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SPINOR AND HERTZIAN DIFFERENTIAL FORMS IN ELECTROMAGNETISM

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Abstract—The purpose of this paper is to extend to spinor electromagnetism the differential forms, based on the Cartan exterior derivative and originally developed for tensor fields, in a very compact way. To this end, differential electromagnetic forms are first compared to conventional tensors. Then, using the local isomorphism between the O(3,C) and SL(2,C) groups supplying the well known connection between complex vectors and traceless second rank spinors, they are generalized to spinor electromagnetism and to Proca fields. These differential forms are finally expressed in terms of Hertz potentials.

1. INTRODUCTION

Differential forms pioneered by Cartan [1] (a differential p-form is a covariant skew-symmetric tensor field of degree p) and introduced some years ago in electromagnetism, may appear as a challenge to the conventional formalism described by scalars, vectors, tensors. All these entities, leaving aside General Relativity, are defined either in the Einstein 4-world with the Minkowski distance $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ or in the Newton $(3+1)$-world with the space Euclidean distance $ds^2 = dx^2 + dy^2 + dz^2$ and the time t apart.

Then, the electromagnetic field in the Minkowski space-time is defined by the skew tensors $F_{\mu\nu}(\mathbf{E},\mathbf{B}), G^{\mu\nu}(\mathbf{D},\mathbf{H})$ solutions of the Maxwell equations [2,3] covariant under the SO(3,1) group locally isomorphic to the Lorentz group SL(2,C)

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{1a}$$

$$\partial_\nu G^{\mu\nu} = J^\mu \tag{1b}$$

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in which the 4-vector J^μ is the electric current, the Greek indices take the values $1, 2, 3, 4$ (with $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = ct$, $\partial_j = \partial/\partial x^j$, $\partial_{4=} 1/c\partial/\partial_t$) and the summation convention is used.

But, this tensor formalism is not well suited to manage electromagnetic processes of practical importance with as consequence to promote electromagnetism in the Newton $(3 + 1)$-world.

Then the components $\mathbf{E}$ (electric field), $\mathbf{B}$ (magnetic induction) of $F_{\mu\nu}$ and $\mathbf{H}$ (magnetic field), $\mathbf{D}$ (dielectric displacement) of $G^{\mu\nu}$ become 3-vectors while the current J^μ splits into a 3-vector $\mathbf{j}$ and a scalar $\boldsymbol{\rho}$ so that the Maxwell equations have the Gibbs representation

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \wedge \mathbf{E} + 1/c\partial_t \mathbf{B} = 0 \tag{2a}$$

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \wedge \mathbf{H} - 1/c\partial_t \mathbf{D} = \mathbf{j} \tag{2b}$$

in which $\wedge$ is the wedge product symbol.

Remark: For a Riemannian space-time with the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, Eq. (1a) is unchanged [4, 5] while Eq. (1b) becomes $\partial_\nu |g|^{1/2} F^{\mu\nu} = J^\mu$ in which $|g|$ is the determinant of $g_{\mu\nu}$.

In a isotropic, homogeneous medium $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$, $\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$ with permittivity $\boldsymbol{\varepsilon}$ and permeability $\boldsymbol{\mu}$, we introduce the complex vector $\boldsymbol{\Lambda} = \sqrt{\boldsymbol{\varepsilon}}\ \mathbf{E} + i\sqrt{\boldsymbol{\mu}}\mathbf{H}$, $i = \sqrt{-1}$, the Λ_j's are in fact the components of the self-dual tensor $F_{\mu\nu} + i/2\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ in which $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor. Then, the Maxwell equations have the self-adjoint representation

$$\boldsymbol{\nabla} \wedge \boldsymbol{\Lambda} - i(n/c)\partial_t \boldsymbol{\Lambda} = i\mathbf{j}\sqrt{\mu}, \quad n\nabla \cdot \boldsymbol{\Lambda} = \rho\sqrt{\boldsymbol{\mu}} \quad , n = (\varepsilon\boldsymbol{\mu})^{1/2} \tag{2c}$$

and, these equations are covariant under the 3D-complex rotation group O(3,C) isomorphic to the SL(2,C) group.

In this work, the differential forms of Eqs. (2a), (2b) are first discussed and then, a generalization of the complex vector $\boldsymbol{\Lambda}$ is used to define spinor electromagnetic differential forms. Finally, the electromagnetic field is expressed in terms of Hertz differential forms.

2. ELECTROMAGNETIC DIFFERENTIAL FORMS

Following De Rham [6] {see [7] for the physical applications} and Meetz-Engl [8], Deschamps [9] introduced in electromagnetism the notion of exterior differential forms, gene-rating numerous works [10–16] on this subject. Exterior means that the algebra of differential forms is endowed with the Cartan exterior derivative d assigning to a p-form $\underline{w}$ (in this Note, differential forms are represented by underlined expressions) a $(p + 1)$-form $\underline{dw}$ such that d is linear and

$$d(\underline{w}_1 \wedge \underline{w}_2) = \underline{dw}_1 \wedge \underline{w}_2 + (-1)^p \underline{w}_1 \wedge \underline{dw}_2, \quad d\underline{dw} = 0 \tag{3}$$

In the (3+1)-world, the Maxwell equations with $d = dx\partial_x + dy\partial_y + dz\partial_z$ have the differential form representation

$$d \wedge \underline{E} + (1/c)\partial_t \underline{B} = 0, \quad d \wedge \underline{B} = 0 \tag{4a}$$
$$d \wedge \underline{H} - (1/c)\partial_t \underline{D} = \underline{j}, \quad d \wedge \underline{D} = \underline{\boldsymbol{\rho}} \tag{4b}$$

$\underline{E}$, $\underline{H}$ are the 1-form

$$\underline{E} = E_x dx + E_y dy + E_z dz, \quad \underline{H} = H_x dx + H_y dy + H_z dz \tag{5a}$$

and $\underline{B}$, $\underline{D}$ the 2-forms

$$\begin{aligned} \underline{B} &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy, \\ \underline{D} &= D_x dy \wedge dz + D_y dz \wedge dx + D_z dx \wedge dy \end{aligned} \tag{5b}$$

while $\underline{J}$ is a 2-form and $\underline{\boldsymbol{\rho}}$ a 3-form

$$\underline{J} = j_x dy \wedge dz + j_y dz \wedge dx + j_z dx \wedge dy, \quad \underline{\rho} = \rho dx \wedge dy \wedge dz \tag{5c}$$

In the Minkowski 4-world, the following 2-forms are defined [13] with the $F_{\mu\nu}$, $G^{\mu\nu}$ tensor fields

$$\underline{F} \;\; = 1/2 F_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{6a}$$
$$\underline{G} \;\; = 1/2 G_{\mu\nu} dx^\mu \wedge dx^\nu \tag{6b}$$

with in (6b) $G_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}\, G^{\alpha\beta}$ where $\varepsilon_{\mu\nu\alpha\beta}$ is the antisymmetric Levi-Civita tensor.

Let $d = dx^\mu \partial_\mu$ be the Cartan exterior derivative. Applying d to (6a), (6b) gives the 3-form

$$\underline{dF} \;=\; 1/2 \partial_\rho F_{\mu\nu} dx^\rho \wedge dx^\mu \wedge dx^\nu, \tag{7a}$$
$$\underline{dG} \;=\; 1/2 \partial_\rho G_{\mu\nu} dx^\rho \wedge dx^\mu \wedge dx^\nu \tag{7b}$$

and, according to the property of the Cartan exterior derivative, the electromagnetic field is solution of the 3-form equations

$$\underline{dF} \;=\; 0, \tag{8a}$$
$$\underline{\mathbf{d}G} \;=\; \underline{J}, \tag{8b}$$

in which $\underline{\mathrm{J}}$ is the 3-form

$$\underline{J} = 1/3! j^\mu \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{9}$$

The constitutive relations between the components ($\mathbf{E}$, $\mathbf{B}$) and ($\mathbf{D}$, $\mathbf{H}$) of the electromagnetic field in the Minkowski space-time have according to Post [17] the covariant expression

$$G^{\mu\nu} = 1/2 \chi^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{10}$$

the tensor $\chi^{\mu\nu\alpha\beta}$ has symmetries reducing to twenty the number of its independent components. Then, the 2-form $\underline{G}$ is changed into ${}^*\underline{F}$ (a notation introduced by analogy with the Hodge star operator [8–16])

$${}^*\underline{F} = 1/2 \chi^{\mu\nu}_{\rho\sigma} F_{\mu\nu} dx^\rho \wedge dx^\sigma \tag{11}$$

so that ${}^*\underline{dF}$ has to be substituted to $\underline{G}$ in (8).

To the self-dual field $\mathbf{\Lambda}$ is associated the differential form $\underline{L} = \sqrt{\varepsilon}\underline{F} + i\sqrt{\mu}\underline{G}$ with according to (6), (9), ε_{ijk} being the Levi-Civita tensor

$$\underline{F} = E_j \left(dx^j \wedge cdt\right) + 1/2\varepsilon_{jkl} B_j \left(dx^k \wedge dx^l\right) \tag{12a}$$

$$\underline{G} = H_j \left(dx^j \wedge cdt\right) - 1/2\varepsilon_{jkl} D_j \left(dx^k \wedge dx^l\right) \tag{12b}$$

$$\underline{J} = 1/2\varepsilon_{jkl} J_j \left(dx^k \wedge dx^l \wedge cdt\right) + \rho \left(dx \wedge dy \wedge dz\right) \tag{12c}$$

so that the differential form L becomes

$$\underline{L} = \Lambda_j \left(dx^j \wedge cdt\right) + in/2\varepsilon_{jkl} \Lambda_j \left(dx^k \wedge dx^l\right) \tag{13}$$

and, the differential form Eq. (8) reduces to $\underline{dL} = \underline{J}\sqrt{\mu}$. So, with differential forms, electromagnetism is written very compactly.

3. SPINOR ELECTROMAGNETIC DIFFERENTIAL FORMS

3.1. Spinors in Electromagnetism

Maxwell's equations, in absence of charge and current, (see Section 4.2 when charge and cur-rent exist) have the Gibbs representation

$$\nabla \wedge \mathbf{E} + (1+c)\partial_t \mathbf{B} = 0, \quad \nabla \wedge \mathbf{H} - (1+c)\partial_t \mathbf{D} = 0 \tag{14a}$$

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0 \tag{14b}$$

and, introducing the complex vectors

$$\mathbf{\Lambda} = \mathbf{E} + i\mathbf{H}, \quad \mathbf{\Omega} = \mathbf{B} + i\mathbf{D} \tag{15}$$

they become

$$\boldsymbol{\nabla} \wedge \mathbf{\Lambda} - (i/c)\partial_t \mathbf{\Omega} = 0, \tag{16a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{\Omega} = 0 \tag{16b}$$

In a homogeneous isotropic medium with permittivity $\boldsymbol{\varepsilon}$ and permeability $\boldsymbol{\mu}$, $\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$, $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$, but more generally $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ are some tensor with components $\boldsymbol{\varepsilon}_{ij}$, $\boldsymbol{\mu}_{ij}$.

Now, first rank spinors [18] and {in [19, 20] the spinors of higher ranks are discussed} are geometrical objects ψ_A defined on a two-dimensional complex space satisfying the transformation law $\psi'_A = t_A^B \psi_B$, $A, B = 1, 2$ with the complex 4 transformation matrix $||t_A^B||$ and a well known connection exists [21, 22] between complex vectors and traceless second rank spinors ψ_r^s, ϕ_r^s, $r, s = 1, 2$:

$$\Lambda_x + i\Lambda_y = \psi_1^2, \quad \Lambda_x - i\Lambda_y = \psi_2^1, \quad \Lambda_z = \psi_1^1 = -\psi_2^2 \quad \text{(17a)}$$
$$\Omega_x + i\Omega_y = \phi_1^2, \quad \Omega_x - i\Omega_y = \phi_2^1, \quad \Omega_z = \phi_1^1 = -\phi_2^2 \quad \text{(17b)}$$

So, D, D^t being the 2×2 matrix derivative operators with components

$$\begin{aligned} &D_{11} = -D_{22} = \partial^z, \quad D_{12} = \partial_x + i\partial_y, \quad D_{21} = \partial_x - i\partial_y \\ &D_{11}^t = D_{22}^t = 1/c\partial_t, \quad D_{12}^t = D_{21}^t = 0 \end{aligned} \tag{18}$$

the Maxwell equations have the Proca representation [19]

$$D\mathbf{\Psi} - D^t\mathbf{\Phi} = 0 \tag{19}$$

In which $\mathbf{\Psi}$, $\mathbf{\Phi}$ are the matrices with components

$$\psi_1^1 =^- \psi_2^2, \quad \psi_1^2, \quad \psi_2^1; \quad \phi_1^1 = -\phi_2^2, \quad \phi_1^2, \quad \phi_2^1 \quad \text{(19a)}$$

Taking into account (17(a), (17b), it is proved in Appendix A that Eqs. (19) imply Eq. (14).

Now, let $\underline{\Lambda}$ be the 1-form and $\underline{\Omega}$ the 2-form

$$\underline{\Lambda} = \Lambda_x dx + \Lambda_y dy + \Lambda_z dz, \quad \text{(20a)}$$
$$\underline{\Omega} = \Omega_x(dy \wedge dz) + \Omega_y(dz \wedge dx) + \Omega_z(dx \wedge dy) \quad \text{(20b)}$$

then, using the exterior derivative operator $d = dx\partial_x + dy\partial_y + dz\partial_z$, we get

$$\begin{aligned} \underline{d\Lambda} =& (\partial_y\Lambda_z - \partial_z\Lambda_y)(dy \wedge dz) + (\partial_z\Lambda_x - \partial_x\Lambda_z)(dz \wedge dx) \\ &+ (\partial_x\Lambda_y - \partial_y\Lambda_x)(dx \wedge dy) \\ \underline{d\Omega} =& (\partial_x\Omega_x + \partial_y\Omega_y + \partial_z\Omega_z)(dx \wedge dy \wedge dz) \end{aligned} \tag{21}$$

so that the Maxwell equations have the differential form representation in terms of Proca field

$$\begin{aligned} \underline{d\Lambda} - (i/c)\partial_t\underline{\Omega} = 0 \\ \underline{d\Omega} = 0 \end{aligned} \tag{22}$$

it is sufficient to observe that making null the coefficients of the differentials $dx^j \wedge dx^k$ gives the Maxwell Eq. (14) which may be considered as strong solutions of Eq. (22).

3.2. Spinor Differential Forms

Let us introduce the complex coordinates $\xi = x + iy$, $\eta = x - iy$ so that

$$\partial_x = \partial_\xi + \partial_\eta, \quad i\partial_y = \partial_\xi - \partial_\eta, \quad dx = 1/2(d\xi + d\eta), \quad idy = 1/2(d\xi - d\eta) \tag{23}$$

Then, substituting (23) into (20a), (20b) and taking into account (17a), (17b), we get

$$\begin{aligned} \underline{\Lambda} &= 1/2\psi_1^2 d\xi + 1/2\psi_2^1 d\eta + \psi_1^1 dz && \text{(24a)} \\ i\underline{\Omega} &= 1/2\phi_2^1(d\xi \wedge dz) + 1/2\phi_1^2(dz \wedge d\eta) + 1/2\phi_2^1(d\xi \wedge dz) \\ &\quad + 1/2\phi_1^1(d\eta \wedge d\xi) && \text{(24b)} \end{aligned}$$

while the exterior derivative operator becomes $d = d\xi\partial_\xi + d\eta\partial_\eta + dz\partial_z$ and a simple calculation gives

$$\begin{aligned} \underline{d\Lambda} &= \left(\partial_\xi\psi_1^1 - 1/2\partial_z\psi_2^1\right)(d\xi \wedge dz) + \left(1/2\partial_z\psi_1^2 - \partial_\eta\psi_1^1\right)(dz \wedge d\eta) \\ &\quad + \left(1/2\partial_\eta\psi_2^1 - 1/2\partial_\xi\psi_1^2\right)(d\eta \wedge d\xi) && \text{(25)} \end{aligned}$$

Substituting (24) and (25) into (22) gives the spinor differential form equation

$$\underline{d\chi} = 0 \tag{26}$$

$$\underline{d\chi} = \chi_2^1(d\xi \wedge dz) + \chi_1^2(dz \wedge d\eta) + \chi_1^1(d\eta \wedge d\xi) \tag{26a}$$

in which with $\partial_\tau = 1/c\partial_t$, the components of the spinor differential form are

$$\begin{aligned} \chi_2^1 &= \partial_\xi\psi_1^1 - 1/2\partial_z\psi_2^1 + 1/2\partial_\tau\phi_2^1 \\ \chi_1^2 &= 1/2\partial_z\psi_1^2 - \partial_\eta\psi_1^1 + 1/2\partial_\tau\phi_1^2 \\ \chi_1^1 &= 1/2\partial_\eta\psi_2^1 - 1/2\partial_\xi\psi_1^2 - 1/2\partial_\tau\phi_1^1 \end{aligned} \tag{27}$$

and

$$\left(\partial_\xi\psi_1^2 - \partial_\eta\psi_2^1 - \partial_z\psi_1^1\right)(d\xi \wedge d\eta \wedge dz) = 0 \tag{26b}$$

The strong solutions $\chi_r^s = 0$ of (26a) are, taking into account (17), the Maxwell Eq. (14). These results translate at once in a isotropic homogeneous medium.

4. HERTZ DIFFERENTIAL FORMS [16, 23]

4.1. ($\underline{\mathbf{E}}$, $\underline{\mathbf{B}}$) Forms

Let $\mathbf{\Pi}$ be a Hertz vector [3] on which no a-priori constraint is imposed except to be differentiable and $\underline{P}$ be the Hertzian 1-form ($\partial_\tau = 1/c\partial_t$)

$$\underline{P} = \partial_\tau\underline{\Pi} + cdt\underline{\Omega} \tag{28}$$

$\underline{\mathbf{\Pi}}$ is a 1-form, and we introduce the 0-form $\underline{\Omega}$ with an arbitrary constant γ

$$\underline{\mathbf{\Pi}} = \Pi_x dx + \Pi_y dy + \Pi_z dz, \quad \underline{\Omega} = \gamma\mathbf{\nabla}\cdot\mathbf{\Pi} \tag{29}$$

Applying to (28) the exterior derivative operator $d = (\partial_x dx + \partial_y dy + \partial_z dz + \partial_t dt)\wedge$ gives the 2-form $\underline{F} = \underline{dP}$

$$\begin{aligned}\underline{F} \;=\; & \left\{(dx \wedge cdt)(\partial_x \underline{\Omega} - \partial_\tau^2 \Pi_x)\right\} + \{\} + \{\} \\ & + c^{-1}\partial_\tau[\{(dx \wedge dy)(\partial_x \Pi_y - \partial_y \Pi_x)\} + \{\} + \{\}] \end{aligned} \tag{30}$$

in which the curly brackets are obtained by a circular permutation of x, y, z. ($\underline{\mathrm{F}}$ corresponds to the Hertz differential form used in [23]). Now, according to the Poincaré lemma [13, 14]

$$\underline{dF} = 0 \tag{31}$$

so that identifying $\underline{\mathrm{F}}$ with the electromagnetic field 2-form

$$\begin{aligned}\underline{F} =& E_x(dx \wedge cdt) + E_y(dy \wedge cdt) + E_z(dz \wedge cdt) + B_z(dx \wedge dy) \\ & + B_x(dy \wedge dz) + B_y(dz \wedge dx) \end{aligned} \tag{31a}$$

gives $\mathbf{E}$ and $\mathbf{B}$ in terms of Hertz potentials

$$\mathbf{E} \;= \gamma \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\Pi}) - \partial_\tau^2 \boldsymbol{\Pi}, \quad \mathbf{B} = \boldsymbol{\nabla} \wedge \partial_\tau \boldsymbol{\Pi} \tag{32}$$

so that ($\mathbf{E}$, $\mathbf{B}$) satisfies the first set of Maxwell's equations: $\nabla \cdot \mathbf{B} = 0$, $\nabla \wedge \mathbf{E} + \partial_\tau \mathbf{B} = 0$. Note that in (32) γ and $\boldsymbol{\Pi}$ are arbitrary scalar and vector.

4.2. ($\underline{\mathbf{D}}$, $\underline{\mathbf{H}}$) Forms

We first suppose that there is neither charge nor current and that the medium is homogeneous and isotropic with the constitutive relations $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$, $\mathbf{H} = \boldsymbol{\mu}^{-1}\mathbf{B}$. So, we get at once $\mathbf{D}$, $\mathbf{H}$ from (32) but we have to look for the conditions to impose on γ and $\boldsymbol{\Pi}$ to make $\mathbf{D}$, $\mathbf{H}$ solutions of the second set of Maxwell's equations $\nabla \cdot \mathbf{D} = 0$, $\nabla \wedge \mathbf{H} - \partial_\tau \mathbf{D} = 0$.

Then, proceeding somewhat backward to the way followed in Section 4.1, we start with the electromagnetic 2-form $\underline{G}(\mathbf{D}, \mathbf{H})$ which becomes for $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{H} = \mu^{-1}\mathbf{B}$

$$\begin{aligned}\underline{G}(\mathbf{D}, \mathbf{H}) =& -\varepsilon[E_x(dy \wedge dz) + E_y(dz \wedge dx) + E_z(dx \wedge dy)] \\ & + \mu^{-1}[B_x(dx \wedge cdt) + B_y(dy \wedge cdt) + B_z(dz \wedge cdt)] \end{aligned} \tag{33}$$

and we impose $\underline{G} =^* \underline{F}$ in which * is the Hodge star operator [6–16]

$$\begin{aligned} &*(dx \wedge cdt) = -\varepsilon(dy \wedge dz), \quad *(dy \wedge cdt) = -\varepsilon(dz \wedge dx), \\ &*(dz \wedge cdt) = -\varepsilon(dx \wedge dy), \end{aligned} \tag{34a}$$

$$\begin{aligned} &*(dy \wedge dz) = \mu^{-1}(dx \wedge cdt), \quad *(dz \wedge dx) = \mu^{-1}(dy \wedge cdt), \\ &*(dx \wedge dy) = \mu^{-1}(dz \wedge cdt) \end{aligned} \tag{34b}$$

Then, applying the *-operator to the 2-form (30) tgives

$$\begin{aligned}\underline{G} \;=\; & \varepsilon\left[(dy \wedge dz)\left(\partial_x \Omega - \partial_\tau^2 \Pi_x\right) + \{\} + \{\}\right] \\ & + 1/\mu c^{-1}\partial_\tau\left[\{(dz \wedge cdt)(\partial_x \Pi_y - \partial_y \Pi_x)\} + \{\} + \{\}\right] \end{aligned} \tag{35}$$

where as previously, the two curly brackets inside each square bracket are obtained from the previous one by a circular permutation of x, y, z.

Applying to (35) the exterior derivative operator d gives with $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$

$$\begin{aligned} \underline{dG} \ &= \ \varepsilon(dx \wedge dy \wedge dz)\left(\Delta\Omega - \partial_\tau^2 \mathbf{\Pi}\right) + c\partial_\tau\Big[(dx \wedge dz \wedge cdt) \\ &\quad \left[\mu^{-1}(\Delta\Pi_y - \partial_y \nabla \cdot \Pi) - \varepsilon(\partial_\tau^2 \Pi_y - \partial_y\Omega)\right] + \{\} + \{\}\Big] \end{aligned} \tag{36}$$

Then, in the same way that $\underline{dF} = 0$ identically when $\underline{F}$ is expressed in terms of Hertz vector, we impose $\underline{dG} = 0$ in the same situation which implies according to (36)

$$\Delta\mathbf{\Omega} - \partial_\tau^2 \mathbf{\nabla} \cdot \mathbf{\Pi} \ = \ 0, \tag{37a}$$

$$\partial_\tau \left[-\boldsymbol{\mu}^{-1}\Delta\mathbf{\Pi} + \boldsymbol{\mu}^{-1}\mathbf{\nabla}\left(\mathbf{\nabla} \cdot \mathbf{\Pi}\right) + \varepsilon\left(\mathbf{\nabla}\Omega - c\partial_\tau^2 \mathbf{\Pi}\right)\right] \ = \ 0 \tag{37b}$$

Taking into account (29), Eq. (37a) becomes

$$\mathbf{\nabla} \cdot \left(\gamma \mathbf{\Delta\Pi} - \partial_\tau^2 \mathbf{\Pi}\right) = 0 \tag{38}$$

Similarly, we may write (37b)

$$\partial_\tau \left[-\Delta\mathbf{\Pi} + (1 - \gamma\varepsilon\mu)\mathbf{\nabla}(\mathbf{\nabla} \cdot \mathbf{\Pi}) + \boldsymbol{\varepsilon\mu\partial}_\tau^2 \mathbf{\Pi}\right] = 0 \tag{39}$$

reducing to (38) for $\gamma = 1/\varepsilon\mu$ provided that the Hertz potential is solution of the wave equation

$$\left(\Delta - \varepsilon\mu\partial_\tau^2\right)\mathbf{\Pi} = 0 \tag{40}$$

When a charge ρ and a current $\mathbf{j}$ exist, we introduce the 3-form $\underline{J}$:

$$\underline{J} = -j_x(dy \wedge dz \wedge dt) - j_y(dz \wedge dx \wedge dt) - j_z(dx \wedge dy \wedge dt) + \boldsymbol{\rho}(dx \wedge dy \wedge dz) \tag{41}$$

and the equation $d\underline{G} = 0$ transforms into $d\underline{G} = \underline{J}$ so that with the vector $\mathbf{p}$ such as

$$(1/c)\partial_t \mathbf{p} = \mathbf{j}\ , \quad \mathbf{\nabla \cdot} \boldsymbol{p} = \boldsymbol{\rho} \tag{42}$$

the Eqs. (37a), (37b) become

$$\nabla \cdot \left(\mathbf{\Delta\Pi} - \varepsilon\mu\partial_\tau^2 \mathbf{\Pi}\right) \ = \ \nabla \cdot \mathbf{p}/\boldsymbol{\varepsilon} \tag{43a}$$

$$\partial_\tau \left(\mathbf{\Delta\Pi} - \boldsymbol{\varepsilon\mu}\partial_\tau^2 \mathbf{\Pi}\right) \ = \ 1/\varepsilon\partial_\tau \mathbf{p} \tag{43b}$$

We get from (43a)

$$\mathbf{\Delta\Pi} - \boldsymbol{\varepsilon\mu}\partial_\tau^2 \mathbf{\Pi} + \mathbf{p}/\varepsilon = \nabla \wedge \mathbf{f} \tag{44}$$

and according to (43b) $\nabla \wedge \partial_\tau \mathbf{f} \ = 0$. So, let $\mathbf{q}$ be a vector satisfying the relations

$$\mathbf{\nabla} \wedge \Delta\mathbf{q} = -\mathbf{p}/\varepsilon, \quad \partial_\tau \mathbf{q} = 0 \tag{45}$$

and $\mathbf{\Pi}^\circ$ be the vector

$$\mathbf{\Pi}^\circ = \mathbf{\Pi} + \boldsymbol{\nabla} \wedge \mathbf{q} \tag{46}$$

then, both Eqs. (37a), (37b) reduce to the nonhomogeneous wave equation

$$\left(\Delta - \varepsilon \boldsymbol{\mu} \partial_\tau^2\right) \mathbf{\Pi} = -\mathbf{p}/\boldsymbol{\varepsilon} \tag{47}$$

and $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{H} = \mu^{-1}\mathbf{B}$, with $\mathbf{E}$, $\mathbf{B}$ is obtained from (32), $\mathbf{\Pi}$ being changed into $\mathbf{\Pi}^\circ$.

As an application we consider a polarized medium. As easy to prove, the second set of Maxwell's equations: $\nabla \cdot \mathbf{D} = 0$, $\nabla \wedge \mathbf{H} - (1/c)\partial_t \mathbf{D} = 0$ has the further solutions in terms of Hertz potentials

$$\mathbf{D}^\dagger = \mathbf{D} - (ß/c)\boldsymbol{\nabla} \cdot \partial_t \mathbf{\Pi}, \quad \mathbf{H}^\dagger = \mathbf{H} - \left(\beta/c^2\right) \partial_t^2 \mathbf{\Pi} - \boldsymbol{\beta} \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{\Pi} \tag{48}$$

that, we write in a isotropic homogeneous medium

$$\mathbf{D}^\dagger = \varepsilon \mathbf{E} + \mathbf{P}, \quad \mathbf{B}/\mu = \mathbf{H}^\dagger + \mathbf{M} \tag{49}$$

in which $\mathbf{P}$, $\mathbf{M}$ are the electric and magnetic polarization of the medium so that according to (48)

$$\mathbf{P} = ß\nabla \wedge \partial_\tau \mathbf{\Pi}, \quad \mathbf{M} = -\beta \partial_\tau^2 \mathbf{\Pi} - \beta \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{\Pi} \tag{50}$$

$\mathbf{\Pi}$ being a solution of the wave equation $(\Delta - \varepsilon\mu \boldsymbol{\partial}_\tau^2)\mathbf{\Pi} = 0$.

Writing $\mathbf{P} = ß\mathbf{B}$, $\mathbf{M} = ß\mathbf{E}$ in agreement with (32) the relations (49) become the Post constitutive relations in a isotropic chiral medium [34], ß being the chirality parameter

$$\mathbf{D}^\dagger = \varepsilon \mathbf{E} - ß\mathbf{B}, \quad \mathbf{H}^\dagger = \mathbf{B}/\mu - ß\mathbf{E} \tag{51}$$

We now have to look for a differential form consistent with (49) but it is easier to work with (51). Then, applying the Hodge star operator

$$\begin{aligned}
*(dx \wedge cdt) &= \varepsilon(dy \wedge dz) + ß(dx \wedge cdt), \\
*(dy \wedge dz) &= \mu^{-1}(dx \wedge cdt) + ß(dy \wedge dz) \\
*(dy \wedge cdt) &= \varepsilon(dz \wedge dx) + ß(dy \wedge cdt), \\
*(dz \wedge dx) &= \mu^{-1}(dy \wedge cdt) + ß(dz \wedge dx) \\
*(dz \wedge cdt) &= \varepsilon(dx \wedge dy) + ß(dz \wedge cdt), \\
*(dx \wedge dy) &= \mu^{-1}(dz \wedge cdt) + ß(dx \wedge dy)
\end{aligned} \tag{52}$$

to the 2-form $\underline{F}$ written according to (31) with the previous definition of the curly brackets

$$\underline{F} = \{E_x(dx \wedge cdt) + B_x(dy \wedge dz)\} + \{\} + \{\} \tag{53}$$

give $\underline{G} =^* \underline{F}$ and according to (53)

$$\begin{aligned}\underline{G} &= \{E_x[\varepsilon(dy\wedge dz)+\text{ß}(dx\wedge cdt)]-B_x[\mu^{-1}(dx\wedge cdt)+\text{ß}(dy\wedge dz)]\}\\ &\quad +\{\}+\{\}=\{(\varepsilon\mathrm{E}_x+\beta\mathrm{B}_x)(dy\wedge dz)+(\mu^{-1}B_x-\beta\mathrm{E}_x)(dx\wedge cdt)\}\\ &\quad +\{\}+\{\}\end{aligned}\tag{54}$$

The comparison of (35) and (54) implies (50).

Incidentally, the Post constitutive relations have the remarkable property to preserve the dissymmetry between the two sets of Maxwell's equations which is a particular feature among the great diversity of proposed constitutive relations This makes calculations easier [37] for radiation from an electric dipole source and for Cherenkov radiation.

Hertz potentials have also been recently used [38] to get a complete description of refraction in a uniaxial anisotropic medium for harmonic plane waves and Gaussian beams. With oz as principal axis and, to work with $\underline{F}$ and $\underline{G}$, one has just to change ε ($\varepsilon_{11}=\varepsilon_{22}$) into $\boldsymbol{\eta}=\varepsilon_{33}$ in the third relation (34a).

5. CONCLUSION

The differential forms give a compact description of Maxwell's equations, see (8), (22), (26), (31), but to get their solutions which are in fact weak solutions of Maxwell's equations requires some integration performed with the help of Stokes's theorem [22]: let ω be a 2-form, then

$$\iint_S d\omega = \int_C \omega \tag{55}$$

in which S is an orientable surface with boundary C.

Many works have been devoted to numerical approximations of (55) for vector fields. Let us consider the Eq. (8b) in the $(3+1)$ Newton space-time, we get from (55)

$$\int_M \underline{dG} = \int_{\partial M} \underline{G} \tag{56}$$

M is a 2D-manifold in R^3 with the boundary ∂M. It has been proved [13] that the finite element technique, largely used in the numerical simulation of partial differential equations [23] may be applied to differential forms. Then, as approximations of $\int \underline{dG} = \int \underline{J}$, we get the integrals

$$\int_M \underline{G}\wedge\underline{d\alpha} = \int_M \underline{J\alpha} \quad \text{for any } \underline{\alpha}\in A \tag{57}$$

α is 0-form belonging to the set A of test functions which are for differential forms the Whit-ney functions [24] while $\underline{d\alpha}$ is a 1-form. So the integrands in both sides of Eq. (57) are 2-forms consistent with the 2D-dimension of M described by a family of triangular surface elements [25, 26].

For spinor fields using (26), we get similarly from (55)

$$\int_M \underline{d\chi} = \int_{\partial M} \underline{\chi} \tag{58}$$

where M is now a 2D-manifold in the complex spin space. So, an important numerical work has still to be made to get approximations of (58).

In some cases, analytical solutions may be obtained in terms of differential forms for fields and currents as for instance for the Wilsons' experiment [10, p. 354], which makes a comparison possible with the conventional explanation [29] of this experiment.

The spinor formalism was precedently used in two different domains: analysis of diffraction patterns of electromagnetic fields by apertures [30] and comparison of the Witten-Penrose [31, 32] and Infeld = Van Der Waerden spinor approaches to General Relativity [33]. But recently, the spinor formalism has appeared as an appropriate tool to explain the superstring theory [34, 35]. Then, the spinor differential forms with their capacity to supply weak solutions could offer a new impetus to this theory.

Hertz potentials known from the start of electromagnetic theory [36] do not seem to have been extensively used in spite of noticeable exception [3, 37] They are a useful tool to analyze electric and magnetic multipole radiation in wave guides [38–42] as well as to cope with anisotropy, bi-anisotropy and chiral isotropy [43–46]. Finally Hertz and Debye potentials intervene to tackle electromagnetism in General Relativity [23, 40]. To check the contribution of hertzian differential forms to these problems will be an interesting job.

Self-dual electromagnetic fields have also been known for a long time [34] and their covariance under the O(3,C) group makes them a useful tool in different situations, for instance to analyze the propagation of Gaussian beams in the atmosphere or radiowave propagation in troposphere [47]. In addition, the self-dual electromagnetism in isotropic media brings a close connection between electric and magnetic fields making easier, according to Synge [48], the physical interpretation of electromagnetic tensor fields. On the other hand, a look to Goggle suggests a new interest [49] for the Proca equation, an interest which could embrace spinor differential forms.

APPENDIX A.

We get explicitly from (19) the equations with $\partial_\tau = 1/\text{c}\partial_t$

$$\partial_z\psi_1^1 + (\partial_x + i\partial_y)\psi_2^1 - \partial_\tau\phi_1^1 = 0 \tag{A1a}$$
$$\partial_z\psi_1^2 - (\partial_x + i\partial_y)\psi_1^1 - \partial_\tau\phi_1^2 = 0 \tag{A1b}$$
$$(\partial_x - i\partial_y)\psi_1^1 - \partial_z\psi_2^1 - \partial_\tau\phi_2^1 = 0 \tag{A1c}$$
$$(\partial_x - i\partial_y)\psi_1^2 + \partial_z\psi_1^1 - +\partial_\tau\phi_1^1 = 0 \tag{A1d}$$

Substituting (17a), (17b) into (A1), it is easily checked that Eqs. (A1a) and (A1d) give the third Eq. (14a) since taking into account (14b),

$$\partial_z\Lambda_a + (\partial_x + i\partial_y)(\Lambda_x - i\Lambda_y) - \partial_\tau\Lambda_z = 0 \tag{A2}$$

(A2) reduces to $\text{i}\partial_y\Lambda_x - i\partial_x\Lambda_y - \partial_\tau\Lambda_z = 0$.

We get from (A1b) and (A1c) the two equations

$$i(\partial_z\Lambda_y - \partial_y\Lambda_z - \partial_\tau\Omega_x + (\partial_z\Lambda_x - \partial_x\Lambda_z - i\partial_\tau\Omega_y) = 0 \tag{A3}$$
$$i(\partial_z\Lambda_y - \partial_y\Lambda_z - \partial_\tau\Omega_x + (\partial_x\Lambda_z - \partial_z\Lambda_x + i\partial_\tau\Omega_y) = 0 \tag{A4}$$

Summing and subtracting (A3) and (A4) gives the first two Eq. (14a).

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