and forth between time and frequency domain at each iteration [12–15]. However, this requires many data points in order to avoid aliasing effects when very fast signals have to be studied. Recently, growing attention has been devoted to wavelets in applied electromagnetics, mainly for the solution of Galerkin-like problems and as shape functions in the moment method to obtain sparse matrices [16]. These methods still require large amount of computation.

This paper presents a novel technique of transient analysis for lossy transmission line with distributed parameters, based on the orthogonal projection method. The discrete propagation function $\{a_L(n)\}$ can be obtained, which is convenient for numerical computation of the traveling waves.

First of all, the differential voltage and current related to variable time can be approximated to be numerical convolution with a fix vector, which is the projection of differential operators by a certain orthogonal basis. The partial differential equations related to both variables time $t$ and location $x$ are subsequently converted to be those only related to variable $x$. By sampling the approximated differential equations, the discrete equation of transmission line is established. The digital coefficients of propagation function are obtained by solving the discrete equation. By employing these propagation coefficients, the traveling wave propagating to the other terminal can be calculated in real time with less amount of computation at a considerably sampling steps. Compared with Electromagnetic Transient Program (EMTP), the proposed algorithm can accurately get the traveling waves in real time with less numerical computation, even in the case of lower sampling frequency.

## 2. APPROXIMATION THEOREM

### 2.1. The Function Approximation [17–19]

A supposed function $f(t)$, by the approximation theorem, can be approximated to be $f_N(t)$ in a certain sampling space $S_N$, with resolution of $N$, i.e., the sampling period $T_S = 1/N$:

$$f(t) \approx f_N = \sum_k c_k \varphi_{N,k}(t) \tag{3}$$

where $\varphi_{N,k}(t) = \sqrt{N}\varphi(Nt - k)$ and $\{\varphi_{N,k}(t) : k \in Z\}$ are a set of orthogonal basis of space $S_N$. Actually, due to the function $f_N \in S_N$, $f_N$ therefore can be linearly combined by the orthogonal basis, and the combination coefficients $c_k$ are also called the projection values to the basis.

Based on the orthogonal properties of the basis, the projection values can be obtained by the following:

$$c_k =< f, \varphi_{N,k} > \tag{4}$$

In which the symbol '<>' means inner product, defined as:

$$< f, g >= \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \tag{5}$$

This is very similar to that in the Euclidean Geometry Space (EG Space), e.g., in 2-dimensional EG Space an arbitrary vector can be linearly combined by the 2-dimensional orthogonal basis with the coefficient of its projection to the basis:

$$\mathbf{A} = A_x\mathbf{i}_x + A_y\mathbf{i}_y$$

where $\mathbf{i}_x = 1$ and $\mathbf{i}_y = j1$ are the basis of the 2-dimensional space; $A_x$ is the projection of vector $A$ to $\mathbf{i}_x$; $A_y$ is that to $\mathbf{i}_y$. Certainly the projection value is obtained by (note: the definition of inner product in EG Space is different from that in Function Space):

$$A_x =< \mathbf{A}, \mathbf{i}_x >, \quad A_y =< \mathbf{A}, \mathbf{i}_y >$$

## 2.2. The Approximation of Differential Function

Considering the differential function $f'(t) = \mathrm{d}f(t)/\mathrm{d}t$, based on formula (3), we obtain:

$$f'(t) \approx f'_N(t) = N\sum_k c_k\psi_{N,k}(t) \tag{6}$$

where $\psi(t) = \mathrm{d}\varphi(t)/\mathrm{d}t$, $\psi_{N,k}(t) = \sqrt{N}\psi(Nt - k)$.

Notice that $f'_N \in S_N$, which means that the differential function also can be linearly combined by the orthogonal basis with its projection values $d_k$:

$$f'(t) \approx f'_N(t) = \sum_k d_k\varphi_{N,k}(t) \tag{7}$$

Based on the previous projection principle, the coefficients $d_k$ should be obtained by the following formula:

$$d_k =< f'_N, \varphi_{N.k} >= N\sum_l c_l < \psi_{N,l}, \varphi_{N,k} >= N\sum_l c_l h_{k-l} \tag{8}$$

where

$$h_l = \int_{-\infty}^{\infty} \varphi(t)\overline{\psi(t-l)}dt$$

The coefficient $h_l$ is the projection of differential operator (PDO). The accurate numerical algorithm for obtaining PDO coefficients is presented in Appendix A.

Substituting formula (8) to (7), the approximation of differential function is shown as:

$$f'(t) \approx \frac{1}{T_S} \sum_k h_k f(t - kT_S) \tag{9}$$

# 3. THE APPLICATION TO LINE TRANSIENTS

## 3.1. Discrete Telegraph Equation of Line (DTEL)

For a single phase transmission line MN, whose length is $L$, associated with the distributed parameters, respectively, series resistance $R_0$, inductance $L_0$, shunt conductance $G_0$ and capacitance $C_0$, the telegraph equation is shown as the following formula in which $u(x,t)$ and $i(x,t)$ are respectively instantaneous voltage and current at location $x$.

$$\begin{cases} -\frac{\partial u(x,t)}{\partial x} = R_0 i(x,t) + L_0 \frac{\partial i(x,t)}{\partial t} \\ -\frac{\partial i(x,t)}{\partial x} = G_0 u(x,t) + C_0 \frac{\partial u(x,t)}{\partial t} \end{cases} \tag{10}$$

Formula (9) is employed to approximate the differential function to variable $t$. The partial differential equations (Equation (10)) are then converted to be equations only related to variable $x$. The numerical equations are subsequently obtained at sampling period $T_S$:

$$\begin{cases} -\frac{\mathrm{d}u(x,n)}{\mathrm{d}x} = R_0 i(x,n) + \frac{L_0}{T_S} \sum\limits_k h_k i(x, n-k) \\ -\frac{\mathrm{d}i(x,n)}{\mathrm{d}x} = G_0 u(x,n) + \frac{C_0}{T_S} \sum\limits_k h_k u(x, n-k) \end{cases} \tag{11}$$

## 3.2. Traveling Wave Differential Equations

Carefully observing the discrete Equation (11), one can find that it can be written as infinite dimensional matrix form:

$$\begin{cases} -\frac{\mathrm{d}\mathbf{V}(x)}{\mathrm{d}x} = \left(R_0\mathbf{E} + \frac{L_0}{T_S}\mathbf{H}\right)\mathbf{I}(x) \\ -\frac{\mathrm{d}\mathbf{I}(x)}{\mathrm{d}x} = \left(G_0\mathbf{E} + \frac{C_0}{T_S}\mathbf{H}\right)\mathbf{V}(x) \end{cases} \tag{12}$$

where $\mathbf{V}(x) = [\cdots, u(x,1), u(x,2), \cdots]^T$ is infinite voltage vector by discrete sampling voltages at location $x$. So does the current vector $\mathbf{I}(x) = [\cdots, i(x,1), i(x,2), \cdots]^T$. Matrix $\mathbf{E}$ is unit diagonal matrix, and Matrix $\mathbf{H}$ is composed by PDOs, shown as following:

$$\mathbf{H} = \begin{bmatrix} h_0 & h_1 & \cdots & \cdots & \cdots \\ h_{-1} & h_0 & h_1 & \cdots & \cdots \\ \cdots & h_{-1} & h_0 & h_1 & \cdots \\ \cdots & \cdots & h_{-1} & h_0 & h_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

The elements of each row of matrix $\mathbf{H}$ are PDOs, and the elements of next row are those coefficients right shifting of the previous row.

Let matrix

$$\mathbf{Z} = R_0\mathbf{E} + \frac{L_0}{T_S}\mathbf{H}, \quad \mathbf{Y} = G_0\mathbf{E} + \frac{C_0}{T_S}\mathbf{H}$$

Define matrix $\mathbf{Z}_C = \sqrt{\mathbf{Z}\mathbf{Y}^{-1}}$ as surge impedance matrix, $\Gamma = \sqrt{\mathbf{Z}\mathbf{Y}}$ as propagation matrix, $\mathbf{F} = \mathbf{V} + \mathbf{Z}_C\mathbf{I}$ as forward traveling wave vector, $\mathbf{B} = \mathbf{V} - \mathbf{Z}_C\mathbf{I}$ as backward traveling wave vector, one can subsequently convert Equation (12) to be the traveling wave equations (Equation (13) is deduced in Appendix B):

$$\begin{cases} \frac{\mathrm{d}\mathbf{F}(x)}{\mathrm{d}x} = -\Gamma x\mathbf{F}(x) \\ \frac{\mathrm{d}\mathbf{B}(x)}{\mathrm{d}x} = \Gamma x\mathbf{B}(x) \end{cases} \tag{13}$$

### 3.3. Solution of Differential Traveling Wave Equation

At the location $x = 0$, forward and backward traveling waves are obtained by voltage and current:

$$\begin{cases} \mathbf{F}(x=0) = \mathbf{F}_M = \mathbf{V}_M + \mathbf{Z}_C\mathbf{I}_M \\ \mathbf{B}(x=0) = \mathbf{B}_M = \mathbf{V}_M - \mathbf{Z}_C\mathbf{I}_M \end{cases}$$

Therefore, the solution of differential traveling wave equation is shown in the follows.

$$\begin{cases} \mathbf{F}(x) = \mathrm{expm}(-\Gamma x)\mathbf{F}_M \\ \mathbf{B}(x) = \mathrm{expm}(\Gamma x)\mathbf{B}_M \end{cases} \tag{14}$$

where symbol "expm" means the matrix exponential operation. Denote $\mathbf{A}_x = \mathrm{expm}(-\Gamma x)$ as Propagation Function matrix.

$$\mathrm{expm}(\mathbf{A}) = \mathbf{P}\begin{bmatrix} e^{-\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{-\lambda_N} \end{bmatrix}\mathbf{P}^{-1}$$

where $\{\lambda_k,\ k\ \in \mathrm{Z}\}$ are eigenvalues of matrix $A$, and $P$ is the transformation matrix composed by eigenvectors.

For a single phase transmission line with length $L$, the propagation relationship of traveling waves between terminals of the line should be written as:

$$\begin{cases} \mathbf{F}_N = \mathrm{expm}(-\Gamma L)\mathbf{F}_M = \mathbf{A}_L\mathbf{F}_M \\ \mathbf{B}_N = \mathrm{expm}(\Gamma L)\mathbf{B}_M = \mathbf{D}_L\mathbf{B}_N \end{cases} \tag{15}$$

where $\mathbf{A}_L$ is matrix of forward propagating function, and $\mathbf{D}_L$ is matrix of backward propagating function.

### 3.4. Matrices of Surge Impedance and Propagation Function

For calculating the propagation of the traveling waves, the matrices of surge impedance and propagation function should be obtained at first.

Observing the impedance matrix $\mathbf{Z}_C$ and propagation function $\mathbf{A}_L$, one can conclude that they are independent of the traveling waves and are determined only by the distributed parameters and projection values of differential operator. That is, the matrices should be calculated first.

Notice the fact that the elements in each row of matrix $\mathbf{H}$ are composed of a set of coefficients $h_l$. However, the elements in the next row is the left shift of the previous row. Matrices of surge impedance and propagation function have the same characteristics.

For example, the distributed parameters of a single phase line are shown in Table 1.

The elements of surge impedance matrix $\mathbf{Z}_C$ and propagation function matrix $\mathbf{A}_L$ are shown in Fig. 1. From Fig. 1 one can see that every row of the matrices have the same elements, but right shifting in next row.

Therefore, the complex matrix computation is not required, and only one row of matrices is required to numerical computation. The elements of the middle row are employed. For example, surge impedance coefficients $Z_C(k)$ is the middle number row of matrix $\mathbf{Z}_C$, and the propagation function coefficients $a_L(k)$ and $d_L(k)$ are respectively the middle number row of matrix $\mathbf{A}_L$ and $\mathbf{D}_L$. Then,

**Table 1.** The distributed parameter.

| Parameter | $R_0$ (Ohm/km) | $L_0$ (mH/km) | $G_0$ (S/km) | $C_0$ (uF/km) |
|---|---|---|---|---|
| Value | 0.0194 | 1.1710 | 0.0001 | 0.0098 |

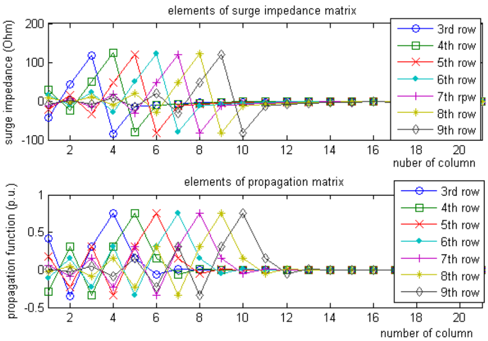


**Figure 1.** Matrices of surge impedance and propagation function.

the traveling waves are calculated by the following formula:

$$\begin{cases} f(n) = u(n) + \sum\limits_{k=-P}^{P} Z_c(k) i(n-k) \\ b(n) = u(n) - \sum\limits_{k=-P}^{P} Z_c(k) i(n-k) \end{cases} \tag{16}$$

The propagating relations in discrete time domain between the two terminals $M$ and $N$are shown in the following:

$$\begin{cases} f_N(n) = \sum\limits_{k=-P}^{P} a_L(k) f_M(n-k) \\ b_N(n) = \sum\limits_{k=-P}^{P} d_L(k) b_M(n-k) \end{cases} \tag{17}$$

### 3.5. Digital Algorithm for Calculating Propagating Traveling Waves

Assume a single phase transmission line MN, whose line length is $L$, and the distributed parameters are known. At one terminal (e.g., terminal M), the discrete voltage and current are measured with the sampling period $T_s$, and the question is how to calculate the forward traveling wave at the other terminal (e.g., terminal N).

First of all, the length (denoted as $P$) of the surge impedance and propagation function coefficients are determined. Obviously, the longer the length is selected, the better the accuracy is, but the larger amount the computation requires.

The matrix of surge impedance $\mathbf{Z}_C$ and the propagating function matrices $\mathbf{A}_L$ and $\mathbf{D}_L$ are calculated off-line. Then, the surge impedance coefficients $Z_C(k)$ and the coefficients of propagating function, respectively $a_L(k)$ and $d_L(k)$, are obtained by choosing the middle row of these matrices.

Formula (16) is employed to compute the forward and backward traveling waves at terminal M, based on the measured discrete voltage and current.

Formula (17) can be utilized to calculate the traveling waves at the other terminal, N.

### 3.6. Algorithm for Three Phase Transmission Lines

The Clark's transform is employed to convert the three phase voltages and currents to mode domain:

$$\begin{cases} u_1 = 2u_a - u_b - u_c \\ u_2 = u_b - u_c \end{cases}$$

$$\begin{cases} i_1 = 2i_a - i_b - i_c \\ i_2 = i_b - i_c \end{cases}$$

where the subscripts "1" and "2" represent modes 1 and 2, "$a$", "$b$" and "$c$" represent phases a, b and c.

## 4. SIMULATION RESULTS

This section presents simulation results of the new algorithm compared with the results of EMTP, associated with real 500 kV transmission network in ShanDong power system, shown in Fig. 2.

The transmission line is that of ZiBo Station to ZouXian Plant. The line length is 328 km, and the distributed parameters are shown in Table 1.

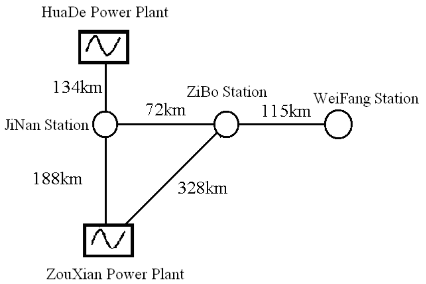


**Figure 2.** The associated 500 kV power network.

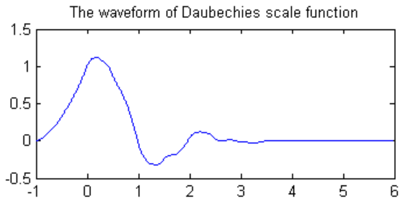


**Figure 3.** The waveform of scale function.

Suppose that the voltages and currents are measured at Zibo Station with 1.25 ms of sampling period. The forward and backward traveling waves at ZouXian terminal are required to calculate.

### 4.1. The Coefficients of Surge Impedance and Propagating Function

The Daubechies's function is selected to be projection basis. Higher order results in better accuracy, but longer length of of propagation function coefficients. For example, the fourth order Daubechies' scale function is employed in the new algorithm, and the coefficients of the scale function are shown as follows.

$$\{p_k\} = \{0.3258, 1.0109, 0.8922, -0.0396, -0.2645, 0.0436, 0.0465, -0.0150\}.$$

The waveform of the orthogonal scale function is presented in Fig. 3.

The coefficients of projection of differential operator are obtained by employing the algorithm shown in Appendix A. Based on these coefficients of differential operator, the coefficients of surge impedance and propagating function are then obtained. These results are shown in Table 2.

**Table 2.** The coefficients of $H$, $Z_c$, $A_L$ and $D_L$.

| No. | $H$ | $Z_c$ | $A_L$ | $D_L$ |
|---|---|---|---|---|
| 5 | −0.0002 | −3.0256 | 0.0040 | −0.0394 |
| 4 | −0.0022 | −7.8444 | −0.0033 | 0.0885 |
| 3 | 0.0336 | −7.3837 | −0.0322 | −0.1840 |
| 2 | −0.1920 | −18.2160 | 0.1571 | 0.3158 |
| 1 | 0.7930 | −75.1322 | 0.6785 | −0.4314 |
| 0 | 0.0000 | 271.7973 | 0.4241 | 0.4289 |
| −1 | −0.7930 | 51.3643 | −0.4236 | 0.6895 |
| −2 | 0.1920 | −32.2644 | 0.3128 | 0.1574 |
| −3 | −0.0336 | 15.4113 | −0.1804 | −0.0333 |
| −4 | 0.0022 | −14.2308 | 0.0894 | −0.0057 |
| −5 | 0.0002 | 4.0785 | −0.0382 | 0.0035 |

### 4.2. The Calculated Results Compared with EMTP

Suppose that a single phase to earth fault has taken place in the line of Zibo Station to Weifang Station. The waveforms of forward and backward traveling waves in mode 1 (the following waveforms of traveling waves are all in mode 1) at terminal Zouxian, which are calculated by the data of Zibo Station using the new algorithm, compared with the waveforms given by EMTP, are shown in Figs. 4 and 5.

From Figs. 4 and 5, one can see that the results calculated by new algorithm are relatively accurate, compared with the results by EMTP. The maximum error is not more than 12% at the sampling period 1.25 ms, and at the same time, with less amount of computation, only have 11 operations of summation and multiplication.

The reason that the error increases when fault takes place is that the waveform of voltage has a sudden change, which may result the larger error of approximation of differential operator at a larger step.

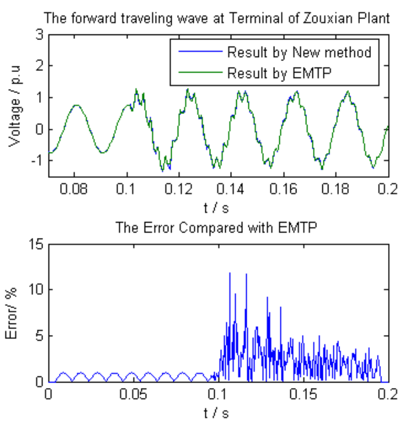


**Figure 4.** The waveforms of forward traveling wave at terminal of ZouXian Plant.

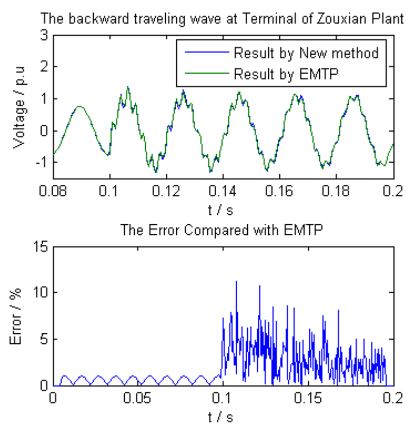


**Figure 5.** The waveforms of backward traveling waves at terminal ZouXian Plant.

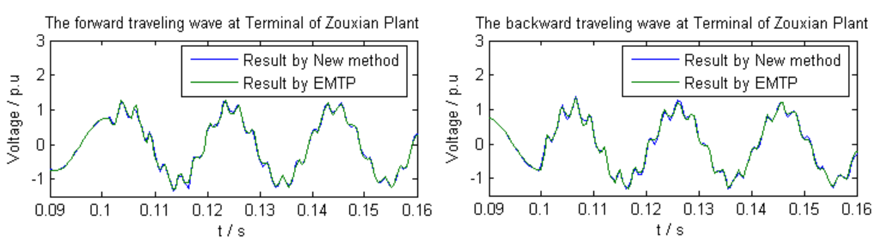


**Figure 6.** The waveform zooms in for closer image.

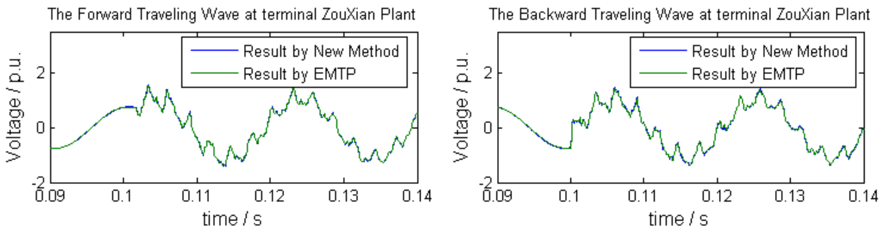


**Figure 7.** The results at the sampling period of 0.5 ms.

**Table 3.** maximum errors under various sampling periods.

| Sampling period (ms) | 1.25 | 1.125 | 1.00 | 0.50 | 0.10 |
|---|---|---|---|---|---|
| Error (%) | 12.3% | 8.9% | 7.1% | 6.3% | 4.8% |
| Length of coefficients | 7 | 11 | 15 | 31 | 91 |

For viewing the waveforms clearly, Fig. 6 is zoomed in for a closer image. One can see that the waveforms computed by the new method compactly coincide with the waveforms given by EMTP. These results show that the algorithm proposed in this paper is accurate with less computation.

### 4.3. The Results at Various Sampling Periods

The waveforms of traveling waves computed by the proposed method at the sampling period of 0.5 ms are shown in Fig. 7.

The maximum errors compared with the results given by EMTP at various sampling periods are shown in Table 3.

From Table 3, one can see that the higher the sampling frequency is, the higher the accuracy of the results is, but the longer length the propagation coefficients require, which means more computations.

## 5. CONCLUSIONS

This paper presents a new algorithm for precisely calculating the propagating traveling waves from one terminal to another with less amount of computation at a considerably large step. The new algorithm is based on the projection theorem in function space, by which the differential functions can be approximated by convolution with projection of differential operators. The surge impedance and propagating coefficients are then obtained by employing these theorems, and at the same time, using these coefficients, the propagating traveling waves are calculated with efficiency and accuracy. Compared with the results given by EMTP, the simulation tests associated with real power network show that the results are accurate with less computation.

## ACKNOWLEDGMENT

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## APPENDIX A.

The numerical algorithm for obtaining the coefficients of PDOs is shown in the following.

Since the projection of the differential operator can be written as the following formula.

$$h_l = \int_{-\infty}^{\infty} \varphi(t-l)\psi(t)\mathrm{d}t \tag{A1}$$

where $\varphi(t)$ is scale function in function space $S_0$, $\psi(t)$ is the derivative function of $\varphi(t)$.

According to the multi-resolution theorem, the scale function can be presented as the linear combination of the higher scale, shown in formula (A2)

$$\varphi(t) = \sum_{k=0}^{L-1} p_k \varphi(2t-k) \tag{A2}$$

In which $\{p_k\}$ are coefficients of the two scales.

Therefore, the differential function $\psi(t)$ can be written as:

$$\psi(t) = 2\sum_{k=0}^{L-1} p_k \psi(2t-k) \tag{A3}$$

Substitute formula (A1) and (A2) to formula (A1), the numerical algorithm for obtaining coefficients of PDOs can be seen in the follows:

$$\begin{aligned} h_l &= 2\int_{-\infty}^{\infty} \sum_k p_k \varphi(2t-k-2l) \sum_n p_n \varphi(2t-n) \\ &= \sum_k p_k \sum_n 2p_n \int_{-\infty}^{\infty} \varphi(2t-k-2l)\varphi(2t-n)dt \end{aligned} \tag{A4}$$

Let $t' = 2t + n$, one can get

$$\begin{aligned} h_l &= \sum_k p_k \sum_n p_n \int_{-\infty}^{\infty} \varphi(t' + n - k - 2l)\varphi(t')dt' \\ &= \sum_k \sum_n p_k p_n h_{2l+k-n} \end{aligned} \tag{A5}$$

Let $n' = k - n$, then

$$\begin{aligned} h_l &= \sum_k \sum_{n'} p_k p_{n'-k} h_{2l+n'} = \sum_n \left( \sum_k p_k p_{n-k} \right) h_{2l+n} \\ &= \sum_n \alpha_n h_{2l+n} \end{aligned} \tag{A6}$$

where $\alpha_n = \sum\limits_k p_k p_{n-k}$.

From formula (A6), one can see that the coefficients of PDOs $h_l$ are the solutions of Equation (A6) relates to vector $h_l$. The solution of the Equation (A6) can be obtained by Gauss iteration method. That is, giving the inception value of $h_l$, substitute it into Equation (A6), the new value could be obtained, repeat the procedure until the error is in the permission range.

## APPENDIX B.

Proof of Equation (13) in Section 3.2 is shown in the following.

Since the discrete time differential equation of the transmission line can be written as:

$$\begin{cases} -\frac{\mathrm{d}\mathbf{V}(x)}{\mathrm{d}x} = \mathbf{Z}\mathbf{I}(x) \\ -\frac{\mathrm{d}\mathbf{I}(x)}{\mathrm{d}x} = \mathbf{Y}\mathbf{V}(x) \end{cases} \tag{B1}$$

Using the first differential equation plus the second equation left multiplied matrix $Z_c$, one can get:

$$\frac{-\mathrm{d}}{\mathrm{d}x}[\mathbf{V}(x) + \mathbf{Z}_C\mathbf{I}(x)] = \mathbf{Z}\mathbf{I}(x) + \mathbf{Z}_C\mathbf{Y}\mathbf{V}(x) \tag{B2}$$

Notice the fact that the transpose of matrix $\mathbf{Z}$ or $\mathbf{Y}$ equals the minus of the matrix, that is $\mathbf{Z}^{\mathbf{T}} = -\mathbf{Z}$, $\mathbf{Y}^{\mathbf{T}} = -\mathbf{Y}$, further more, $(\mathbf{YZ})^T = \mathbf{YZ}$, due to the elements of the PDOs matrix $H$ has the character of $h_l = -h_{-l}$. One can conclude that:

$$\mathbf{YZ} = (\mathbf{YZ})^T = \mathbf{Z}^T\mathbf{Y}^T = (-\mathbf{Z})(-\mathbf{Y}) = \mathbf{ZY} \tag{B3}$$

Since $\mathbf{Z}_C = \sqrt{\mathbf{Z}\mathbf{Y}^{-1}}$, so we have:

$$\mathbf{Z}_C\mathbf{Y} = \sqrt{\mathbf{Z}\mathbf{Y}^{-1}\mathbf{Y}^2} = \sqrt{\mathbf{Z}\mathbf{Y}} = \Gamma \tag{B4}$$

Based on formula (B3), one can get that

$$\mathbf{Z} = \sqrt{\mathbf{Z}^2\mathbf{Y}\mathbf{Y}^{-1}} = \sqrt{\mathbf{Z}\mathbf{Y}\mathbf{Z}\mathbf{Y}^{-1}} = \Gamma\mathbf{Z}_C \tag{B5}$$

Substitute formula (B4) and (B5) into Equation (B2), one can get:

$$\frac{-\mathrm{d}}{\mathrm{d}x}[\mathbf{V}(x) + \mathbf{Z}_C\mathbf{I}(x)] = \Gamma[\mathbf{V}(x) + \mathbf{Z}_C\mathbf{I}(x)] \tag{B6}$$

Then the forward traveling wave differential equation in Equation (13) has been proved. The proof of the backward traveling wave differential equation is similar to that of forward traveling wave equation.

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