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CLASS OF ELECTROMAGNETIC SQ-MEDIA

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Abstract—A novel class of electromagnetic media called that of SQ-media is defined in terms of compact four-dimensional differential-form formalism. The medium class lies between two known classes, that of Q-media and SD-media (also called self-dual media). Eigenfields for the defined medium dyadic are derived and shown to be uncoupled in a homogeneous medium. However, energy transport requires their interaction. The medium shares the nonbirefringence property of the Q-media (not shared by the SD media) and the eigenfield decomposition property of the SD media (not shared by the Q-media). Comparison of the three medium classes is made in terms of their three-dimensional medium dyadics.

1. INTRODUCTION

The most general linear electromagnetic medium (bi-anisotropic medium) can be expressed in terms of four medium dyadics in the three-dimensional Gibbsian vector representation as [1, 2]

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{1}$$

and the maximum number of free parameters is $4 \times 9 = 36$. The four-dimensional differential-form representation of electromagnetic fields as two-forms [3–5], elements of the space $\mathbb{F}_2$

$$\mathbf{\Phi} = \mathbf{B} + \mathbf{E} \wedge \mathbf{d}\tau, \qquad \mathbf{\Psi} = \mathbf{D} - \mathbf{H} \wedge \mathbf{d}\tau, \tag{2}$$

allows one to write the constitutive Equation (1) in the more compact form [5]

$$\mathbf{\Psi} = \overline{\overline{\mathrm{M}}}|\mathbf{\Phi}, \tag{3}$$

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or

$$\mathbf{e}_N\lfloor\mathbf{\Psi} = \overline{\overline{\mathsf{M}}}_g|\mathbf{\Phi}. \tag{4}$$

Here, $\mathbf{e}_N = \mathbf{e}_{1234} \in \mathbb{E}_4$ denotes the quadrivector in the basis of vectors $\mathbf{e}_i \in \mathbb{E}_1$ and $\lfloor$ denotes the contraction operation. The reciprocal basis one-forms $\boldsymbol{\varepsilon}_j \in \mathbb{F}_1$ satisfy $\mathbf{e}_i|\boldsymbol{\varepsilon}_j = \delta_{ij}$ and $\boldsymbol{\varepsilon}_N = \boldsymbol{\varepsilon}_{1234}$, where $\boldsymbol{\varepsilon}_4 = \mathbf{d}\tau$ corresponds to the temporal one-form. The medium dyadic $\overline{\overline{\mathsf{M}}} \in \mathbb{F}_2\mathbb{E}_2$ maps two-forms to two-forms and the modified medium dyadic $\overline{\overline{\mathsf{M}}}_g = \mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}} \in \mathbb{E}_2\mathbb{E}_2$ maps two-forms to bivectors. Basis expansions of both dyadics correspond to the same 6×6 matrix. Definitions and operational rules for differential forms, multivectors and dyadics applied in this study have been summarized in the Appendices of [6, 7] and, more extensively, in the book [5].

The most general medium dyadic can be uniquely decomposed in three components as introduced by Hehl and Obukhov [4],

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{M}}}_1 + \overline{\overline{\mathsf{M}}}_2 + \overline{\overline{\mathsf{M}}}_3, \tag{5}$$

called principal (1), skewon (2) and axion (3) parts of $\overline{\overline{\mathsf{M}}}$. The axion part $\overline{\overline{\mathsf{M}}}_3$ is a multiple of the unit dyadic $\overline{\overline{\mathsf{I}}}^{(2)T}$ mapping any two-form to itself and the other parts are trace free. The skewon part is defined so that the corresponding modified medium dyadic $\overline{\overline{\mathsf{M}}}_{g2}$ is antisymmetric, while the principal part $\overline{\overline{\mathsf{M}}}_1$ is trace free and $\overline{\overline{\mathsf{M}}}_{g1}$ is symmetric. Media with vanishing components can be called accordingly, e.g., a medium defined by $\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{M}}}_1$ is called a principal medium and one with $\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{M}}}_2 + \overline{\overline{\mathsf{M}}}_3$ is called a skewon-axion medium.

Four-dimensional formalism allows simple definition of important classes of electromagnetic media. For example, if the modified medium dyadic can be expressed in terms of some dyadic $\overline{\overline{\mathsf{Q}}} \in \mathbb{E}_1\mathbb{E}_1$ mapping one-forms to vectors as

$$\overline{\overline{\mathsf{M}}}_g = \frac{1}{2}\overline{\overline{\mathsf{Q}}}{}^\wedge_\wedge\overline{\overline{\mathsf{Q}}} = \overline{\overline{\mathsf{Q}}}^{(2)}, \tag{6}$$

it is called a Q-medium, which has the property of being non-birefringent to propagating waves [5, 8]. Thus, media in this class can be conceived as generalizations of isotropic media. A more general medium class was called that of generalized Q-media and defined by medium dyadics of the form [14]

$$\overline{\overline{\mathsf{M}}}_g = \overline{\overline{\mathsf{Q}}}^{(2)} + \mathbf{A}\mathbf{B}, \tag{7}$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{E}_2$ are two bivectors. Such media were shown to coincide with the class of decomposable media, defined in terms of Gibbsian three-dimensional dyadics [2, 9] in a more complicated way [10]. In such a medium any field can be decomposed in two noncoupling parts, which is a generalization of the well-known TE/TM decomposition of fields in simpler media.

Other important classes of media arise from simple conditions satisfied by the medium condition. It is known that any medium dyadic $\overline{\overline{\mathrm{M}}}$ like any 6×6 matrix satisfies an algebraic equation of the sixth order. Medium dyadics satisfying equations of lower order define certain classes of media. First-order equations,

$$\overline{\overline{\mathrm{M}}} + A\overline{\overline{\mathrm{I}}}^{(2)T} = 0, \tag{8}$$

obviously define axion media. Axion media were also called by the name perfect electromagnetic conductor (PEMC), because they unify and generalize the concepts of perfect electric and magnetic conductor (PEC and PMC). Another interesting class of media is defined by medium dyadics satisfying the second-order equation,

$$\overline{\overline{\mathrm{M}}}^2 + A\overline{\overline{\mathrm{M}}} + B\overline{\overline{\mathrm{I}}}^{(2)T} = 0, \tag{9}$$

for some parameters A and B. Excluding the axion media, such a class was called that of SD-media, or self-dual media [12, 13], because one can define a linear (duality) transformation of electromagnetic fields in which the medium appears invariant.

It is the purpose of the present paper to introduce a class of media which appears to be in between those of Q-media and SD-media in sharing properties of both of these medium classes.

2. SQ-MEDIA

2.1. Properties of Some Dyadics

The contraction dyadic transforming two-forms to bivectors

$$\begin{aligned}\mathbf{e}_N\lfloor\overline{\overline{\mathrm{I}}}^{(2)T} &= \mathbf{e}_N\lfloor\left(\sum_i^4 \mathbf{e}_i\boldsymbol{\varepsilon}_i\right)^{(2)T} = \mathbf{e}_N\lfloor\sum_{i<j}\boldsymbol{\varepsilon}_{ij}\mathbf{e}_{ij} = \overline{\overline{\mathrm{I}}}^{(2)}\rfloor\mathbf{e}_N \\ &= \mathbf{e}_{12}\mathbf{e}_{34} + \mathbf{e}_{23}\mathbf{e}_{14} + \mathbf{e}_{31}\mathbf{e}_{24} + \mathbf{e}_{14}\mathbf{e}_{23} + \mathbf{e}_{24}\mathbf{e}_{31} + \mathbf{e}_{34}\mathbf{e}_{12}\end{aligned} \tag{10}$$

is symmetric:

$$\left(\mathbf{e}_N\lfloor\overline{\overline{\mathrm{I}}}^{(2)T}\right)^T = \mathbf{e}_N\lfloor\overline{\overline{\mathrm{I}}}^{(2)T} = \overline{\overline{\mathrm{I}}}^{(2)}\rfloor\mathbf{e}_N, \tag{11}$$

and its inverse can be expressed as

$$\left(\mathbf{e}_N\lfloor\overline{\overline{\mathrm{I}}}^{(2)T}\right)^{-1} = \overline{\overline{\mathrm{I}}}^{(2)T}\rfloor\boldsymbol{\varepsilon}_N = \boldsymbol{\varepsilon}_N\lfloor\overline{\overline{\mathrm{I}}}^{(2)} = \boldsymbol{\varepsilon}_N\lfloor\sum_{i<j}\mathbf{e}_{ij}\boldsymbol{\varepsilon}_{ij}. \tag{12}$$

The inverse $\bar{\bar{\mathrm{Q}}}^{-1} \in \mathbb{F}_1\mathbb{F}_1$ of a dyadic $\bar{\bar{\mathrm{Q}}} \in \mathbb{E}_1\mathbb{E}_1$ can be expressed as [5]

$$\bar{\bar{\mathrm{Q}}}^{-1} = \frac{1}{\Delta_Q}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N \lfloor\lfloor \bar{\bar{\mathrm{Q}}}^{(3)T}. \tag{13}$$

This rule requires that the determinant-like quantity

$$\Delta_Q = \boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N || \bar{\bar{\mathrm{Q}}}^{(4)} \tag{14}$$

be nonzero. In the following we assume that the $\bar{\bar{\mathrm{Q}}}$ dyadic is normalized by assuming

$$\Delta_Q = 1. \tag{15}$$

The inverse of the double-wedge square dyadic $\bar{\bar{\mathrm{Q}}}^{(2)} \in \mathbb{E}_2\mathbb{E}_2$, denoted by

$$\left(\bar{\bar{\mathrm{Q}}}^{(2)}\right)^{-1} = \left(\bar{\bar{\mathrm{Q}}}^{-1}\right)^{(2)} = \bar{\bar{\mathrm{Q}}}^{(-2)}, \tag{16}$$

can then be obtained through the rule [5]

$$\bar{\bar{\mathrm{Q}}}^{(-2)} = \frac{1}{\Delta_Q}\boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N \lfloor\lfloor \bar{\bar{\mathrm{Q}}}^{(2)T} = \boldsymbol{\varepsilon}_N\boldsymbol{\varepsilon}_N \lfloor\lfloor \bar{\bar{\mathrm{Q}}}^{(2)T}. \tag{17}$$

The converse rule is

$$\bar{\bar{\mathrm{Q}}}^{(2)} = \mathbf{e}_N\mathbf{e}_N \lfloor\lfloor \bar{\bar{\mathrm{Q}}}^{(-2)T}. \tag{18}$$

Introducing a dyadic $\bar{\bar{\mathrm{K}}} \in \mathbb{F}_2\mathbb{E}_2$ mapping two-forms to two-forms by

$$\bar{\bar{\mathrm{K}}} = \boldsymbol{\varepsilon}_N \lfloor \bar{\bar{\mathrm{Q}}}^{(2)} = \left(\boldsymbol{\varepsilon}_N \lfloor \bar{\bar{\mathrm{I}}}^{(2)}\right) | \bar{\bar{\mathrm{Q}}}^{(2)}. \tag{19}$$

we have

$$\begin{aligned}\bar{\bar{\mathrm{K}}}^{-1} &= \bar{\bar{\mathrm{Q}}}^{(-2)} | \left(\boldsymbol{\varepsilon}_N \lfloor \bar{\bar{\mathrm{I}}}^{(2)}\right)^{-1} = \bar{\bar{\mathrm{Q}}}^{(-2)} \rfloor \mathbf{e}_N = \boldsymbol{\varepsilon}_N \lfloor \left(\mathbf{e}_N\mathbf{e}_N \lfloor\lfloor \bar{\bar{\mathrm{Q}}}^{(-2)}\right) \\ &= \boldsymbol{\varepsilon}_N \lfloor \bar{\bar{\mathrm{Q}}}^{(2)T} = \boldsymbol{\varepsilon}_N\mathbf{e}_N \lfloor\lfloor \bar{\bar{\mathrm{K}}}^T, \end{aligned} \tag{20}$$

which gives rise to the rule

$$\bar{\bar{\mathrm{K}}}^T | \left(\mathbf{e}_N \lfloor \bar{\bar{\mathrm{K}}}\right) = \left(\bar{\bar{\mathrm{K}}}^T \rfloor \mathbf{e}_N\right) | \bar{\bar{\mathrm{K}}} = \left(\mathbf{e}_N \lfloor \bar{\bar{\mathrm{K}}}^{-1}\right) | \bar{\bar{\mathrm{K}}} = \mathbf{e}_N \lfloor \bar{\bar{\mathrm{I}}}^{(2)T}. \tag{21}$$

A natural dot product of two-forms $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ yielding a scalar, defined as

$$\boldsymbol{\Gamma}_1 \cdot \boldsymbol{\Gamma}_2 = \mathbf{e}_N | (\boldsymbol{\Gamma}_1 \wedge \boldsymbol{\Gamma}_2) = \boldsymbol{\Gamma}_1 | \left(\mathbf{e}_N \lfloor \boldsymbol{\Gamma}_2\right) = \boldsymbol{\Gamma}_2 \cdot \boldsymbol{\Gamma}_1, \tag{22}$$

satisfies

$$\left(\bar{\bar{\mathrm{K}}} | \boldsymbol{\Gamma}_1\right) | \mathbf{e}_N \lfloor \left(\bar{\bar{\mathrm{K}}} | \boldsymbol{\Gamma}_2\right) = \boldsymbol{\Gamma}_1 | \bar{\bar{\mathrm{K}}}^T | \left(\mathbf{e}_N \lfloor \bar{\bar{\mathrm{K}}}\right) | \boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_1 | \left(\mathbf{e}_N \lfloor \boldsymbol{\Gamma}_2\right), \tag{23}$$

or

$$\left(\bar{\bar{\mathrm{K}}} | \boldsymbol{\Gamma}_1\right) \cdot \left(\bar{\bar{\mathrm{K}}} | \boldsymbol{\Gamma}_2\right) = \boldsymbol{\Gamma}_1 \cdot \boldsymbol{\Gamma}_2. \tag{24}$$

Thus, mapping by the dyadic $\bar{\bar{\mathrm{K}}}$ does not change the dot product of two-forms. In this, it resembles a rotation or reflection operation. In particular, a simple two-form $\mathbf{\Gamma}$ satisfying $\mathbf{\Gamma}\cdot\mathbf{\Gamma}=0$, is mapped to a simple two-form $\bar{\bar{\mathrm{K}}}|\mathbf{\Gamma}$.

2.2. Definition of SQ-media

Let us now assume that $\bar{\bar{\mathrm{Q}}}^{(2)}$ is a symmetric dyadic. In terms of symmetric and antisymmetric parts, $\bar{\bar{\mathrm{Q}}}=\bar{\bar{\mathrm{Q}}}_s+\bar{\bar{\mathrm{Q}}}_a$, this requires that the condition $\bar{\bar{\mathrm{Q}}}_s{}^{\wedge}_{\wedge}\bar{\bar{\mathrm{Q}}}_a=0$ be satisfied. For symmetric $\bar{\bar{\mathrm{Q}}}^{(2)}$ we have

$$\boldsymbol{\varepsilon}_N\mathbf{e}_N\lfloor\lfloor\bar{\bar{\mathrm{K}}}^T=\boldsymbol{\varepsilon}_N\mathbf{e}_N\lfloor\lfloor(\boldsymbol{\varepsilon}_N\lfloor\bar{\bar{\mathrm{Q}}}^{(2)})^T=\boldsymbol{\varepsilon}_N\lfloor\bar{\bar{\mathrm{Q}}}^{(2)}=\bar{\bar{\mathrm{K}}}, \tag{25}$$

whence (20) becomes

$$\bar{\bar{\mathrm{K}}}^{-1}=\bar{\bar{\mathrm{K}}},\quad\Rightarrow\quad\bar{\bar{\mathrm{K}}}^2=\bar{\bar{\mathrm{I}}}^{(2)T}. \tag{26}$$

Thus, the dyadic $\bar{\bar{\mathrm{K}}}$ acts as a square root of the unit dyadic. Of course, there are other square roots as well, like the unit dyadic itself.

Let us consider an extension to Q-media with symmetric $\bar{\bar{\mathrm{Q}}}^{(2)}$ by adding a multiple of the unit dyadic in the medium dyadic, i.e., defining

$$\bar{\bar{\mathrm{M}}}=\alpha\bar{\bar{\mathrm{I}}}^{(2)T}+\beta\bar{\bar{\mathrm{K}}}. \tag{27}$$

Here, we exclude the axion medium special case by assuming $\beta\neq 0$. Any medium defined by a medium dyadic of the form (27), based by symmetric $\bar{\bar{\mathrm{Q}}}^{(2)}$, will now called an SQ-medium for brevity. The product of two SQ-medium dyadics satisfies

$$\begin{aligned}\bar{\bar{\mathrm{M}}}_1|\bar{\bar{\mathrm{M}}}_2 &= \left(\alpha_1\bar{\bar{\mathrm{I}}}^{(2)T}+\beta_1\bar{\bar{\mathrm{K}}}\right)\Big|\left(\alpha_2\bar{\bar{\mathrm{I}}}^{(2)T}+\beta_2\bar{\bar{\mathrm{K}}}\right)\\ &= (\alpha_1\alpha_2+\beta_1\beta_2)\,\bar{\bar{\mathrm{I}}}^{(2)T}+(\alpha_1\beta_2+\alpha_2\beta_1)\,\bar{\bar{\mathrm{K}}}=\bar{\bar{\mathrm{M}}}_2|\bar{\bar{\mathrm{M}}}_1. \end{aligned}\tag{28}$$

Defining

$$\alpha_i=M_i\cosh\theta_i,\qquad\beta_i=M_i\sinh\theta_i,\quad i=1,2, \tag{29}$$

the rule can be cast in the more suggestive form

$$\bar{\bar{\mathrm{M}}}_1|\bar{\bar{\mathrm{M}}}_2=M_1M_2\left(\cosh(\theta_1+\theta_2)\bar{\bar{\mathrm{I}}}^{(2)T}+\sinh(\theta_1+\theta_2)\bar{\bar{\mathrm{K}}}\right). \tag{30}$$

Thus, the medium dyadic of any SQ-medium acts as a hyperbolic rotation dyadic multiplied by a magnitude coefficient. Actually, we can write more compactly

$$\bar{\bar{\mathrm{M}}}=Me^{\theta\bar{\bar{\mathrm{K}}}}, \tag{31}$$

when the expotential function of a dyadic is understood in terms of its power series. The multiplication rule now appears as

$$\overline{\overline{\mathrm{M}}}_1|\overline{\overline{\mathrm{M}}}_2 = M_1 e^{\theta_1 \overline{\overline{\mathrm{K}}}}|M_2 e^{\theta_2 \overline{\overline{\mathrm{K}}}} = M_1 M_2 e^{(\theta_1+\theta_2)\overline{\overline{\mathrm{K}}}}, \tag{32}$$

and the inverse of the medium dyadic can be expressed as

$$\overline{\overline{\mathrm{M}}}^{-1} = Me^{-\theta\overline{\overline{\mathrm{K}}}} = \frac{1}{M}\left(\cosh\theta\overline{\overline{\mathrm{I}}}^{(2)T} - \sinh\theta\overline{\overline{\mathrm{K}}}\right) = \frac{1}{\alpha^2-\beta^2}\left(\alpha\overline{\overline{\mathrm{I}}}^{(2)} - \beta\overline{\overline{\mathrm{K}}}\right). \tag{33}$$

Because there is no antisymmetric part in $\overline{\overline{\mathrm{Q}}}^{(2)}$, there is no skewon component in the medium dyadic $\overline{\overline{\mathrm{M}}}$, whence it consists of principal and axion parts, only [4]. Such medium dyadics are defined by $36-15 = 21$ parameters. For a given $\overline{\overline{\mathrm{Q}}}$ dyadic the present medium dyadics $\overline{\overline{\mathrm{M}}}$ define a two-dimensional subspace in the 21 dimensional space of principal-axion medium dyadics.

2.3. Relation to Q-media and SD-media

From

$$\overline{\overline{\mathrm{M}}}^2 = \left(\alpha^2+\beta^2\right)\overline{\overline{\mathrm{I}}}^{(2)} + 2\alpha\beta\overline{\overline{\mathrm{K}}} = \left(\alpha^2+\beta^2\right)\overline{\overline{\mathrm{I}}}^{(2)} + 2\alpha\left(\overline{\overline{\mathrm{M}}} - \alpha\overline{\overline{\mathrm{I}}}^{(2)T}\right) \tag{34}$$

we see that the SQ-medium dyadic satisfies

$$\overline{\overline{\mathrm{M}}}^2 - 2\alpha\overline{\overline{\mathrm{M}}} + \left(\alpha^2-\beta^2\right)\overline{\overline{\mathrm{I}}}^{(2)T} = 0, \tag{35}$$

or

$$\overline{\overline{\mathrm{M}}}^2 - 2M\cosh\theta\overline{\overline{\mathrm{M}}} + M^2\overline{\overline{\mathrm{I}}}^{(2)T} = 0, \tag{36}$$

which is an algebraic dyadic equation of the second order. Since the pure axion medium corresponding to $\beta = 0$ or $\theta = 0$ was excluded, the medium is seen to belong to the class of SD media, defined by (9), as a special case.

Let us compare the general SQ-medium, Q-medium and SD-medium in terms of their three-dimensional (spatial) medium-dyadics in the representation (1) where we must now replace the Gibbsian dot product by the multivector product $|$.

- The three-dimensional medium dyadics of the Q-medium have the general form [5, 8]

$$\overline{\overline{\epsilon}} = \epsilon\overline{\overline{\mathrm{D}}}, \quad \overline{\overline{\mu}} = \mu\overline{\overline{\mathrm{D}}}^T, \quad \overline{\overline{\xi}} = \mathbf{X}\rfloor\overline{\overline{\mathrm{I}}}^T, \quad \overline{\overline{\zeta}} = \mathbf{Z}\rfloor\overline{\overline{\mathrm{I}}}^T, \tag{37}$$

  where $\overline{\overline{\mathrm{D}}} \in \mathbb{E}_1\mathbb{E}_1$ is any spatial dyadic and $\mathbf{X}$, $\mathbf{Z}$ are any spatial bivectors. Thus, $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ may be any antisymmetric dyadics and $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ satisfy a relation of the form $\mu\overline{\overline{\epsilon}} - \epsilon\overline{\overline{\mu}}^T = 0$.

- The three-dimensional medium dyadics of the SD-medium dyadics have the general form [12, 13]

$$\bar{\bar{\epsilon}} = \epsilon \bar{\bar{\rm D}}, \quad \bar{\bar{\mu}} = \mu \bar{\bar{\rm D}}, \quad \bar{\bar{\xi}} = \xi \bar{\bar{\rm D}} + \xi' \bar{\bar{\rm B}}, \quad \bar{\bar{\zeta}} = \zeta \bar{\bar{\rm D}} + \zeta' \bar{\bar{\rm B}}, \tag{38}$$

  where $\bar{\bar{\rm D}}$ and $\bar{\bar{\rm B}}$ are any two spatial dyadics. Thus, $\bar{\bar{\epsilon}}$, $\bar{\bar{\mu}}$ and $\bar{\bar{\xi}} + \bar{\bar{\zeta}}$ are multiples of the same dyadic $\bar{\bar{\rm D}}$ while $\bar{\bar{\xi}} - \bar{\bar{\zeta}}$ may be any other dyadic.
- The three-dimensional medium dyadics of the SQ-medium dyadic (27) can be found along the procedure given for the Q-medium in [5, 8]. Omitting the details, the result can be expressed in the form

$$\bar{\bar{\epsilon}} = \epsilon \bar{\bar{\rm S}}, \quad \bar{\bar{\mu}} = \mu \bar{\bar{\rm S}}, \quad \bar{\bar{\xi}} = \xi \bar{\bar{\rm S}} + \mathbf{A}\rfloor \bar{\bar{\rm I}}^T, \quad \bar{\bar{\zeta}} = \zeta \bar{\bar{\rm S}} - \mathbf{A}\rfloor \bar{\bar{\rm I}}^T, \tag{39}$$

  where $\bar{\bar{\rm S}} \in \mathbb{E}_1\mathbb{E}_1$ is a symmetric spatial dyadic and $\mathbf{A} \in \mathbb{E}_2$ is a spatial bivector. Thus, the dyadics $\bar{\bar{\epsilon}}$, $\bar{\bar{\mu}}$ and $\bar{\bar{\xi}} + \bar{\bar{\zeta}}$ are multiples of the same symmetric dyadic $\bar{\bar{\rm S}}$ while $\bar{\bar{\xi}} - \bar{\bar{\zeta}} = 2\mathbf{A}\lfloor \bar{\bar{\rm I}}^T$ is any antisymmetric spatial dyadic.

When comparing (39) with the conditions of the Q-medium (37), those of the SQ-medium are more restricted in requiring that $\bar{\bar{\rm D}}$ be a symmetric dyadic and more general in allowing $\bar{\bar{\xi}}$ and $\bar{\bar{\zeta}}$ to possess symmetric components in addition to the antisymmetric components, related by $\mathbf{X} = -\mathbf{Z}$. On the other hand, the SQ-medium appears as a special case of the SD medium (38) with symmetric dyadic $\bar{\bar{\rm D}}$, antisymmetric dyadic $\bar{\bar{\rm B}}$ and with $\xi' = -\zeta'$.

3. FIELDS IN SQ-MEDIA

3.1. Eigenfield Decomposition

The dyadic second-order Equation (36) can be written factorized form as

$$\begin{aligned} &\left(\bar{\bar{\rm M}} - Me^{\theta}\bar{\bar{\rm I}}^{(2)T}\right) \Big| \left(\bar{\bar{\rm M}} - Me^{-\theta}\bar{\bar{\rm I}}^{(2)T}\right) \\ =\ &\left(\bar{\bar{\rm M}} - Me^{-\theta}\bar{\bar{\rm I}}^{(2)T}\right) \Big| \left(\bar{\bar{\rm M}} - Me^{\theta}\bar{\bar{\rm I}}^{(2)T}\right) = 0. \end{aligned} \tag{40}$$

Multiplying by an arbitrary two-form $|\mathbf{\Phi}$ yields

$$\left(\bar{\bar{\rm M}} - Me^{\pm\theta}\bar{\bar{\rm I}}^{(2)T}\right) \Big| \left[\left(\bar{\bar{\rm M}} - Me^{\mp\theta}\bar{\bar{\rm I}}^{(2)T}\right) \Big| \mathbf{\Phi}\right] = 0, \tag{41}$$

whence there are two solutions for the eigenproblem

$$\bar{\bar{\rm M}}|\mathbf{\Phi}_\pm = M_\pm \mathbf{\Phi}_\pm, \qquad M_\pm = Me^{\pm\theta}. \tag{42}$$

The eigen-two-forms can be expressed in the form

$$\mathbf{\Phi}_\pm = \bar{\bar{\mathrm{P}}}_\pm|\mathbf{\Phi}, \tag{43}$$

for any two-form $\mathbf{\Phi}$ yielding nonzero results. The two normalized dyadics

$$\bar{\bar{\mathrm{P}}}_\pm = \pm\frac{1}{2M\sinh\theta}\left(\bar{\bar{\mathrm{M}}} - Me^{\mp\theta}\bar{\bar{\mathrm{I}}}^{(2)T}\right) = \frac{1}{2}\left(\bar{\bar{\mathrm{I}}}^{(2)T} \pm \bar{\bar{\mathrm{K}}}\right) \tag{44}$$

serve as orthogonal projection dyadics because they satisfy

$$\bar{\bar{\mathrm{P}}}_\pm^2 = \frac{1}{4}\left(\bar{\bar{\mathrm{I}}}^{(2)T} \pm \bar{\bar{\mathrm{K}}}\right)^2 = \frac{1}{4}\left(2\bar{\bar{\mathrm{I}}}^{(2)T} \pm 2\bar{\bar{\mathrm{K}}}\right) = \bar{\bar{\mathrm{P}}}_\pm, \tag{45}$$

$$\bar{\bar{\mathrm{P}}}_+|\bar{\bar{\mathrm{P}}}_- = \bar{\bar{\mathrm{P}}}_-|\bar{\bar{\mathrm{P}}}_+ = 0, \qquad \bar{\bar{\mathrm{P}}}_+ + \bar{\bar{\mathrm{P}}}_- = \bar{\bar{\mathrm{I}}}^{(2)T}, \tag{46}$$

and the symmetry conditions

$$\mathbf{e}_N\lfloor\bar{\bar{\mathrm{P}}}_\pm = \frac{1}{2}\left(\mathbf{e}_N\lfloor\bar{\bar{\mathrm{I}}}^{(2)T} \pm \bar{\bar{\mathrm{Q}}}^{(2)}\right) = \left(\mathbf{e}_N\lfloor\bar{\bar{\mathrm{P}}}_\pm\right)^T. \tag{47}$$

Thus, any two-form $\mathbf{\Phi}$ can be uniquely split in two two-form components $\mathbf{\Phi}_\pm$ as

$$\mathbf{\Phi} = \left(\bar{\bar{\mathrm{P}}}_+ + \bar{\bar{\mathrm{P}}}_-\right)|\mathbf{\Phi} = \mathbf{\Phi}_+ + \mathbf{\Phi}_-, \tag{48}$$

defined by

$$\mathbf{\Phi}_\pm = \bar{\bar{\mathrm{P}}}_\pm|\mathbf{\Phi}, \qquad \bar{\bar{\mathrm{P}}}_\mp|\mathbf{\Phi}_\pm = 0. \tag{49}$$

Existence of the projection dyadics requires $\theta \neq 0$, i.e., that the medium is not a pure axion medium, which was assumed above.

The dot product of two eigen-two-forms yields

$$\begin{aligned}\mathbf{\Phi}_+\cdot\mathbf{\Phi}_- &= \mathbf{e}_N|(\mathbf{\Phi}_+\wedge\mathbf{\Phi}_-) = \mathbf{\Phi}|\left(\bar{\bar{\mathrm{P}}}_+^T|\mathbf{e}_N\lfloor\bar{\bar{\mathrm{P}}}_-\right)|\mathbf{\Phi}\\ &= \frac{1}{4}\mathbf{\Phi}|\left(\mathbf{e}_N\lfloor\bar{\bar{\mathrm{I}}}^{(2)T} - \bar{\bar{\mathrm{K}}}^T|\mathbf{e}_N\lfloor\bar{\bar{\mathrm{K}}}\right)|\mathbf{\Phi}\\ &= \frac{1}{4}\mathbf{\Phi}|\left(\mathbf{e}_N\lfloor\bar{\bar{\mathrm{I}}}^{(2)T} - \bar{\bar{\mathrm{Q}}}^{(2)}|\boldsymbol{\varepsilon}_N\lfloor\bar{\bar{\mathrm{Q}}}^{(2)}\right)|\mathbf{\Phi} = 0,\end{aligned} \tag{50}$$

where at the last step we have applied the inverse rule (17). This implies a set of orthogonality relations for the eigenfields,

$$\mathbf{\Phi}_+\cdot\mathbf{\Phi}_- = 0, \quad \mathbf{\Phi}_+\cdot\mathbf{\Psi}_- = \mathbf{\Psi}_+\cdot\mathbf{\Phi}_- = 0, \quad \mathbf{\Psi}_+\cdot\mathbf{\Psi}_- = 0, \tag{51}$$

whence

$$\mathbf{\Phi}\cdot\mathbf{\Phi} = \mathbf{\Phi}_+\cdot\mathbf{\Phi}_+ + \mathbf{\Phi}_-\cdot\mathbf{\Phi}_-. \tag{52}$$

As a summary one can state that, for an SQ-medium, and unlike for the Q-medium in general, the fields can be decomposed in two simple eigenfields. Actually, the eigenfields are similar to those in the so-called Bohren decomposition [15], valid for isotropic chiral media.

3.2. Eigenfield Equations

Electromagnetic fields in a homogeneous SQ-medium defined by the medium dyadic (27) satisfy the two Maxwell equations,

$$\mathbf{d} \wedge \left(\bar{\bar{\mathrm{M}}}|\mathbf{\Phi}\right) = \boldsymbol{\gamma}_e, \quad \mathbf{d} \wedge \mathbf{\Phi} = \boldsymbol{\gamma}_m, \tag{53}$$

where $\boldsymbol{\gamma}_e$ and $\boldsymbol{\gamma}_m$ denote electric and magnetic source three-forms

$$\boldsymbol{\gamma}_e = \boldsymbol{\varrho}_e - \mathbf{J}_e \wedge \boldsymbol{\varepsilon}_4, \quad \boldsymbol{\gamma}_m = \boldsymbol{\varrho}_m - \mathbf{J}_m \wedge \boldsymbol{\varepsilon}_4. \tag{54}$$

Substituting the decomposition (49) in terms of the eigenfields (43), the equations

$$M_+\mathbf{d} \wedge \mathbf{\Phi}_+ + M_-\mathbf{d} \wedge \mathbf{\Phi}_- = \boldsymbol{\gamma}_e, \tag{55}$$

$$\mathbf{d} \wedge \mathbf{\Phi}_+ + \mathbf{d} \wedge \mathbf{\Phi}_- = \boldsymbol{\gamma}_m \tag{56}$$

can be split in two uncoupled equations

$$\mathbf{d} \wedge \mathbf{\Phi}_+ = \boldsymbol{\gamma}_+, \quad \mathbf{d} \wedge \mathbf{\Phi}_- = \boldsymbol{\gamma}_- \tag{57}$$

where the decomposed sources are

$$\boldsymbol{\gamma}_\mp = \frac{\pm 1}{M_+ - M_-}(M_\pm \boldsymbol{\gamma}_m - \boldsymbol{\gamma}_e). \tag{58}$$

Thus, the eigenfields see the medium as an axion or PEMC medium with effective PEMC admittance values $M_\pm$ [11]. However, the pure axion medium corresponding to $M_+ - M_- = 0$ was originally excluded from our analysis.

Because the stress-energy dyadic $\bar{\bar{\mathrm{T}}}(\mathbf{\Psi}, \mathbf{\Phi}) \in \mathbb{F}_3\mathbb{F}_1$ [5],

$$\bar{\bar{\mathrm{T}}}(\mathbf{\Psi}, \mathbf{\Phi}) = \frac{1}{2}\left(\mathbf{\Psi} \wedge \bar{\bar{\mathrm{I}}}^T \rfloor \mathbf{\Phi} - \mathbf{\Phi} \wedge \bar{\bar{\mathrm{I}}}^T \rfloor \mathbf{\Psi}\right) = -\bar{\bar{\mathrm{T}}}(\mathbf{\Phi}, \mathbf{\Psi}), \tag{59}$$

is obtained by the operation $\frac{1}{2}() \wedge \bar{\bar{\mathrm{I}}}^T \rfloor ()$ from the antisymmetric dyadic $\bar{\bar{\mathrm{A}}}(\mathbf{\Psi}, \mathbf{\Phi}) \in \mathbb{F}_2\mathbb{F}_2$

$$\bar{\bar{\mathrm{A}}}(\mathbf{\Psi}, \mathbf{\Phi}) = \mathbf{\Psi}\mathbf{\Phi} - \mathbf{\Phi}\mathbf{\Psi} = -\bar{\bar{\mathrm{A}}}(\mathbf{\Phi}, \mathbf{\Psi}) = -\bar{\bar{\mathrm{A}}}^T(\mathbf{\Psi}, \mathbf{\Phi}), \tag{60}$$

it obviously vanishes for each eigenfield:

$$\bar{\bar{\mathrm{T}}}(\mathbf{\Psi}_\pm, \mathbf{\Phi}_\pm) = M_\pm \bar{\bar{\mathrm{T}}}(\mathbf{\Phi}_\pm, \mathbf{\Phi}_\pm) = 0. \tag{61}$$

Thus, the eigenfields alone do not carry any energy. For total fields the energy transportation is possible through the interaction of both eigenfields:

$$\begin{aligned}\bar{\bar{\mathrm{T}}}(\mathbf{\Psi}, \mathbf{\Phi}) &= \bar{\bar{\mathrm{T}}}(\mathbf{\Psi}_+, \mathbf{\Phi}_-) + \bar{\bar{\mathrm{T}}}(\mathbf{\Psi}_-, \mathbf{\Phi}_+) = M_+\bar{\bar{\mathrm{T}}}(\mathbf{\Phi}_+, \mathbf{\Phi}_-) + M_-\bar{\bar{\mathrm{T}}}(\mathbf{\Phi}_-, \mathbf{\Phi}_+) \\ &= (M_+ - M_-)\bar{\bar{\mathrm{T}}}(\mathbf{\Phi}_+, \mathbf{\Phi}_-),\end{aligned} \tag{62}$$

recalling, again, the non-axion assumption $M_+ \neq M_-$.

3.3. Potential Equation

For no magnetic sources the Maxwell equation

$$\mathbf{d} \wedge \boldsymbol{\Phi}(\mathbf{x}) = 0, \tag{63}$$

is satisfied when the field two-form is expressed in terms of a potential one-form $\boldsymbol{\phi}$ as

$$\boldsymbol{\Phi}(\mathbf{x}) = \mathbf{d} \wedge \boldsymbol{\phi}(\mathbf{x}). \tag{64}$$

The equation for $\boldsymbol{\phi}$ is obtained from the other Maxwell equation as

$$\mathbf{d} \wedge \boldsymbol{\Psi}(\mathbf{x}) = \mathbf{d} \wedge \left(\bar{\bar{\mathrm{M}}} \middle| (\mathbf{d} \wedge \boldsymbol{\phi}(\mathbf{x}))\right) = \beta \mathbf{d} \wedge \left(\bar{\bar{\mathrm{K}}} \middle| (\mathbf{d} \wedge \boldsymbol{\phi}(\mathbf{x}))\right) = \boldsymbol{\gamma}_e(\mathbf{x}), \tag{65}$$

because the axion term falls off. Operating by $\mathbf{e}_N \rfloor$ yields

$$\beta \mathbf{d} \rfloor \left(\mathbf{e}_N \lfloor \bar{\bar{\mathrm{K}}} \middle| (\mathbf{d} \wedge \boldsymbol{\phi}(\mathbf{x}))\right) = \beta \mathbf{d} \rfloor \left(\bar{\bar{\mathrm{Q}}}^{(2)} \middle| (\mathbf{d} \wedge \boldsymbol{\phi}(\mathbf{x}))\right)$$
$$= \beta \left(\bar{\bar{\mathrm{Q}}} | \mathbf{d}\right) \left(\mathbf{d} | \bar{\bar{\mathrm{Q}}} | \boldsymbol{\phi}(\mathbf{x})\right) - \beta \left(\mathbf{d} | \bar{\bar{\mathrm{Q}}} | \mathbf{d}\right) \left(\bar{\bar{\mathrm{Q}}} | \boldsymbol{\phi}(\mathbf{x})\right) = \mathbf{e}_N \lfloor \boldsymbol{\gamma}_e(\mathbf{x}). \tag{66}$$

Since the potential is not unique, we can assume the Lorenz condition for the potential in the form

$$\mathbf{d} | \bar{\bar{\mathrm{Q}}} | \boldsymbol{\phi}(\mathbf{x}) = 0, \tag{67}$$

whence the equation is reduced to

$$\left(\mathbf{d} | \bar{\bar{\mathrm{Q}}} | \mathbf{d}\right) \left(\bar{\bar{\mathrm{Q}}} | \boldsymbol{\phi}(\mathbf{x})\right) = \frac{1}{\beta} \mathbf{e}_N \lfloor \boldsymbol{\gamma}_e(\mathbf{x}). \tag{68}$$

Applying (13) this finally becomes

$$\left(\mathbf{d} | \bar{\bar{\mathrm{Q}}} | \mathbf{d}\right) \boldsymbol{\phi}(\mathbf{x}) = \frac{1}{\beta} \boldsymbol{\varepsilon}_N \lfloor \bar{\bar{\mathrm{Q}}}^{(3)} | \boldsymbol{\gamma}_e(\mathbf{x}), \tag{69}$$

which is a second-order differential equation for the one-form potential $\boldsymbol{\phi}$. Its nature depends on the signature of the metric dyadic $\bar{\bar{\mathrm{Q}}}$.

As an example we can consider the plane-wave field in the SQ-medium,

$$\boldsymbol{\Phi}(\mathbf{x}) = \boldsymbol{\Phi} e^{\boldsymbol{\nu}|\mathbf{x}}, \tag{70}$$

where $\boldsymbol{\nu} \in \mathbb{F}_1$ is the wave one-form. Representing the field in terms of the potential one-form as

$$\boldsymbol{\Phi}(\mathbf{x}) = \boldsymbol{\nu} \wedge \boldsymbol{\phi}(\mathbf{x}) = \boldsymbol{\nu} \wedge \boldsymbol{\phi} e^{\boldsymbol{\nu}|\mathbf{x}}, \tag{71}$$

and since the sources of the plane wave are outside the finite region, from (69) the wave one-form $\boldsymbol{\nu}$ must satisfy the dispersion equation

$$\boldsymbol{\nu} \left| \bar{\bar{\mathrm{Q}}} \right| \boldsymbol{\nu} = 0. \tag{72}$$

This coincides with that of the general Q-medium [5, 8]. Thus, we see that the extension by the axion term and the restriction by symmetry of the dyadic $\bar{\bar{\mathrm{Q}}}^{(2)}$ do not change form of the dispersion equation of the Q-medium. In particular, this means that there is no birefringence in the SQ-medium and both eigenfields obey the same equation. It is known that there is no birefringence in any Q-medium, either, while the general SD medium is birefringent.

4. SUMMARY

In the present study, we have defined a class of media with properties somewhat between the previously known classes of Q-media and SD-media. The novel class was dubbed that of SQ-media. Since the medium dyadic of any SQ-medium satisfies a dyadic equation of the second order, it belongs as a special case to the class of SD-media by definition. However, SQ-media share the property of no birefringence of the Q-media which is not shared by the general SD-media. SQ-medium dyadic has certain interesting properties. It acts as a hyperbolic rotation multiplied by a magnitude coefficient for the electromagnetic two-form $\mathbf{\Phi}$. On the other hand, any field two-form can be decomposed in two eigencomponents Finally, comparisons between the three-dimensional (spatial) medium-dyadic definitions are given for the three medium classes which show their difference.

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