THE WAVE EQUATION AND GENERAL PLANE WAVE SOLUTIONS IN FRACTIONAL SPACE

M. Zubair and M. J. Mughal

Faculty of Electronic Engineering
GIK Institute of Engineering Sciences and Technology
Topi, Swabi 23640, Khyber Pakhtunkhwa, Pakistan

Q. A. Naqvi

Electronics Department
Quaid-i-Azam University
Islamabad 45320, Pakistan

Abstract—This work presents the analytical solution of vector wave equation in fractional space. General plane wave solution to the wave equation for fields in source-free and lossless media is obtained in fractional space. The obtained solution is a generalization of wave equation from integer dimensional space to a non-integer dimensional space. The classical results are recovered when integer-dimensional space is considered.

1. INTRODUCTION

Fractional-dimensional space concept is effectively used in many areas of physics to describe the physical description of confinement in low dimensional systems [1–6]. This approach is applied to replace the real confining structure with an effective space, where the measurement of its confinement is given by non-integer dimension [2, 3]. This confinement can be described in low dimensional system which can have different degree of confinement in different orthogonal directions, e.g., if we have system that is confined as 1.8 dimensional, then it could be described as $1 + 0.8$ dimensional in two coordinates and as $1 + 0.2 + 0.6$ dimensional in three coordinates, if dimensions add linearly [6]. Fractional calculus (a generalization of differentiation

and integration to fractional order) is used by different authors to describe fractional solutions to many electromagnetic problems as well as fractional dimensional space [6–10, 16, 17].

Axiomatic basis for the concept of fractional space and formulation of Schrödinger wave mechanics in D-dimensional fractional space is provided in [1]. Also it has been pointed out that the experimental measurement of the dimension D of our real world is given by $D = 3 \pm 10^{-6}$, not exactly 3 [1, 11]. D-dimensional generalization of Laplacian operator in different coordinate systems is provided in [1, 6].

Applications of the idea of fractional space in electromagnetic research include the derivation of Gauss law in D-dimensional fractional space [3], solution of electrostatic problem in D-dimensional fractional space ($2 < D \leq 3$) by solving Poisson's equation in fractional space [3], and solution of Laplacian equation in fractional space which describes potential of charge distribution in fractional space using Gegenbauer polynomials [4]. Multipoles and magnetic field of charges in fractional space have also been obtained in [4]. The fractional electrodynamics on fractals is reported in [5]. Also the scattering of electromagnetic waves in fractal media is described in [15].

The vector wave equation in fractional space can describe complex phenomenon of wave propagation in any non-integer-dimensional space. For example, if we consider plane wave solution in integer dimensional space with $D = 2$ as "Case 1" and for $D = 3$ as "Case 2", then by problem of fractional space solution shown in Figure 1, we mean intermediate solution for D-dimensional space, where $2 < D \leq 3$. It is worthwhile to mention that clouds, turbulence in fluids, rough surfaces, snow, etc., can be described as fractional dimensional. The study of wave propagation and scattering phenomenon in such media
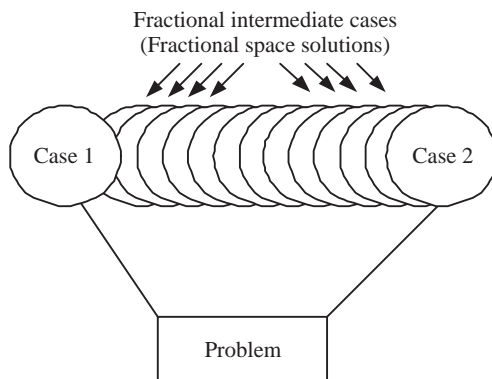

Figure 1. Block diagram symbolizing the fractional space solutions.

is important in practical applications, such as communications, remote sensing, navigation and even bioengineering [15]. The phenomenon of wave propagation in such fractal media can be studied by replacing these fractal confining structures with an effective space of non-integer dimension D. The plane wave solutions investigated in this paper have potential applications in electromagnetic wave propagation problems in fractional space. In Section 2, we investigate full analytical solution of wave equation in D-dimensional fractional space, where three parameters are used to describe the measure distribution of space. In Section 3, solution of wave equation in integer-dimensional space is obtained from the results of previous section. Finally, in Section 4, conclusions are drawn.

2. GENERAL PLANE WAVE SOLUTIONS IN FRACTIONAL SPACE

For source-free and lossless media, the vector wave equations for the complex electric and magnetic field intensities are given by the Helmholtz equation as follows [12].

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0 \tag{1a}$$

$$\nabla^2 \mathbf{H} + \beta^2 \mathbf{H} = 0 \tag{1b}$$

where, $\beta^2 = \omega^2 \mu \varepsilon$. Time dependency e^{jwt} has been suppressed throughout the discussion. Here ∇^2 is the scalar Laplacian operator in D dimensional fractional space and is defined as follows [6].

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z} \tag{2}$$

Equation (2) uses three parameters ($0 < \alpha_1 \leq 1, 0 < \alpha_2 \leq 1$ and $0 < \alpha_3 \leq 1$) to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $D = \alpha_1 + \alpha_2 + \alpha_3$. Once the solution to any one of Equations (1a) and (1b) in fractional space is known, the solution to the other can be written by an interchange of $\mathbf{E}$ with $\mathbf{H}$ or $\mathbf{H}$ with $\mathbf{E}$ due to duality. We will examine the solution for $\mathbf{E}$.

In rectangular coordinates, a general solution for $\mathbf{E}$ can be written as

$$\mathbf{E}(x, y, z) = \hat{a}_x E_x(x, y, z) + \hat{a}_y E_y(x, y, z) + \hat{a}_z E_z(x, y, z) \tag{3}$$

Substituting (3) into (1a) we can write that

$$\nabla^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) + \beta^2(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) = 0 \tag{4}$$

which reduces to three scalar wave equations as follows:

$$\nabla^2 E_x(x, y, z) + \beta^2 E_x(x, y, z) = 0 \qquad (5a)$$

$$\nabla^2 E_y(x, y, z) + \beta^2 E_y(x, y, z) = 0 \qquad (5b)$$

$$\nabla^2 E_z(x, y, z) + \beta^2 E_z(x, y, z) = 0 \qquad (5c)$$

Equation (5a) through (5c) are all of the same form; solution for any one of them in fractional space can be replicated for others by inspection. We choose to work first with E_x as given by (5a).

In expanded form (5a) can be written as

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\alpha_1 - 1}{x} \frac{\partial E_x}{\partial x} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\alpha_2 - 1}{y} \frac{\partial E_x}{\partial y} + \frac{\partial^2 E_x}{\partial z^2}$$

$$+ \frac{\alpha_3 - 1}{z} \frac{\partial E_x}{\partial z} + \beta^2 E_x = 0 \qquad (6)$$

Equation (6) is separable using separation of variables. We consider

$$E_x(x, y, z) = f(x)g(y)h(z) \qquad (7)$$

the resulting ordinary differential equations are obtained as follows:

$$\left[\frac{d^2}{dx^2} + \frac{\alpha_1 - 1}{x} \frac{d}{dx} + \beta_x^2 \right] f = 0 \qquad (8a)$$

$$\left[\frac{d^2}{dy^2} + \frac{\alpha_2 - 1}{y} \frac{d}{dy} + \beta_y^2 \right] g = 0 \qquad (8b)$$

$$\left[\frac{d^2}{dz^2} + \frac{\alpha_3 - 1}{z} \frac{d}{dz} + \beta_z^2 \right] h = 0 \qquad (8c)$$

where, in addition,

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 \qquad (9)$$

Equation (9) is referred to as constraint equation. In addition $\beta_x, \beta_y, \beta_z$ are known as wave constants in the x, y, z directions, respectively, which will be determined using boundary conditions.

Equation (8a) through (8c) are all of the same form; solution for any one of them can be replicated for others by inspection. We choose to work first with $f(x)$. We write (8a) as

$$\left[x \frac{d^2}{dx^2} + a \frac{d}{dx} + \beta_x^2 x \right] f = 0 \qquad (10)$$

where, $a = \alpha_1 - 1$. Equation (10) is reducible to Bessel's equation under substitution $f = x^n \xi$ as follows:

$$\left[x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (\beta_x^2 x^2 - n^2) \right] \xi = 0, \quad n = \frac{|1 - a|}{2} \qquad (11)$$

The solution of Bessel's equation in (11) is given as [13]

$$\xi = C_1 J_n(\beta_x x) + C_2 Y_n(\beta_x x) \tag{12}$$

where, $J_n(\beta_x x)$ is referred to as Bessel function of the first kind of order n, $Y_n(\beta_x x)$ as the Bessel function of the second kind of order n. Finally the solution of (8a) becomes

$$f(x) = x^{n_1}[C_1 J_{n_1}(\beta_x x) + C_2 Y_{n_1}(\beta_x x)], \quad n_1 = 1 - \frac{\alpha_1}{2} \tag{13}$$

Similarly, the solutions to (8b) and (8c) are obtained as

$$g(y) = y^{n_2}[C_3 J_{n_2}(\beta_y y) + C_4 Y_{n_2}(\beta_y y)], \quad n_2 = 1 - \frac{\alpha_2}{2} \tag{14}$$

$$h(z) = z^{n_3}[C_5 J_{n_3}(\beta_z z) + C_6 Y_{n_3}(\beta_z z)], \quad n_3 = 1 - \frac{\alpha_3}{2} \tag{15}$$

From (7) and (13) through (15), the solution of (5a) have the form

$$\begin{aligned}
E_x(x, y, z) = {} & x^{n_1} y^{n_2} z^{n_3}[C_1 J_{n_1}(\beta_x x) + C_2 Y_{n_1}(\beta_x x)] \\
& \times [C_3 J_{n_2}(\beta_y y) + C_4 Y_{n_2}(\beta_y y)] \\
& \times [C_5 J_{n_3}(\beta_z z) + C_6 Y_{n_3}(\beta_z z)]
\end{aligned} \tag{16}$$

where, C_1 through C_6 are constant coefficients. Similarly, the solutions to (5b) and (5c) are obtained as

$$\begin{aligned}
E_y(x, y, z) = {} & x^{n_1} y^{n_2} z^{n_3}[D_1 J_{n_1}(\beta_x x) + D_2 Y_{n_1}(\beta_x x)] \\
& \times [D_3 J_{n_2}(\beta_y y) + D_4 Y_{n_2}(\beta_y y)] \\
& \times [D_5 J_{n_3}(\beta_z z) + D_6 Y_{n_3}(\beta_z z)]
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
E_z(x, y, z) = {} & x^{n_1} y^{n_2} z^{n_3}[F_1 J_{n_1}(\beta_x x) + F_2 Y_{n_1}(\beta_x x)] \\
& \times [F_3 J_{n_2}(\beta_y y) + F_4 Y_{n_2}(\beta_y y)] \\
& \times [F_5 J_{n_3}(\beta_z z) + F_6 Y_{n_3}(\beta_z z)]
\end{aligned} \tag{18}$$

where, D_1 through D_6 and F_1 through F_6 are constant coefficients.

For e^{jwt} time variations, the instantaneous form $\mathcal{E}(x, y, z; t)$ of the vector complex function $\mathbf{E}(x, y, z)$ in (3) takes the form

$$\begin{aligned}
\mathcal{E}(x, y, z; t) = {} & \Re e[\{\hat{a}_x E_x(x, y, z) + \hat{a}_y E_y(x, y, z) \\
& + \hat{a}_z E_z(x, y, z)\} e^{jwt}]
\end{aligned} \tag{19}$$

where $E_x(x, y, z), E_y(x, y, z)$ and $E_z(x, y, z)$ are given by (16) through (18).

Equation (19) provides a general plane wave solution in fractional space. This solution can be used to study the phenomenon of electromagnetic wave propagation in any non-integer dimensional space.

3. DISCUSSION ON FRACTIONAL SPACE SOLUTION

Equation (19) is the generalization of the concept of wave propagation in integer dimensional space to the wave propagation in non-integer dimensional space. As a special case, for three-dimensional space, this problem reduces to classical wave propagation concept; i.e., if we set $\alpha_1 = 1$ in Equation (13) then $n_1 = \frac{1}{2}$ and it gives

$$f(x) = x^{\frac{1}{2}} \left[C_1 J_{\frac{1}{2}}(\beta_x x) + C_2 Y_{\frac{1}{2}}(\beta_x x) \right] \tag{20}$$

Using Bessel functions of fractional order [14]:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \tag{21a}$$

$$Y_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \cos(x) \tag{21b}$$

Equation (13) can be reduced to

$$f(x) = C_1' \sin(\beta_x x) + C_2' \cos(\beta_x x) \tag{22}$$

where, $C_i' = C_i \sqrt{\frac{2}{\pi \beta_x}}$, $i = 1, 2$.

Similarly, we set $\alpha_2 = 1$ and $\alpha_3 = 1$ in (14) and (15) respectively and using Bessel functions of fractional order in (21a) through (21b), we get

$$g(y) = C_3' \sin(\beta_y y) + C_4' \cos(\beta_y y) \tag{23}$$
$$h(z) = C_5' \sin(\beta_z z) + C_6' \cos(\beta_z z) \tag{24}$$

From (22) through (24), we get $E_x(x, y, z)$ in three-dimensional space $(D = 3)$ as follows

$$\begin{aligned} E_x(x, y, z) &= [C_1' \sin(\beta_x x) + C_2' \cos(\beta_x x)] \\ &\times [C_3' \sin(\beta_y y) + C_4' \cos(\beta_y y)] \\ &\times [C_5' \sin(\beta_z z) + C_6' \cos(\beta_z z)] \end{aligned} \tag{25}$$

which is comparable to the solution of wave equation in integer dimensional space obtained by Balanis [12]. Similarly, field components $E_y(x, y, z)$ and $E_z(x, y, z)$ can also be reduced for three-dimensional case.

As another special case, if we choose a single parameter for non-integer dimension D where $2 < D \le 3$, i.e., we take $\alpha_1 = \alpha_2 = 1$ so $D = \alpha_3 + 2$. In this case from Equation (6) we obtain

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{D-3}{z} \frac{\partial E_x}{\partial z} + \beta^2 E_x = 0 \tag{26}$$

Solving this equation by separation of variables leads to the following result

$$
\begin{aligned}
E_x(x, y, z) = {}& z^n[G_1 \cos(\beta_x x) + G_2 \sin(\beta_x x)] \\
& \times [G_3 \cos(\beta_y y) + G_4 \sin(\beta_y y)] \\
& \times [G_5 J_n(\beta_z z) + G_6 Y_n(\beta_z z)]
\end{aligned}
\tag{27}
$$

where, $n = 2 - \frac{D}{2}$. Here if we set $D = 3$, and using (21a) and (21b), we get

$$
\begin{aligned}
E_x(x, y, z) = {}& \sqrt{\frac{2}{\pi \beta_z}} [G_1 \cos(\beta_x x) + G_2 \sin(\beta_x x)] \\
& \times [G_3 \cos(\beta_y y) + G_4 \sin(\beta_y y)] \\
& \times [G_5 \sin(\beta_z z) + G_6 \cos(\beta_z z)]
\end{aligned}
\tag{28}
$$

where, G_1 through G_6 are constant coefficients. The result obtained in (28) is comparable to that obtained by Balanis [12] for 3-dimensional space.

As an example, an infinite sheet of surface current can be considered as a source of plane waves in D-dimensional fractional space. We assume that an infinite sheet of electric surface current density $J_s = J_{s_0}\hat{x}$ exists on the $z = 0$ plane in free space. Since the sources do not vary with x or y, the fields will not vary with x or y but will propagate away from the source in $\pm z$ direction. The boundary conditions to be satisfied at $z = 0$ are $\hat{z} \times (\mathbf{E_2} - \mathbf{E_1}) = 0$ and $\hat{z} \times (\mathbf{H_2} - \mathbf{H_1}) = J_{s_0}\hat{x}$, where $\mathbf{E_1}$, $\mathbf{H_1}$ are the fields for $z < 0$, and $\mathbf{E_2}$, $\mathbf{H_2}$ are the fields for $z > 0$. To satisfy the later boundary condition, $\mathbf{H}$ must have a $\hat{y}$ component. Then for $\mathbf{E}$ to be normal to $\mathbf{H}$ and $\hat{z}$, $\mathbf{E}$ must have an $\hat{x}$ component. Thus, the corresponding wave equation for $\mathbf{E}$ and $\mathbf{H}$ fields in D-dimensional fractional space where $2 < D \leq 3$ can be written by modifying (26) as

$$
\frac{d^2 E_x}{dz^2} + \frac{D-3}{z} \frac{dE_x}{dz} + \beta^2 E_x = 0
\tag{29a}
$$

$$
\frac{d^2 H_y}{dz^2} + \frac{D-3}{z} \frac{dH_y}{dz} + \beta^2 H_y = 0
\tag{29b}
$$

Solution of (29a) and (29b) takes the similar form as (27) and under above mentioned boundary conditions the fields will have the following form:

$$
\mathbf{E_1} = -\hat{x} \frac{J_{s_0}}{2} J_n(\beta_z z), \qquad \mathbf{H_1} = \hat{y} \frac{J_{s_0}}{2\eta_0} J_n(\beta_z z); \quad z < 0
\tag{30a}
$$

$$
\mathbf{E_2} = -\hat{x} \frac{J_{s_0}}{2} Y_n(\beta_z z), \qquad \mathbf{H_2} = -\hat{y} \frac{J_{s_0}}{2\eta_0} Y_n(\beta_z z); \quad z > 0
\tag{30b}
$$

where, η_0 is wave impedance in free space. Assuming a time dependency e^{jwt} and $J_{s_0} = -2A/m$, the solution for the usual wave for $z > 0$ with $D = 3$ is shown in Figure 2, which is comparable to well known plane wave solutions in 3-dimensional space [12]. Similarly, for $D = 2.5$ we have fractal medium wave for $z > 0$ as shown in Figure 3, where amplitude variations are described in terms of Bessel functions.
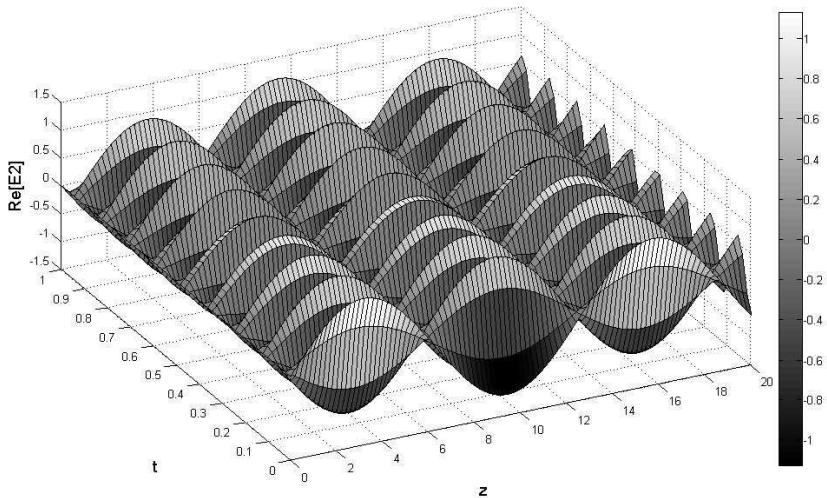

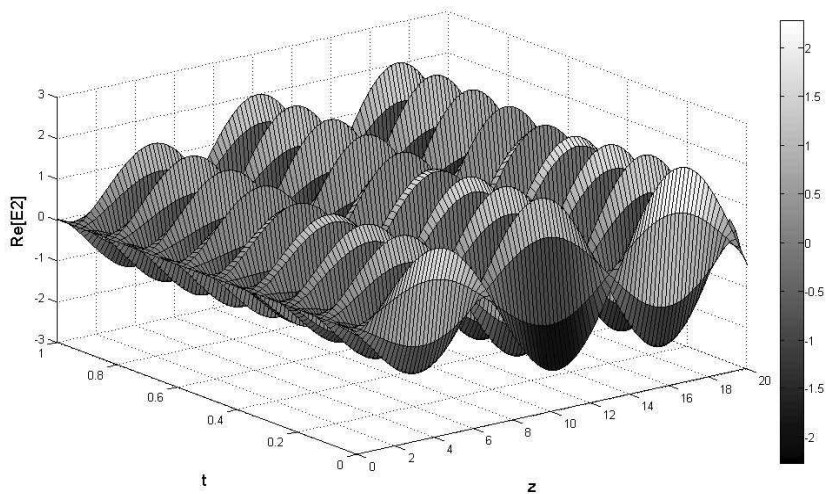**Figure 2.** Usual wave propagation ($D = 3$).

Figure 3. Wave propagation in fractional space ($D = 2.5$).

4. CONCLUSION

General plane wave solution in source-free and lossless media in fractional space is presented by solving vector wave equation in D-dimensional fractional space. When the wave propagates in fractional space, the amplitude variations are described by Bessel functions. The obtained general plane wave solution is a generalization of integer-dimensional solution to a non-integer dimensional space. For all investigated cases when D is an integer-dimension, the classical results are recovered.

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