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ON THE FUNDAMENTAL EQUATIONS OF ELECTROMAGNETISM IN FINSLERIAN SPACETIMES

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Abstract—In spaces with Finslerian geometry, the metric tensor depends on the directional variable, which leads to a dependence on this variable of the electromagnetic tensor and of the 4-potential. In this paper, we investigate some of the consequences of this fact, regarding the basic notions and equations of classical electromagnetic field theory.

1. INTRODUCTION

Finsler geometry and its generalizations have found applications to a wide range of domains, such as: theory of anisotropic media ([7, 9]), Lagrangian mechanics, statistical physics and thermodynamics, theory of evolution of biological systems, theory of space-time and gravitation and even in attempts of unifying gravity and electromagnetism.

In what concerns Finslerian space-time models (e.g., in [1, 2, 4, 5, 10, 15, 16, 19, 22]), they arose from a series of questions of modern astrophysics. Problems like, for instance ([16]): rotation curves of spiral galaxies, the 3D-problem for spiral galaxies (usual gravity theory does not work in the plane of the galaxy but works in the orthogonal direction) or the location of globular clusters (which is close to the center of the galaxy and not at its periphery, as expected), indicate that, though classical General Relativity properly works at the scale of our solar system, still, for larger scales, some extra assumptions or modifications are needed. Among other hypotheses, Finsler geometry, as a generalization of Riemannian one, could provide a viable framework for solving such questions.

But, passing to spaces with Finslerian geometry, we can expect that there appear modifications in the fundamental notions and basic

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equations of electromagnetic field theory. The metric tensor depends not only on the coordinates on the spacetime manifold M, but also on a directional variable and more generally, the involved geometric objects are no longer defined on M, but on the tangent bundle TM. In the present paper, we are going to investigate some mathematical aspects of these changes.

A first (and beautiful) model for electromagnetism in Finsler spaces was developed starting with the late 80's — early 90's by Miron and collaborators [10, 12–14]. There, the electromagnetic tensor is regarded as arising from deflection tensors attached to a distinguished linear connection on TM. Still, in this approach, it is not necessarily related to a potential, which hinders the use of variational approaches and makes interpretation of the newly appeared objects more difficult.

In 2008–2009, in two joint papers with Siparov [3, 20], we provided a new approach (and with different results), based exclusively on variational calculus and differential form language. Trying to extend to Finsler spaces the idea of total action attached to the electromagnetic field together with a system of particles

$$S = -\sum mc \int ds - \sum \frac{q}{c} \int A_k dx^k - \frac{1}{16\pi c} \int F_{ij} F^{ij} d\Omega,$$

we need a generalization of the notion of 4-potential A. Since Maxwell equations involve the components $g_{ij}(x, y)$ of the metric tensor, in the Finslerian case, the dependence of g_{ij} on the fiber coordinates y^i on TM (i.e., on the directional variable) generally leads to solutions depending on these. It appears as reasonable the idea that the electromagnetic tensor and accordingly, the potential A, depend on both the base and on the fiber coordinates. Thus, we regarded the potential as a horizontal 1-form

$$A = A_i(x, y) dx^i$$

on TM, satisfying certain restrictions, and the electromagnetic tensor, as its exterior derivative:

$$F = dA.$$

The obtained electromagnetic tensor F is a 2-form on TM, consisting of two blocks — a horizontal $(dx^i \wedge dx^j)$ one, which is similar to the usual one, and a new, mixed $(dx^i \wedge \delta y^{\bar{j}})$ one, which appears in the Maxwell equations and in the equations of motion of charged particles.

The generalized Maxwell equations we proposed on TM have a similar form to the usual ones in pseudo-Riemannian spaces:

$$dF = 0, \quad (\delta F)^\sharp = -\frac{4\pi}{c} J,$$

where δ denotes codifferential, $\sharp$: $T^*M \to TM$ is the musical isomorphism (raising indices of 1-forms) and J is a vector field on TM, whose horizontal component is the usual 4-current (plus a correction due to the anisotropy of the space, which reminds the idea of bound current in a material). The vector field J has identically vanishing divergence, which provides an analogue of the usual continuity equation.

In this paper, we bring clarifications and improvements to these ideas.

First of all, we relax the definition of the 4-potential in [3, 20], namely, we only impose that its components A_i be 0-homogeneous in the fiber coordinates, which insures that the second term in the total action does not depend on the choice of the parameter on the integration path.

Also, we provide new details regarding the TM-current J, such as: its link to charge density, the relation between the continuity equation on TM and gauge invariance.

A problem in Finsler spaces M with metrics of Lorentz signature $(-,+,+,+)$, is having a well-defined volume form on M. The usual notions of volume form in Finsler spaces with positive definite metrics — Busemann-Hausdorff volume and Holmes-Thompson volume, [17] — would lead in our case to improper integrals (since they involve integration on indicatrices $\|y\| = 1$ at considered points of the base manifold, and these indicatrices are no longer compact). By using an appropriate completion of our metric up to a metric on TM, we solve this problem and adapt the idea of Holmes-Thompson volume to our case. A direct application is being able to write the total charge in a region of space as an integral of charge density.

In the last two sections, we propose a generalization of the notion of stress-energy tensor of the electromagnetic field to Finsler spaces. In flat (locally Minkowski) Finsler spaces, this generalization is obtained by symmetrizing the Noether current given by the invariance of the field Lagrangian to transformations on TM induced by spacetime translations. We obtain a tensor consisting of two blocks

$$T = T_{ij}dx^i \otimes dx^j + T_{i\bar{j}}dx^i \otimes \delta y^{\bar{j}}.$$

The obtained tensor identically satisfies a TM-analogue of the usual energy-momentum conservation law.

In curved Finsler spaces, the horizontal block $T_{ij}dx^i \otimes dx^j$ of the generalized energy-momentum tensor (which corresponds to the usual stress-energy tensor) can be obtained by varying the field Lagrangian with respect to the metric tensor and the mixed block $T_{i\bar{j}}dx^i \otimes \delta y^{\bar{j}}$, by varying the same Lagrangian with respect to the nonlinear connection.

2. THE RIEMANNIAN CASE — A BRIEF OVERVIEW

In this section, we will present in brief some basic ideas and methods in classical electromagnetic field theory. We will adopt the language of differential forms (see [8, 26–28]), which provides elegant, concise equations and is tightly related to variational calculus.

Let (M, g) be a Lorentzian manifold of dimension 4 (and class $\mathcal{C}^\infty$), regarded as spacetime manifold. We denote local coordinates on M by $x = (x^i)_{i=\overline{0,3}}$; the first coordinate is regarded as the time coordinate and $\mathbf{x} = (x^\alpha)_{\alpha=\overline{1,3}}$, as spatial coordinates. By $_{;k}$, we mean Levi-Civita covariant derivative with respect to $\frac{\partial}{\partial x^k}$, by $*$, the Hodge dual, by δ, the codifferential of p-forms and by $\flat$: $TM \to T^*M$, $\sharp$: $T^*M \to TM$, the musical isomorphisms (lowering/raising indices).

The 4-potential is described as a 1-form

$$A = A_i(x)dx^i. \tag{1}$$

The electromagnetic tensor (or *Faraday 2-form*) is its exterior derivative:

$$F = dA = \frac{1}{2}F_{jk}dx^j \wedge dx^k, \tag{2}$$

(where $F_{jk} = A_{k;j} - A_{j;k}$). In the language of differential forms, homogeneous Maxwell equations

$$F_{ij;k} + F_{ki;j} + F_{jk;i} = 0 \tag{3}$$

become [8],

$$dF = 0. \tag{4}$$

If the electromagnetic tensor F is given, then (4) implies (on a contractible domain) the existence of a 1-form A, such that $F = dA$. Conversely, if one considers the potential 1-form A as a priori given and *define* F as its exterior differential, then the homogeneous Maxwell equation is obtained as an identity.

An important property of the electromagnetic field is *gauge invariance*. Namely, the field strength tensor F is invariant to transformations

$$A \mapsto A + d\psi,$$

where $\psi : M \to \mathbb{R}$ is a differentiable function.

Inhomogeneous Maxwell equations and the equations of motion of charged particles are obtained by variational methods, namely, from the *total action* attached to the field and to a system of particles:

$$S = \underbrace{-\sum mc \int ds}_{S_p} \underbrace{- \sum \frac{q}{c} \int A_k(x)dx^k}_{S_{int}} \underbrace{- \frac{1}{16\pi c} \int F_{ij}F^{ij}d\Omega,}_{S_f} \tag{5}$$

(here, m, q, c are constants: m — the mass of a particle, q, its charge, c, the speed of light in vacuum and the sums are taken over the particles in the system; $d\Omega = \sqrt{|g|}dx$ is the invariant volume element on M). The first term S_p characterizes free particles, the third one S_f characterizes the electromagnetic field and the second one S_{int}, the interaction between the field and the particles.

Variation of the action S with respect to the 4-potential A (which actually means varying A in $S_1 := S_{int} + S_f$), leads to the inhomogeneous Maxwell equations; by varying S (written for a single particle) with respect to the trajectory, i.e., varying $S_2 := S_p + S_{int}$, one obtains the equations of motion of charged particles in a given electromagnetic field.

Charge density $\rho = \rho(x)$ is the amount of electric charge in a given spatial volume; its integral over a certain region of space provides the total charge situated inside that region:

$$q = \int \rho dV, \tag{6}$$

where $dV = \dfrac{\sqrt{|g|}}{\sqrt{g_{00}}} d\mathbf{x}$ is the spatial volume element [11]. Here, the charge distribution is regarded as continuous; for a discrete distribution of charges $q_1, ..., q_n$, the writing (6) is achieved by means of the Dirac delta function.

Relation (6) allows us to write S_{int} as a volume integral

$$S_{int} = -\frac{1}{c} \int A_i J^i d\Omega,$$

where the quantities

$$J^i := \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0} \tag{7}$$

are the components of a vector field J, called the *4-current*.

Thus, the sum $S_1 := S_{int} + S_f$ can be written as a single integral:

$$S_1 = -\int \left(\frac{1}{c} A_i J^i + \frac{1}{16\pi c} F_{ij} F^{ij} \right) d\Omega,$$

thus leading to *inhomogeneous Maxwell equations*

$$F^{ij}{}_{;j} = -\frac{4\pi}{c} J^i, \tag{8}$$

i.e., in a coordinate-free writing [8],

$$(\delta F)^\sharp = -\frac{4\pi}{c} J. \tag{9}$$

From (9), it follows that $-\frac{4\pi}{c}\delta J_\flat = \delta\delta F = 0$, i.e., the 4-current J identically satisfies the *continuity equation* (which is equivalent to the *charge conservation law*):

$$\operatorname{div}(J) = 0. \tag{10}$$

Equations of motion of charged particles are obtained as:

$$\frac{D\dot{x}^i}{ds} = \frac{q}{c} F^i{}_j \dot{x}^j, \quad i = \overline{0,3}, \tag{11}$$

where $\frac{D\dot{x}^i}{ds} = \frac{d\dot{x}^i}{ds} + \gamma^i{}_{jk}\dot{x}^j\dot{x}^k$ denotes Levi-Civita covariant derivative.

Finally, let us say a few words about the stress-energy-momentum tensor of the electromagnetic field.

A. In special relativity (where $M = \mathbb{R}^4$ and $g = \operatorname{diag}(-1,1,1,1)$ is the Minkowski metric), the energy-momentum tensor is obtained [11], by symmetrizing the Noether current given by the invariance of the field action S_f to spacetime translations $x \mapsto x + a$ (where $a = const.$); its expression is

$$T = T_{ij} dx^i \otimes dx^j, \qquad T^l{}_i = \frac{1}{4\pi}\left(-F^{lk}F_{ik} + \frac{1}{4}\delta^l_i F_{jk}F^{jk}\right). \tag{12}$$

The energy-momentum tensor satisfies the identities:

$$\frac{\partial T^j{}_i}{\partial x^j} = -\frac{1}{c}F_{ij}J^j \quad \leftrightarrow \quad \operatorname{div}(T) = \frac{1}{c} i_J F. \tag{13}$$

The quantity $\frac{1}{c}F_{ij}J^j$ provides the *density of Lorentz force* and the above relation is interpreted as conservation of the total energy and momentum of the system (field+particles).

B. In general relativity, the energy-momentum tensor T, (12), is obtained by varying S_f with respect to the metric

$$\delta_g S_f = \frac{1}{2c}\int T_{ik}\delta g^{ik} d\Omega = -\frac{1}{2c}\int T^{ik}\delta g_{ik} d\Omega. \tag{14}$$

In this case, the stress-energy tensor satisfies the identities:

$$T^j{}_{i;j} = -\frac{1}{c}F_{ij}J^j. \tag{15}$$

3. SOME GEOMETRIC STRUCTURES IN FINSLER SPACES

In the following, we will present some usual notions in Finsler geometry, such as: Ehresmann (nonlinear) connection (which provides invariant frames on TM), linear connection (producing a covariant derivation law which generalizes the Levi-Civita one) and introduce a volume form on Finslerian spacetimes.

Let now, for a 4-dimensional differentiable manifold M of class $\mathcal{C}^\infty$, (TM, π, M) be its tangent bundle and $(x, y) = (x^i, y^i)_{i=\overline{0,3}}$, the coordinates in a local chart on TM. The base coordinates $x = (x^i)$ will be called positional variables and the fiber ones $y = (y^i)$, directional variables.

We suppose that $(M, \mathcal{F})$ is a *Finsler space,* i.e., the function $\mathcal{F} : TM \to \mathbb{R}$ has the properties:

1) $\mathcal{F} = \mathcal{F}(x, y)$ is $\mathcal{C}^\infty$-smooth for $y \neq 0$;
2) positive homogeneity of degree 1 in the directional variable: $\mathcal{F}(x, \lambda y) = \lambda \mathcal{F}(x, y)$ for all $\lambda > 0$;
3) the *Finslerian metric tensor*:

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}, \tag{16}$$

is nondegenerate: $\det(g_{ij}(x, y)) \neq 0$, $\forall x \in M$, $y \in T_x M \backslash \{0\}$.

In the following, we will consider that the metric has signature $(-, +, +, +)$[†].

In a Finsler space, the squared element of arc length along a curve $t \mapsto x(t)$ is

$$ds^2 = \mathcal{F}^2 \left(x, \frac{dx}{dt} \right) dt^2 = g_{ij}(x, dx) dx^i dx^j.$$

Finsler spaces are a generalization of pseudo-Riemannian manifolds, in which the coefficients g_{ij} of the metric tensor are no longer functions defined on M, but on the tangent bundle TM. If on a usual Lorentzian manifold, the tangent space at each point carries a pseudo-Euclidean metric structure, in a Finsler space, at each fixed point x^0, the "norm" $\mathcal{F}(x_0, y)$ is generally not given by a quadratic form.[‡]

The 1-homogeneity of $\mathcal{F}$ in the fiber coordinates insures that the integral $\int ds$ does not depend on eventual changes of the parameter along the curve. Given a Finslerian metric tensor $g_{ij} = g_{ij}(x, y)$, the corresponding spatial metric is defined similarly to the pseudo-Riemannian case: $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}$, $\alpha, \beta \in \{1, 2, 3\}$ and its determinant is $\det(\gamma_{\alpha\beta}) = \frac{\sqrt{|g|}}{\sqrt{g_{00}}}$.

† Strictly speaking, it would be more rigorous to call these spaces "pseudo-Finsler". But since a lot of authors already use in the latter case the term *Finsler*, we will also adopt this more relaxed terminology.

‡ A very interesting insight on the interrelations between a non-Euclidean (though, still quadratic) geometry and electromagnetism, is given in [23].

With respect to coordinate changes on the tangent bundle TM, [6, 10]:

$$\tilde{x}^i = \tilde{x}^i(x), \qquad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j \tag{17}$$

the quantities $\frac{\partial}{\partial y^i}$ have a tensorial rule of transformation: $\frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}$, while the quantities $\frac{\partial}{\partial x^i}$ have a more complicated transformation law. Hence, if one wants to work with invariant blocks only, then one needs Ehresmann (nonlinear) connections, i.e., *adapted frames* on TM.

Let $(N^{\bar{j}}{}_i)$ be the coefficients of a nonlinear connection $TTM = HTM \oplus VTM$, [10, 17], and by

$$\left\{\delta_i = \frac{\partial}{\partial x^i} - N^{\bar{j}}{}_i \frac{\partial}{\partial y^{\bar{j}}}, \quad \partial_{\bar{\imath}} = \frac{\partial}{\partial y^{\bar{\imath}}}\right\}, \quad \left\{dx^i, \ \delta y^{\bar{\imath}} = dy^{\bar{\imath}} + N^{\bar{\imath}}{}_j dx^j\right\} \tag{18}$$

the elements of the corresponding *adapted basis* and of its *dual cobasis* respectively. With respect to coordinate changes (17), δ_i and $\partial_{\bar{\imath}}$ have tensorial rules of transformation, i.e.,

$$\delta_i = \frac{\partial \tilde{x}^j}{\partial x^i}\delta_j, \quad \partial_{\bar{\imath}} = \frac{\partial \tilde{x}^j}{\partial x^i}\partial_{\bar{j}}.$$

In the adapted basis, any vector field V on TM can be written as $V = V^i\delta_i + V^{\bar{\imath}}\partial_{\bar{\imath}}$; the component $hV = V^i\delta_i$ is a vector field, called the *horizontal* component of V, while $vV = V^{\bar{\imath}}\partial_{\bar{\imath}}$ is also a vector field, called its *vertical* component. Similarly, a 1-form ω on TM can be decomposed into invariant blocks as $\omega = \omega_i dx^i + \omega_{\bar{\imath}}\delta y^{\bar{\imath}}$, with $h\omega = \omega_i dx^i$ called the *horizontal* component, and $v\omega = \omega_{\bar{\imath}}\delta y^{\bar{\imath}}$ the *vertical* one [10]. Accordingly, any tensor field on TM is decomposed with respect to the Ehresmann connection into invariant blocks.

Whenever needed to make a clear distinction, we will denote by $i, j, k, \ldots$ indices corresponding to horizontal geometric objects, $\bar{\imath}$, $\bar{j}$, $\bar{k}, \ldots$ (with bars), indices corresponding to vertical ones and by capital letters $A, B, C, \ldots$ indices which take values corresponding to both distributions.

In order to make sense of the notion of volume form on TM and to be able to raise/lower indices of tensors on this space, we complete g up to a metric (G_{AB}) (an *hv-metric*, [10]) on TM:

$$G(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + v_{\bar{\imath}\bar{j}}(x, y)\delta y^{\bar{\imath}} \otimes \delta y^{\bar{j}}. \tag{19}$$

where v is a positive definite metric tensor[§].

[§] Assuming that the topological space M is metrizable, then a natural choice would be, for instance, a metric v which provides the topology of M. In the case when (M, g) is the Minkowski space, the manifold topology of $M = \mathbb{R}^4$ is the Euclidean one, hence we can choose v as the Euclidean metric.

It appears as convenient to express the results in terms of covariant derivatives given by the following linear connection D (inspired from [10]):

$$D_{\delta_k}\delta_j = L^i{}_{jk}\delta_i, \quad D_{\delta_k}\partial_{\bar{j}} = L^{\bar{\imath}}{}_{\bar{j}k}\partial_{\bar{\imath}}, \quad D_{\partial_{\bar{k}}}\delta_j = 0, \quad D_{\partial_{\bar{k}}}\partial_{\bar{j}} = 0. \tag{20}$$

where

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2}g^{ih}\left(\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}\right), \\ L^{\bar{\imath}}{}_{\bar{j}k} &= N^{\bar{\imath}}{}_{k\cdot\bar{j}} + \frac{1}{2}v^{\bar{\imath}\bar{h}}\left(\delta_k v_{\bar{h}\bar{j}} - \left(\partial_{\bar{j}}N^{\bar{l}}{}_k\right)v_{\bar{l}\bar{h}} - \left(\partial_{\bar{h}}N^{\bar{l}}{}_k\right)v_{\bar{l}\bar{j}}\right). \end{aligned} \tag{21}$$

We denote by $_{|k}$ covariant derivation by δ_k and by $\cdot_{\bar{k}}$, covariant derivation by $\partial_{\bar{k}}$.

The above linear connection preserves the distributions generated by the Ehresmann connection and it is h-metrical, i.e., $g_{ij|k} = 0$, $v_{\bar{\imath}\bar{j}|k} = 0$. The only nonvanishing components of its torsion tensor T are

$$\begin{aligned} R^{\bar{\imath}}{}_{jk} &= \delta y^{\bar{\imath}}(T(\delta_k, \delta_j)) = \delta_k N^{\bar{\imath}}{}_j - \delta_j N^{\bar{\imath}}{}_k; \\ P^{\bar{\imath}}{}_{j\bar{k}} &= \delta y^{\bar{\imath}}(T(\partial_{\bar{k}}, \delta_j)) = N^{\bar{\imath}}{}_{j\cdot\bar{k}} - L^{\bar{\imath}}{}_{\bar{k}j}. \end{aligned}$$

Having a metric structure (19) on TM, it makes sense the invariant volume element on TM :

$$d\Omega = \sqrt{|G|}dx \wedge dy.$$

where $G = \det(G_{AB})$. This volume element defines a volume element $d\Omega_M$ on M by:

$$d\Omega_M = \sigma(x)dx, \qquad \sigma(x) = \int\limits_{D_x}\sqrt{|G|}dy, \tag{22}$$

where $D_x = \{y \in T_xM \mid v_{ij}(x,y)y^iy^j \leq r^2\}$ and $r = \sqrt[4]{2/\pi^2}$ is the radius of a 3-sphere of volume 1 in the 4-dimensional Euclidean space. Then, for a function on a domain $\Delta \subset M$, f: $\Delta \to \mathbb{R}$, we have

$$\int\limits_{\Delta} f(x)d\Omega_M = \int\limits_{\Delta'} f(x)d\Omega,$$

where $\Delta' = \{(x,y) \in TM \mid x \in \Delta,\ y \in D_x\}$.

Similarly to ([11]), we can build the corresponding spatial volume element

$$dV = \tilde{\sigma}(x)d\mathbf{x}, \qquad \text{where } \tilde{\sigma}(x) = \int\limits_{D_x}\frac{\sqrt{|G|}}{\sqrt{g_{00}}}dy, \quad d\mathbf{x} = dx^1 \wedge dx^2 \wedge dx^3.$$

In the subsequent field theory considerations, we will integrate by y on domains D_x as above and with respect to x, on a "large enough" compact domain in M (assuming that far away from sources, the field is negligible and the considered time interval is a bounded one).

The *divergence* of a vector field $X = X^i\delta_i + X^{\bar{\imath}}\partial_{\bar{\imath}}$ on TM, [18], is locally expressed as

$$\begin{aligned}\mathrm{div}X &= \frac{1}{\sqrt{|G|}}\left[\delta_i\left(X^i\sqrt{|G|}\right) + \partial_{\bar{\imath}}\left(X^{\bar{\imath}}\sqrt{|G|}\right)\right] - N^{\bar{\jmath}}{}_{i\cdot\bar{\jmath}}X^i \\ &= X^i{}_{|i} + X^{\bar{\imath}}{}_{\cdot\bar{\imath}} - X^j P_j + X^{\bar{\imath}}\mathbf{C}_{\bar{\imath}}, \qquad (23)\end{aligned}$$

where

$$P_j = P^{\bar{\imath}}{}_{j\bar{\imath}}, \qquad \mathbf{C}_{\bar{\imath}} = \partial_{\bar{\imath}}\left(\ln\sqrt{|G|}\right).$$

The codifferential of a 2-form on TM:

$$\xi = \frac{1}{2}\xi_{ij}dx^i\wedge dx^j + \xi_{i\bar{\jmath}}dx^i\wedge\delta y^{\bar{\jmath}} + \frac{1}{2}\xi_{\bar{\imath}\bar{\jmath}}\delta y^{\bar{\imath}}\wedge\delta y^{\bar{\jmath}}$$

can be calculated from the relation $\langle\eta,\delta\xi\rangle = \langle d\eta,\xi\rangle$, where $\langle\ ,\ \rangle$ is the inner product of p-forms on TM. We get, in terms of covariant derivatives:

$$\begin{aligned}(\delta\xi)^i &= \xi^{ij}{}_{|j} + \xi^{i\bar{\jmath}}{}_{\cdot\bar{\jmath}} - \xi^{ij}P_j + \xi^{i\bar{\jmath}}\mathbf{C}_{\bar{\jmath}} \\ (\delta\xi)^{\bar{\imath}} &= \xi^{\bar{\imath}j}{}_{|j} + \xi^{\bar{\imath}\bar{\jmath}}{}_{\cdot\bar{\jmath}} - \xi^{\bar{\imath}j}P_j + \xi^{\bar{\imath}\bar{\jmath}}\mathbf{C}_{\bar{\jmath}} - \frac{1}{2}\xi^{jk}R^i{}_{jk} - \xi^{j\bar{k}}P^{\bar{\imath}}{}_{j\bar{k}}.\end{aligned}$$

4. FARADAY 2-FORM, HOMOGENEOUS MAXWELL EQUATIONS

Definition 1 By 4-potential, we will understand a horizontal 1-form

$$A = A_i(x,y)dx^i \qquad (24)$$

on TM, with the property that the components A_i are 0-homogeneous in y:

$$A_i(x,\lambda y) = A_i(x,y), \qquad \forall\lambda\in\mathbb{R}.$$

Remarks: 1) This is a more general definition than in [3, 20], where we also required that $A_i = (L_1)_{\cdot i}$ for some function L_1. In this paper, this supplementary restriction will be regarded as a "gauge" (Section 7), and not as a part of the definition.

2) Examples of anisotropic potentials (24) are given in [3].

The generalized Faraday 2-form (the *electromagnetic tensor*) is then defined as in [20]:

$$F = dA; \qquad (25)$$

in local coordinates, we get:

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j + F_{i\bar{j}} dx^i \wedge \delta y^{\bar{j}}, \tag{26}$$

where

$$F_{ij} = A_{j|i} - A_{i|j}, \quad F_{i\bar{j}} = -A_{i\cdot\bar{j}}. \tag{27}$$

Relation (25) leads to the generalized *homogeneous Maxwell equation*:

$$dF = 0, \tag{28}$$

or, locally:

$$\begin{aligned} &F_{ij|k} + F_{ki|j} + F_{jk|i} = -\sum_{(i,j,k)} R^{\bar{h}}{}_{jk} F_{i\bar{h}}; \\ &F_{\bar{i}j|k} + F_{k\bar{i}|j} + F_{jk\cdot\bar{i}} = P^{\bar{h}}_{j\bar{i}} F_{k\bar{h}} - P^{\bar{h}}_{k\bar{i}} F_{j\bar{h}}, \quad F_{k\bar{i}\cdot\bar{j}} + F_{\bar{j}k\cdot\bar{i}} = 0. \end{aligned} \tag{29}$$

In the above, we have started from A as an *a priori* given object and defined F as its exterior derivative. Conversely, on a contractible domain in TM, if it is given a closed 2-form

$$F := \frac{1}{2} F_{ij} dx^i \wedge dx^j + F_{i\bar{j}} dx^i \wedge \delta y^{\bar{j}}$$

then there exists, [20], a horizontal form A such that $F = dA$.

5. INHOMOGENEOUS MAXWELL EQUATIONS

In the following, let us see how the terms of the action (5) transform on TM.

The interaction term of the total action becomes

$$S_{int} = -\sum \frac{q}{c} \int A_i(x, \dot{x}) dx^i.$$

We write total charge as an integral:

$$q = \int \rho(x) dV = \int \frac{\rho(x)}{\sqrt{g_{00}}} \sqrt{G} d\mathbf{x} \wedge dy.$$

Then, with the notation

$$J^i = \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0}, \tag{30}$$

S_{int} is written as a volume integral on a domain in TM :

$$-\frac{q}{c} \int A_k dx^k = -\frac{1}{c} \int A_i J^i d\Omega \tag{31}$$

and the action $S_1 = S_{int} + S_f$ becomes:

$$S_1 = -\frac{1}{c}\int A_i J^i + \frac{1}{16\pi} F_{AB} F^{AB} d\Omega. \tag{32}$$

By varying it with respect to A, we get [20]:

$$F^{ij}{}_{|j} + F^{i\bar{j}}{}_{\cdot\bar{j}} - F^{ij}P_j + F^{i\bar{j}}C_{\bar{j}} = -\frac{4\pi}{c}J^i. \tag{33}$$

Equations (33) gave the idea to generalize the *inhomogeneous Maxwell equation* as

$$(\delta F)^\sharp = -\frac{4\pi}{c}J. \tag{34}$$

The above equation is formally similar to Equation (9); the difference is that, in the Finslerian case, the involved quantities are no longer defined on the spacetime manifold, but on its tangent bundle.

In local coordinates, this is:

$$F^{ij}{}_{|j} + F^{i\bar{j}}{}_{\cdot\bar{j}} + Q^i = -\frac{4\pi}{c}J^i, \qquad F^{\bar{\imath}j}{}_{|j} + Q^{\bar{\imath}} = -\frac{4\pi}{c}J^{\bar{\imath}}, \tag{35}$$

where[‖]

$$Q^i = -F^{ij}P_j + F^{i\bar{j}}\mathbf{C}_{\bar{j}}, \quad Q^{\bar{\imath}} = -F^{\bar{\imath}j}P_j - \frac{1}{2}F^{jk}R^{\bar{\imath}}{}_{jk} - F^{j\bar{k}}P^{\bar{\imath}}{}_{j\bar{k}}. \tag{36}$$

Thus, we have obtained a vector field on TM :

$$J = J^i\delta_i + J^{\bar{\imath}}\partial_{\bar{\imath}}$$

which we will call the *TM-current.*

6. CONTINUITY EQUATION AND GAUGE INVARIANCE

From (34), we have $-\frac{4\pi}{c}\delta J_\flat = \delta\delta F = 0$. In other words, [20]:

Proposition 2 There holds the generalized continuity equation:

$$\mathrm{div}(J) = 0. \tag{37}$$

The 2-form F remains invariant under *gauge transformations*

$$A(x,y) \mapsto A(x,y) + d\lambda(x), \tag{38}$$

where λ: $M \to \mathbb{R}$ is a scalar function. Consequently, in the general action (5), the first term S_p and the third one S_f will also be invariant.

Proposition 3 Transformations (38) do not affect the action $S_{int} = -\frac{1}{c}\int A_i J^i d\Omega$.

‖ Here is an erratum to the local expression of the inhomogeneous Maxwell equation in [20] – namely, in the term corresponding to $Q^{\bar{\imath}}$.

Proof. Let $\tilde{A} = A + d\lambda$. We have:

$$\int \tilde{A}_i J^i d\Omega = \int A_i J^i d\Omega + \int \frac{\partial \lambda}{\partial x^i} J^i d\Omega.$$

Since $\lambda = \lambda(x)$, we have $\frac{\partial \lambda}{\partial x^i} = \delta_i \lambda$ and

$$\int \frac{\partial \lambda}{\partial x^i} J^i d\Omega = \int \delta_i \left(\lambda J^i \sqrt{|G|} \right) dx \wedge dy - \int \lambda \delta_i \left(J^i \sqrt{|G|} \right) dx \wedge dy.$$

Adding and subtracting a $\int \lambda J^i N^{\bar{j}}_{i \cdot \bar{j}} d\Omega$, the right hand side becomes $\int \mathrm{div}(\lambda J^H) - \lambda \mathrm{div}(J^H) d\Omega$. Taking into account the continuity equation and $\lambda = \lambda(x)$, the latter integral is actually $\int \mathrm{div}(\lambda J^H) + div(\lambda J^V) d\Omega = \int \mathrm{div}(\lambda J) d\Omega$, i.e., it can be written as a boundary term. When performing variations of the action (and assuming, as in the classical case, that variations vanish on the boundary), this term will vanish.

7. EQUATIONS OF MOTION OF CHARGED PARTICLES

Let us consider the case of a single particle. The equations of motion are obtained by varying the trajectory $x = x(t)$ (where t is a parameter) in the first two terms of (5), which in our case become:

$$S_2 = -\int \left(mc\sqrt{g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j} + \frac{q}{c} A_k(x, \dot{x}) \dot{x}^k \right) dt. \tag{39}$$

The 0-homogeneity of A insures that the action S_2 does not depend on the choice of the parameter on the integration path, hence we can choose this parameter according to our wish. By choosing the arclength s as a parameter (i.e., $g_{ij} \dot{x}^i \dot{x}^j = 1$), the action S_2 is equivalent to the one provided by the Lagrangian

$$L = \frac{1}{2} mc g_{ij}(x, y) y^i y^j + \frac{q}{c} A_k(x, y) y^k, \quad y = \frac{dx}{ds}. \tag{40}$$

The canonical momentum of L is given by

$$p_i = \frac{\partial L}{\partial y^i} = mc y_i + \frac{q}{c} \left(A_{k \cdot i} y^k + A_i \right).$$

A further restriction can be imposed on the y-dependence of A in order to make all the approach more elegant and provide a simple relation of A with the canonical 4-momentum and simple equations of motion.

In pseudo-Riemannian spaces, where $A = A(x)$, then there exists only one potential 1-form providing a given interaction Lagrangian

$L_{int} = A_i(x)y^i$. But in Finsler spaces (where $A_i = A_i(x,y)$), $L_{int} = A_i(x,y)y^i$ can be given by infinitely many functions A_i; the relation $A_i y^i = \tilde{A}_i y^i$ is an equivalence, consequently, we can choose from each class the representative for which satisfies the *y-gradient condition:*

$$A_{k\cdot i}y^k = 0. \tag{41}$$

The y-gradient condition actually means:

$$A_i = \frac{\partial L_{int}}{\partial y^i}$$

Under this condition, the canonical 4-momentum is given by

$$p_i = \frac{\partial L}{\partial y^i} = mcy_i + \frac{q}{c}A_i$$

and the Euler-Lagrange equations for (40) are [20]:

$$mc\frac{Dy^i}{ds} = \frac{q}{c}F^i{}_j y^j + \frac{q}{c}F^i{}_{\bar{j}}\frac{\delta y^{\bar{j}}}{ds}, \quad y^i = \frac{dx^i}{ds}, \tag{42}$$

where $\frac{Dy^i}{ds} = \frac{dy^i}{ds} + L^i{}_{jk}y^j y^k$. Here, both the usual Lorentz force term given by $F^i = \frac{q}{c}F^i{}_h y^h$ and the correction given by $\tilde{F}^i = \frac{q}{c}F^i{}_{\bar{j}}\frac{\delta y^{\bar{j}}}{ds}$ are orthogonal to the velocity 4-vector $y = \dot{x}$, i.e., $g_{ij}F^i y^j = 0, \;\; g_{ij}\tilde{F}^i y^j = 0$.

8. STRESS-ENERGY TENSOR

8.1. In Flat Finsler Spaces

Let us consider, on the vector space $M = \mathbb{R}^4$, a flat (*locally Minkowskian* [6]) Finsler metric

$$g_{ij} = g_{ij}(y).$$

Assuming that coordinate transformations are linear (as in special relativity), we can choose the trivial Ehresmann connection $N^{\bar{i}}{}_j = 0$, hence $\delta_i = \frac{\partial}{\partial x^i}$ and $\delta y^{\bar{i}} = dy^{\bar{i}}$.

Spacetime translations $\bar{x}^i = x^i + \varepsilon^i$, $i = \overline{0,3}$ induce the following transformation on TM :

$$\bar{x}^i = x^i + \varepsilon^i, \;\; \bar{y}^i = y^i. \tag{43}$$

The action

$$S_f = -\frac{1}{16\pi c}\int F_{AB}F^{AB}d\Omega, \tag{44}$$

is invariant to transformations (43) (actually, this invariance reduces to the absence of an explicit dependence on x of the Lagrangian).

Definition 4 By *generalized stress-energy-momentum tensor of the electromagnetic field on* TM, we understand the symmetrized Noether current given by the invariance of the action (44) to transformations (43).

Generally, for an action on TM,

$$S = \frac{1}{c}\int \Lambda(q_{(k)}, \frac{\partial q_{(k)}}{\partial x^i}, \frac{\partial q_{(k)}}{\partial y^{\bar{\imath}}})d\Omega, \tag{45}$$

where $\Lambda = L\sqrt{|G|}$ is a Lagrangian density on TM and $q_{(k)} = q_{(k)}(x,y)$ are the field variables, the Euler-Lagrange equations are:

$$\frac{\partial}{\partial x^i}\left(\frac{\partial \Lambda}{\partial q_{(k),i}}\right) + \frac{\partial}{\partial y^{\bar{\imath}}}\left(\frac{\partial \Lambda}{\partial q_{(k)\cdot\bar{\imath}}}\right) - \frac{\partial \Lambda}{\partial q_{(k)}} = 0. \tag{46}$$

The absence of explicit dependence on x^i of Λ means

$$\frac{\partial \Lambda}{\partial x^i} = \frac{\partial \Lambda}{\partial q_{(k)}} q_{(k),i} + \frac{\partial \Lambda}{\partial q_{(k),l}} q_{(k),li} + \frac{\partial \Lambda}{\partial q_{(k)\cdot\bar{l}}} q_{(k)\cdot\bar{l}\ ,i}.$$

(where we understood summation over k). Substituting $\frac{\partial\Lambda}{\partial q_{(k)}}$ from (46), we get

$$\frac{\partial \tilde{T}^l{}_i}{\partial x^l} + \frac{\partial \tilde{T}^{\bar{l}}{}_i}{\partial y^{\bar{l}}} = 0,$$

where:

$$\tilde{T}^l{}_i = \left(q_{(k),i}\frac{\partial \Lambda}{\partial q_{(k),l}} - \delta^l_i \Lambda\right), \qquad \tilde{T}^{\bar{l}}{}_i = q_{(k),i}\frac{\partial \Lambda}{\partial q_{(k)\cdot\bar{l}}} \tag{47}$$

In order to "guess" the form of the generalized energy-momentum tensor for the electromagnetic field, we assume for the beginning that $J = 0$ and apply the above to :

$$\Lambda = -\frac{1}{16\pi}F_{BC}F^{BC}\sqrt{|G|}, \qquad q_{(k)} = A_k.$$

We get $\frac{\partial\Lambda}{\partial A_{k,l}} = -\frac{1}{4\pi}F^{lk}\sqrt{|G|}$, $\frac{\partial\Lambda}{\partial A_{k\cdot\bar{l}}} = -\frac{1}{4\pi}F^{\bar{l}k}\sqrt{|G|}$ and

$$\tilde{T}^l{}_i = \frac{1}{4\pi}\left(-F^{lk}A_{k,i} + \frac{1}{4}\delta^l_i F_{BC}F^{BC}\right)\sqrt{|G|}, \qquad \tilde{T}^{\bar{l}}{}_i = -\frac{1}{4\pi}F^{\bar{l}k}A_{k,i}\sqrt{|G|}.$$

For $J = 0$, it follows from the inhomogeneous Maxwell equations that the following terms

$$\frac{1}{4\pi}\left(F^{lk}A_{i,k} + F^{l\bar{k}}A_{i\cdot\bar{k}}\right)\sqrt{|G|} = \frac{1}{4\pi}\left(F^{lk}A_i\sqrt{|G|}\right)_{,k} + \frac{1}{4\pi}\left(F^{l\bar{k}}A_i\sqrt{|G|}\right)_{\cdot\bar{k}}$$

$$\frac{1}{4\pi}F^{\bar{l}k}A_{i,k}\sqrt{|G|} = \frac{1}{4\pi}\left(F^{\bar{l}k}A_i\sqrt{|G|}\right)_{,k}$$

provide divergences. By adding them to $\tilde{T}^l{}_i$ and $\tilde{T}^{\bar{l}}{}_i$ respectively and dividing by $\sqrt{|G|}$, we get the symmetrized tensor

$$T^l{}_i = \frac{1}{4\pi}\left(-F^{lB}F_{iB} + \frac{1}{4}\delta^l_i F_{BC}F^{BC}\right),\ T^{\bar{l}}{}_i = -\frac{1}{4\pi}F^{\bar{l}k}F_{ik}; \tag{48}$$

in the case $J = 0$, the divergence of T is identically 0.

This suggests the following

Definition 5 We call generalized energy-momentum tensor in the flat Finsler space $(\mathbb{R}^4, \mathcal{F}(y))$, the symmetric tensor

$$T = T_{ij}dx^i \otimes dx^j + T_{i\bar{j}}dx^i \otimes dy^{\bar{j}} \tag{49}$$

with local components given by (48).

If the TM-current J is arbitrary, then by using Maxwell equations, we get:

$$\frac{1}{\sqrt{|G|}}\left[\frac{\partial}{\partial x^j}\left(T^j{}_i\sqrt{|G|}\right) + \frac{\partial}{\partial y^{\bar{j}}}\left(T^{\bar{j}}{}_i\sqrt{|G|}\right)\right] = -\frac{1}{c}\left(F_{ij}J^j + F_{i\bar{j}}J^{\bar{j}}\right). \tag{50}$$

In brief, T identically satisfies the equality:

$$\operatorname{div}(T) = \frac{1}{c}h(i_J F), \tag{51}$$

where h denotes projection to the horizontal distribution HTM.

8.2. In General Finsler Spaces

In general (pseudo-) Finsler spaces, we define the generalized energy-momentum tensor of the electromagnetic field as above:

$$\begin{gathered} T = T_{ij}dx^i \otimes dx^j + T_{i\bar{j}}dx^i \otimes \delta y^{\bar{j}}, \\ T_{iA} = \frac{1}{4\pi}\left(-F_A{}^B F_{iB} + \frac{1}{4}g_{iA}F_{BC}F^{BC}\right), \end{gathered} \tag{52}$$

(where $g_{i\bar{j}} = 0$). By a direct computation, it follows that:

Proposition 6 The horizontal components T_{ij} of the generalized energy-momentum tensor can be obtained by varying the action S_f with respect to the spacetime metric g (i.e., with respect to the *horizontal* part of the metric (G_{AB})):

$$\delta_g S_f = \frac{1}{2c}\int T_{ik}\delta g^{ik}d\Omega,$$

while the mixed components $T_{i\bar{j}}$ are obtained by varying S_f with respect to the nonlinear connection N :

$$\delta_N S_f = \frac{1}{c}\int T^j{}_{\bar{\imath}}\delta N^{\bar{\imath}}{}_j d\Omega.$$

In curved Finsler spaces, the generalized energy-momentum tensor T satisfies some more complicated identities, involving the torsion of the linear connection D. The situation reminds the one in Riemann-Cartan geometry [21].

9. CONCLUSION

For a 4-dimensional pseudo-Finsler space $(M, \mathcal{F})$, we have generalized the basic notions and results in classical electromagnetic field theory. The 4-potential is defined as a horizontal 1-form $A = A_i(x, y)dx^i$ on the tangent bundle TM, having its components A_i homogeneous of degree 0 in y. The generalized electromagnetic tensor is the 2-form $F = dA$. Maxwell's equations on TM are then written as:

$$dF = 0, \quad \delta F = -\frac{4\pi}{c} J_\flat.$$

The TM-current $J = J^i\delta_i + J^{\bar{\imath}}\partial_{\bar{\imath}}$ is a vector field on TM satisfying identically $\operatorname{div} J = 0$. Its horizontal component $J^i\delta_i$ provides the usual notion of 4-current (plus a correction term due to the anisotropy of the space).

Further, for flat Finsler spaces $(M, \mathcal{F}(y))$, the generalized energy-momentum tensor is defined as the symmetrized Noether current corresponding to invariance of the field Lagrangian to spacetime translations. We obtained

$$T = T_{ij}dx^i \otimes dx^j + T_{i\bar{\jmath}}dx^i \otimes dy^{\bar{\jmath}}, \tag{53}$$

$$T_{\ iA} = \frac{1}{4\pi}\left(-F_A^B F_{iB} + \frac{1}{4} g_{iA} F_{BC} F^{BC}\right), \tag{54}$$

(where $g_{i\bar{\jmath}} = 0$ and A, B, C take all values corresponding to both horizontal and vertical components). The generalized energy-momentum tensor satisfies the identity:

$$\operatorname{div}(T) = \frac{1}{c} h(i_J F),$$

which is a generalization to TM of the usual energy-momentum conservation law.

In curved Finsler spaces, the components of the generalized stress-energy tensor can be obtained by varying the field action with respect to the metric $g_{ij}(x, y)$ (thus getting T_{ij}) and with respect to the Ehresmann connection N (which provides the components, $T_{i\bar{\jmath}}$).

We estimate that differential form language, combined with tangent bundle (in particular, Riemann-Finsler) geometry methods could also offer a very useful tool for the study of topical problems (e.g., in [24, 25, 29]) of electromagnetics in anisotropic media.

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