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PROPAGATION OF ELECTROMAGNETIC WAVES IN MATERIAL MEDIA WITH MAGNETIC MONOPOLES

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Abstract—The objective of this paper is to establish the properties of the electromagnetic wave propagation in a diversity of situations in material media with magnetic monopoles and even in the situations of entities simultaneously containing electric and magnetic charges. This analysis requires the knowledge and solutions of the "Maxwell" equations in material media compatible with the existence of magnetic monopoles and the extended concepts of linear responses (conductivity, split-charge susceptibility, kinetic susceptibility, permittivity and magnetic permeability) in the case of presence of electric and magnetic charges. This analysis can facilitate insights and suggestions for electrical and optical experiments affording a better knowledge of the materials whose behaviour can be analyzed under the consideration of the existence of entities with equivalent properties of the magnetic monopoles.

1. INTRODUCTION

The existence of a total symmetry of the fundamental electrodynamics equations is an objective even previous to the formulation of Maxwell's equations in 1861. The lack of the complete symmetry is due to the non-existence of particles with magnetic charge. The consideration of magnetic charge currents has been used for the determination of the electromagnetic fields in scattering, diffraction and aperture antennas [1–3]. This is carried out in order to obtain more efficacy and simplicity in the solutions of the standard Maxwell equations in complex electromagnetic problems. In this case, a real problem is mathematically substituted by an equivalent system with the same solution in a space region in which the **E** and **B** fields should

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be determined. This equivalent system is constituted of superficial electric and magnetic currents located in the closed surface. The consideration of these currents of magnetic charges without physical assigned phenomenology is an excellent mathematical procedure of calculation in order to improve the solution of complex electromagnetic problems. On the other hand, analyses that present certain similarities with those of antenna radiations based on the electrostatic image method are considered in recent experiments carried out in topological insulators [4].

In any case, the first time that the magnetic monopoles appears in scientific literature was in a comment of P. Curie [5] in which he makes a parallelism between the electricity and magnetic conductivity. However, the magnetic monopole concept was quantitatively introduced by Dirac [6, 7] in 1931 by means of a rigid string such as a long solenoid. This Dirac monopole is an unobservable [8] whose magnetic charge, g, is quantized and related with universal constants, $g = 2\pi\hbar/(e\mu_0)$, where $\hbar$ is the Planck constant, e the electronic charge, and μ_0 the vacuum permeability. The brilliant arguments used by Dirac are oriented to justify the quantization of the electric charge. 51 years after Dirac's monopole definition, Cabrera [9] seems to have detected in a well known cosmic radiation experiment an induced current peak in a superconducting squid which was only possible when free magnetic monopoles crossed the superconducting ring. However, to our knowledge, no new cosmic event similar to that of Cabrera's experiment seems to have been repeated.

Recent experiments in certain materials, such as some above cited exotic insulators, and, above all in spin ice materials [8, 10–15] are giving new impulses to the analysis about the existence of composite and emergent entities which have behaviours compatible with those of the magnetic monopoles. Spin-ices are crystal structures of holmium titanate and dysprosium titanate that have a honeycomb aspect [8, 16] in whose vertexes there are magnetic ions located in crystal configurations in a tetrahedric form. These tetrahedra are connected to equal adjacent ones by a vertex. In the vertexes of each tetrahedron there are, in the ground state of the system, two inward magnetic moments and two outward magnetic moments in such a way that each tetrahedron is magnetic neutral (i.e., their total magnetic moment is zero).

When an external agent, such as magnetic field, axial stress, temperature variation or other causes is applied, an excitation of the system can produce a spin flip in one of the vertexes of two contiguous tetrahedra. The result of this spin inversion of a shared magnetic spin moment is the breakdown of the magnetic neutrality, and then a net magnetic moment appears in each of the two contiguous tetrahedra. These net magnetic moments have opposite directions in each component of the contiguous tetrahedron pair. The magnetic field created by these net opposite magnetic moments is equivalent to that which would produce a pair constituted of one monopole and one antimonopole whose magnetic charges, obviously, are opposite. The equivalent monopoles formed via spin-flip action in spin-ice, in contrast to the Dirac's monopoles, are observable and there is no reason for requiring the quantization of their charge [8]. On the other hand, the monopole pair created by a spin inversion is an effective object which does not contain the elementary particle properties. However, these monopoles differs from those introduced in the antenna theory in which these effective magnetic charges of the spin-ice structures cause a rich phenomenology that has influence in the conductivity, magnetic and thermal properties, and can even produce phase transitions. In a starting phase, these monopole pairs are confined forming a magnetic charge dipole whose charge splitting is a minimal lattice parameter. The Coulomb binding energy of the monopole-antimonopole pair is $-\frac{\mu_0 g^2}{4\pi a}$, with a being the average separation between the two equivalent magnetic charges. The external magnetic field with its action over the magnetic charge exerts the Lorentz force, gB, which competes with the coulomb force among monopoles and antimonopoles. This competence between the two forces can provoke the dissociations of the monopole pair in a similar image to that of the dissociation of a molecule in an electrolyte [11]. The proliferation of deconfined monopoles pairs and the thermal generation of moment defects can produce a new phase constituted of a more or less dense gas of magnetic charges [11] spread within the solid. The dynamic of this magnetic charge gas can be assimilated to a slightly concentrated electrolyte which can be analyzed via the Onsager theory of the Wien effect [11, 17]. The neutron scattering experiments [12, 15] have unequivocally elucidated the magnetic structure of these spin-ice materials and the conductance measurements analyzed in the light of Onsager's theory have obtained the determination of the magnetic charge.

The existence of these experimental analyses and the interest generated by the study of these fascinating magnetic materials induce the necessity of revisiting the classical electrodynamics by including the magnetic monopoles as sources of electromagnetic fields. One of the most important applications could be the creation and manipulation of "magnetronic" circuits in similar way to that which, at the present time, is being established by the spintronic [18] devices. Obviously, this extended classical electrodynamics should be completed rethinking the quantum electrodynamic and the solid state physics by considering this magnetic charges. However, in a first stage, it is necessary to known the consequences of the existence of spatial points with divergence of **B** different from zero in order to be able to apply the quantum correspondence principle to the microscopic phenomena.

The main mathematical consequence of the assumption of magnetic monopole existence is the complete symmetry of the fundamental electrodynamics equations [19] which can be obtained by the consideration of the most pristine fundamentals of physics. The existence of monopoles and its formalization as a possible model has been considered in some unification theories for analyzing phenomenologies and behaviours of interacting hadronic systems [20]. Therefore, these models consider the evolutions of possible magnetic charges in the vacuum and without existence of continuum matter which mediates and conditions the interactions among the different electric and magnetic charges. In this paper, we consider a plausible theory about the classical electromagnetic wave propagation within matter, such as, for instance, the spin-ice materials, where there are effective magnetic monopoles. A new construction of the classical electrodynamics is necessary, since, while the formulation of the "Maxwell" equations considering magnetic monopoles in the empty space is well established [19], an extended electrodynamics within the matter with electric and magnetic charges is poorly treated in specialized literature. Maybe, this absence of scientific discussion in this matter is due to the non-existence of any concrete experiment (until 2009) which had detected entities whose behaviour is compatible with the magnetic monopole idea in an incontrovertible way. The author of the previous experiment carried out in 1982 [9] presented some clues which could signify the existence of gauge monopoles (i.e., elemental free particles in empty space) but, this experiment is, in our opinion, not sufficiently conclusive and no similar experiment has been presented since then in literature.

2. FIELD EQUATIONS WITH MAGNETIC MONOPOLES

It is well known [19, 21–24] that with the assumption of existence of magnetic monopoles the "Maxwell" equations should be:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -K\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}, \tag{1}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = K\rho_m, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{\mathbf{J}_e}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}, \tag{2}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, ρ_e the electric charge density, ρ_m the magnetic monopole density, $\mathbf{J}_e$ the electric current density, and $\mathbf{J}_m$ the magnetic current density. The speed of light in vacuum is $c = \sqrt{(\varepsilon_0\mu_0)^{-1}}$. In the International System of Units, $\mu_0 \equiv 4\pi \times 10^{-7}$ NA^{-2}, and ε_0 the vacuum permittivity.

Equations which present compatibilities with (1) and (2) in the main variant with respect to the standard Maxwell equations (i.e., $\boldsymbol{\nabla} \cdot \mathbf{B} \neq 0$) are found in the analysis of electromagnetic scattering, diffraction and aperture antenna problems [1–3], and in memristive media [25].

On the other hand, the Lorentz force for point charges can be written as:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{K}{\mu_0} g \left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right). \tag{3}$$

where the particle has both electric (q) and magnetic (g) charges, and $\mathbf{v}$ is their velocity (obviously if the particle only has either electric or magnetic charge one of the two terms between parentheses should be zero). This hypothetical particle with magnetic and electric charge is called *dyon* [26]. The K-parameter should be fixed by the definition of the magnetic charge which should be adjusted by means of experimental force between particles and fields through the force of Lorentz. Obviously for $K = 0$, we have the standard equations of electromagnetism.

The "Maxwell" Equations (1) and (2) can be expressed in a unified way by making the following definition

$$\kappa \equiv \varepsilon_0 c K = \frac{K}{\mu_0 c}. \tag{4}$$

Reasonable choices for K and κ are

K	1	μ_0	$\mu_0 c$
κ	$(\mu_0 c)^{-1}$	c^{-1}	1

(5)

In the first column, the unit of magnetic charge is the weber used in Jackson's book [19] and in the antenna theory [2, 3]. In he second column the unit of magnetic charge is ampere·meter, [8, 22, 23]. The third is, in our opinion, the best choice, because it simplifies our equations slightly. Obviously, all columns yield equivalent "Maxwell" equations each of them in its respective system of units.

The four Eqs. (1) and (2) are reduced to the following two composed equations

$$\boldsymbol{\nabla}\cdot\begin{pmatrix}\mathbf{E}\\ c\mathbf{B}\end{pmatrix}=\frac{1}{\varepsilon_0}\begin{pmatrix}\rho_e\\ \kappa\rho_m\end{pmatrix}, \tag{6}$$

$$\boldsymbol{\nabla}\times\begin{pmatrix}\mathbf{E}\\ c\mathbf{B}\end{pmatrix}=\frac{1}{\varepsilon_0 c}\,\Omega\begin{pmatrix}\mathbf{J}_e\\ \kappa\mathbf{J}_m\end{pmatrix}+\frac{1}{c}\frac{\partial}{\partial t}\Omega\begin{pmatrix}\mathbf{E}\\ c\mathbf{B}\end{pmatrix}; \tag{7}$$

where the Ω-matrix is defined as:

$$\Omega\equiv\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{8}$$

As a consequence, the Lorentz force, in coherence with these compacting "Maxwell" equations, can be written as:

$$\mathbf{F}=\begin{pmatrix}q, & \kappa g\end{pmatrix}\left(\mathbb{1}-\frac{\mathbf{v}}{c}\times\Omega\right)\begin{pmatrix}\mathbf{E}\\ c\mathbf{B}\end{pmatrix}. \tag{9}$$

We define an electromagnetic charge density ϱ, as a two dimensional vector, which includes the electric and magnetic charge,

$$\varrho\equiv\begin{pmatrix}\rho_e\\ \kappa\rho_m\end{pmatrix},\quad Q\equiv\begin{pmatrix}q\\ \kappa g\end{pmatrix}\equiv|Q|\begin{pmatrix}\cos\zeta\\ \sin\zeta\end{pmatrix}, \tag{10}$$

where $|Q|\equiv\sqrt{q^2+\kappa^2g^2}$ and $\tan\zeta\equiv q/(\kappa g)$. On the other hand, we define two six-dimension vectors $\mathbf{J}$ and $\mathbf{G}$ arranged as follows:

$$\mathbf{J}\equiv\begin{pmatrix}\mathbf{J}_e\\ \kappa\mathbf{J}_m\end{pmatrix},\quad \mathbf{G}\equiv\begin{pmatrix}\mathbf{E}\\ c\mathbf{B}\end{pmatrix}. \tag{11}$$

Note that a electromagnetic charge Q has two components, an electric charge, q, and a magnetic charge, κg. Similarly, the other field vectors have two components: the electrical component is the upper subspace component, and the another component will be the magnetic component.

The Maxwell Eqs. (1) and (2) can be also written as follows

$$\boldsymbol{\nabla}\cdot\mathbf{G}=\frac{1}{\varepsilon_0}\,\varrho, \tag{12}$$

$$\boldsymbol{\nabla}\times\mathbf{G}=\frac{1}{\varepsilon_0 c}\,\Omega\mathbf{J}+\frac{1}{c}\frac{\partial}{\partial t}\Omega\mathbf{G}. \tag{13}$$

and the Lorentz force takes the form:

$$\mathbf{F}=\begin{pmatrix}q, & \kappa g\end{pmatrix}\left(\mathbb{1}-\frac{\mathbf{v}}{c}\times\Omega\right)\mathbf{G}. \tag{14}$$

We want to remark the complete symmetry and similarity of Eqs. (12) and (13) with respect to each of the components of the $\mathbf{G}$ field, therefore, the electric field generated by electric and magnetic charges and the magnetic field produced by the electric and magnetic currents are two constitutive fields of the this extended electromagnetism.

Another equivalent representation (complex notation) of the above "Maxwell" and "Lorentz" equations is obtained by multiplying Eq. (10) by the row matrix $(1, i)$, where $i \equiv \sqrt{-1}$ is the imaginary unit. We define:

$$\begin{aligned} &Q \equiv q + i\kappa g \equiv |Q|e^{i\zeta} = |Q|\cos\zeta + i|Q|\sin\zeta, \\ &\varrho \equiv \rho_e + i\kappa\rho_m, \qquad \mathbf{J} \equiv \mathbf{J}_e + i\kappa\mathbf{J}_m, \\ &\mathbf{G} \equiv \mathbf{E} + ic\mathbf{B}. \end{aligned} \tag{15}$$

In this notation, ζ is the polar angle in a complex plane of the complex charge Q. The dyon charge Q has an electric charge represented in the real axis, and the magnetic charge represented in the imaginary axis. Then, the four Maxwell equations are converted in the two following equations

$$\boldsymbol{\nabla}\cdot\mathbf{G} = \frac{\varrho}{\varepsilon_0}, \qquad \boldsymbol{\nabla}\times\mathbf{G} = \frac{i}{c}\left(\frac{\mathbf{J}}{\varepsilon_0} + \frac{\partial\mathbf{G}}{\partial t}\right), \tag{16}$$

obtaining again a total symmetry between the field $\mathbf{E}$ and $\mathbf{B}$. Therefore, there is now a single electromagnetic field, $\mathbf{G} \equiv \mathbf{E} + ic\mathbf{B}$, with two "flavours", where the real part is an electric field, $\mathbf{E}$, and the imaginary part is a magnetic field, $c\mathbf{B}$. The Lorentz force is

$$\mathbf{F} = \mathrm{Re}\,Q^*\left(1 - i\frac{\mathbf{v}}{c}\times\right)\mathbf{G} = |Q|\,\mathrm{Re}\left(\cos\zeta - i\sin\zeta\right)\left(1 - i\frac{\mathbf{v}}{c}\times\right)\mathbf{G}. \tag{17}$$

Note that considering all the force, and not only the real part, the Eq. (17) is not the standard Lorentz force when $\zeta = 0$.

The so-called duality transformation is, in the complex notation, to rotate the function in the complex plane, i.e., multiplying by a phase $e^{i\zeta'}$. Therefore, the duality transformation over positive charge q becomes $Q = qe^{i\zeta'}$, and an electric charge becomes a dyon with electric and magnetic component. The electromagnetic complex vector, $\mathbf{G}$, have invariant module ($|\mathbf{G}| \equiv \mathbf{G}\cdot\mathbf{G}^*$) in this transformation and the "Maxwell" equations remain invariant before this duality operation, since both fields of $\mathbf{G}$ rotate such as the charge density, ϱ. Besides, the Lorentz force is also invariant, since the product $\varrho^*\mathbf{G}$ does not suffer any change by multiplying by $e^{-i\zeta'}e^{i\zeta'}$.

2.1. Field Equations in Matter

The main objective of this paper is the determination and analysis of the electromagnetic fields within the matter in which the magnetic monopoles is not discarded. In this second part of this paper, we comment the formulation of a plausible theory about the extended classical electrodynamics within the matter, in which several magnetic monopoles have been detected in recent experiments [11, 15]. The interpretation of the experimental results, as said above, requires the new construction of the classical electrodynamics within the continuum matter constituted of magnetic and electric charges.

Hence, we have to analyze the Maxwell equations where the multipole expansion of the potentials is developed as a function of electric and magnetic polarizations which in this case depend on the spatial distribution of two different kind of dipoles. The expressions of these electric and magnetic polarizations are:

$$\mathbf{P} \equiv \begin{pmatrix} \mathbf{P}_e \\ \kappa\mathbf{P}_m \end{pmatrix} \equiv \frac{1}{\Delta V} \int_{\Delta V} \mathbf{r} \begin{pmatrix} \rho_e \\ \kappa\rho_m \end{pmatrix} d^3r, \tag{18}$$

$$\mathbf{M} \equiv \begin{pmatrix} \mathbf{M}_e \\ \kappa\mathbf{M}_m \end{pmatrix} \equiv \frac{1}{2\Delta V} \int_{\Delta V} \mathbf{r} \times \begin{pmatrix} \mathbf{J}_e \\ \kappa\mathbf{J}_m \end{pmatrix} d^3r. \tag{19}$$

where, ΔV is a volume that approaches zero, and $\mathbf{P}_e$ ($\mathbf{P}_m$) is the electric (magnetic) polarization due to a split in the gravity centers of the electric (magnetic) charges, and $\mathbf{M}_e$ ($\mathbf{M}_m$) is the magnetization due to the movements of electric (magnetic) charges which constitute the electric and magnetic currents. In order to clarify the duplicity of the polarization, dipoles, currents and charges, we will, henceforth, call $\mathbf{P}$ *split-charge polarization* and $\mathbf{M}$ *kinetic polarization.* Following a coherent and parallel way to the the standard classical electrodynamics, we establish that the divergence equation of the electromagnetic field, $\mathbf{G}$, becomes

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \frac{1}{\varepsilon_0} \left(\varrho - \boldsymbol{\nabla} \cdot \mathbf{P}\right) \tag{20}$$

and this expression allows us to define the corresponding field within the matter $\mathbf{d}$, defined as

$$\mathbf{d} \equiv \varepsilon_0 \mathbf{G} + \mathbf{P}. \tag{21}$$

This is an extended concept of the electric displacement vector and whose divergence expression becomes:

$$\boldsymbol{\nabla} \cdot \mathbf{d} = \varrho. \tag{22}$$

By utilizing the same analogy and similar procedure to that of the divergence case, the determination of the rotational equation allows us to define the extended concept of the corresponding magnetic intensity $\mathbf{h}$:

$$\boldsymbol{\nabla} \times \mathbf{G} = \mu_0 c\, \Omega \left(\mathbf{J} + \boldsymbol{\nabla} \times \mathbf{M}\right), \tag{23}$$

then

$$\nabla \times \mathbf{h} = \Omega \mathbf{J}, \tag{24}$$

where one has to define that

$$\mathbf{h} \equiv \frac{1}{\mu_0 c}\mathbf{G} - \Omega \mathbf{M}. \tag{25}$$

An important and epistemological principle of Physics of the higher hierarchy is the charge conservation and therefore, it is necessary to maintain it in the case of a theory in extended electrodynamics. As a consequence of this principle, if the time derivatives of the density of electric and magnetic charges are not zero, we must impose the continuity equation for both electric and magnetic charges and then, the equivalent equations, in non static conditions, are:

$$\nabla \times \mathbf{h} = \Omega \left(\mathbf{J} + \frac{\partial \mathbf{d}}{\partial t} \right). \tag{26}$$

It must be remembered that the coherence with the standard classical electrodynamics should always exist when there are not magnetic monopoles, and therefore, for $\kappa = 0$, we have

$$\mathbf{d} = \begin{pmatrix} \mathbf{D} \\ \varepsilon_0 c \mathbf{B} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} \varepsilon_0 c \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{27}$$

which are the fields $\mathbf{D}$ (electric displacement field) and $\mathbf{H}$ (magnetic field intensity or magnetic field strength) of the standard electromagnetism.

3. LINEAR RESPONSES OF THE MATERIALS

The existence of electric and magnetic charges (q and g), the corresponding electric and magnetic currents ($\mathbf{J}_e$ and $\mathbf{J}_m$), which result from their movements, and the interaction forces of these charges and currents with the generated fields ($\mathbf{G}$, $\mathbf{d}$ and $\mathbf{h}$) generates changes in the particle velocity and in the polarizations: split-charge polarizations ($\mathbf{P}_e$, $\mathbf{P}_m$) and kinetic polarizations ($\mathbf{M}_e$, $\mathbf{M}_m$). In this section, we give an improved version of the relationships among the different electromagnetic fields and the dynamic responses induced in the material [21]. Fundamentally, these responses are: i) the current of charges, which, in this case, can be due to movements of electric and magnetic charges, which constitute the electricity and "magneticity", ii) changes of the positions of the charges in function of their signs, which constitute the split-charge polarization and iii) changes of the localized charge currents, and changes in the angular momentum of the particles, which constitute the kinetic polarization.

3.1. Conductivity or Generalized Ohm's law

In standard electromagnetism, the classical theory of conductivity establishes that due to the Lorentz force acting on the electrons, the electric field accelerates them till they are scattered with ions, and then, their velocities change. In an average estimation, and considering that the characteristic velocities within the matter are clearly non-relativistic, the velocity of a given and generic electron just after the collision is zero, therefore, the velocity starts to increase after collision and during an interval time between two successive collisions, this interval being an average time, τ. If one considers instead of electrons which have only electric charge, particles with both electric and magnetic charge, it is plausible to accept that the average velocity will also be given by the product of the acceleration during the average time between collisions, τ,

$$\mathbf{v} = \frac{q\mathbf{E} + \kappa g c\mathbf{B}}{m}\tau = \frac{\tau}{m}\,\mathrm{Re}\left(Q^{*}\mathbf{G}\right), \tag{28}$$

where the force is the sum of electric and magnetic forces. We have assumed that the velocity and τ are small enough (these conditions are valid for non-relativistic velocities and with relatively large density of particles within matter) so that the term of the Lorentz force proportional to $\mathbf{v}\times$ in Eq. (3) does not have to be considered. Therefore, we can define the extended Ohm law as

$$\mathbf{J} = \varrho\mathbf{v} = \frac{n\tau}{m}Q\mathrm{Re}\left(Q^{*}\mathbf{G}\right), \tag{29}$$

where n is the density of particles, i.e., $\varrho = nQ$.

This latter expression of Ohm's law, in matrix form, can be written

$$\mathbf{J} = \frac{n\tau}{m}\begin{pmatrix} q \\ \kappa g \end{pmatrix}(q, \kappa g)\begin{pmatrix} \mathbf{E} \\ c\mathbf{B} \end{pmatrix} = \overleftrightarrow{\sigma}\,\mathbf{G}. \tag{30}$$

As a consequence, the conductivity in a linear and isotropic material, equivalent to a simple function in the standard electromagnetism, is now a matrix defined by

$$\overleftrightarrow{\sigma} \equiv \sigma\Theta, \tag{31}$$

where σ and the Θ matrix are

$$\begin{aligned} \sigma &= \frac{n|Q|^2\tau}{m}, \\ \Theta &\equiv \frac{1}{|Q|^2}\begin{pmatrix} q^2 & q\kappa g \\ q\kappa g & \kappa^2 g^2 \end{pmatrix} = \begin{pmatrix} \cos^2\zeta & \frac{1}{2}\sin 2\zeta \\ \frac{1}{2}\sin 2\zeta & \sin^2\zeta \end{pmatrix}. \end{aligned} \tag{32}$$

In the cases, in which there are several classes of particles (we define that two particles are of the same class if they have the same value of

q/g, i.e., if the ζ values are equal or differ by π), then, the conductivity becomes the addition of all contributions, and it can be written as

$$\overleftrightarrow{\sigma} = \sum_p \sigma_p \Theta_p \,, \qquad \Theta_p \equiv \begin{pmatrix} \cos^2\zeta_p & \frac{1}{2}\sin 2\zeta_p \\ \frac{1}{2}\sin 2\zeta_p & \sin^2\zeta_p \end{pmatrix}. \tag{33}$$

The index of the summation runs through all the classes of particles. This summation can be effective in both doped semiconductors or insulators and the different situations of energy condensed states such as superconducting materials, Bose-Einstein condensates, polaron-polariton condensed states and excitonic bubble.

3.2. Split-charge Susceptibility

The split-charge susceptibility is defined by means of a relationship between the density of split-charge dipole moments and the electromagnetic field $\mathbf{G}$. The split-charge dipole moment of a electric dipole [see Eq. (18)] is defined by a vector that, in the most simplified case, is the charge multiplied by the vector whose origin is the negative point charge and its final is in the positive point charge. In standard electromagnetism and linear approximation, the electric polarization is proportional to the electric field which tends to split and modify the locations of the charges, moving away the positive charges and approaching the negative ones. When instead of electrical charges, we have dyons, we can consider that this splitting between positive and negative particles is also proportional to the force, $\mathbf{F}_0 = q\mathbf{E} + \kappa g c\mathbf{B}$, and the corresponding split-charge polarization will be proportional to the charge q and g. Then, it seems reasonable to write:

$$\begin{pmatrix} \mathbf{P}_e \\ \mathbf{P}_m \end{pmatrix} = n\alpha \begin{pmatrix} q \\ g \end{pmatrix} \mathbf{F}_0, \tag{34}$$

where α is a characteristic constant of each material. The last equation can be written, at not too much density of particles, as

$$\begin{pmatrix} \mathbf{P}_e \\ \kappa\mathbf{P}_m \end{pmatrix} = n\alpha \begin{pmatrix} q \\ \kappa g \end{pmatrix} (q, \kappa g) \begin{pmatrix} \mathbf{E} \\ c\mathbf{B} \end{pmatrix} = n|Q|^2\alpha\Theta \begin{pmatrix} \mathbf{E} \\ c\mathbf{B} \end{pmatrix}, \tag{35}$$

and as a consequence, we can define the function of the split-charge susceptibility as $\chi_s \equiv n|Q|^2\alpha/\varepsilon_0$, and then

$$\mathbf{P} = \varepsilon_0 \chi_s \Theta \mathbf{G}. \tag{36}$$

Therefore, having in mind Eq. (21), we can establish the relationship between vectors $\mathbf{G}$ and $\mathbf{d}$ which we name extended permittivity matrix and whose expression is:

$$\mathbf{d} = \varepsilon_0 \mathbf{G} + \mathbf{P} = \varepsilon_0 (\mathbb{1} + \chi_s \Theta)\mathbf{G} = \overleftrightarrow{\epsilon}\, \mathbf{G}, \tag{37}$$

where the permittivity matrix is:

$$\overleftrightarrow{\epsilon} \equiv \varepsilon_0(\mathbb{1} + \chi_s\Theta) = \varepsilon_0 \begin{pmatrix} 1+\chi_s\cos^2\zeta & \frac{1}{2}\chi_s\sin 2\zeta \\ \frac{1}{2}\chi_s\sin 2\zeta & 1+\chi_s\sin^2\zeta \end{pmatrix}. \quad (38)$$

From this latter expression, we can obtain the standard electric susceptibility in absence of magnetic monopoles ($\zeta = 0$). Actually,

$$\overleftrightarrow{\epsilon} = \varepsilon_0 \begin{pmatrix} 1+\chi_e & 0 \\ 0 & 1 \end{pmatrix}, \quad (39)$$

which obviously reduces to the expression of standard electromagnetism ($\chi_e = \chi_s$ is the electric susceptibility). In the cases with different particles of the same class, it is only necessary to modify the value of χ_s and with several classes of particles (groups of particles whose ζ is different from either 0 or π) it is also valid, $\mathbf{d} = \overleftrightarrow{\epsilon}\,\mathbf{G}$, but

$$\overleftrightarrow{\epsilon} = \varepsilon_0 \begin{pmatrix} 1+\sum_p \chi_{s,p}\cos^2\zeta_p & \frac{1}{2}\sum_p \chi_{s,p}\sin 2\zeta_p \\ \frac{1}{2}\sum_p \chi_{s,p}\sin 2\zeta_p & 1+\sum_p \chi_{s,p}\sin^2\zeta_p \end{pmatrix} = \varepsilon_0 \sum_p (\mathbb{1}+\chi_{s,p}\Theta_p). \quad (40)$$

We can define the permittivity as

$$\varepsilon \equiv \frac{\det|\overleftrightarrow{\epsilon}|}{\varepsilon_0}, \quad (41)$$

and when there is a unique class of dyons, this permittivity becomes

$$\varepsilon = \varepsilon_0 \det|1 + \chi_s\Theta| = \varepsilon_0(1+\chi_s). \quad (42)$$

Note that this expression for the permittivity is invariant by a duality transformation.

3.3. Kinetic Susceptibility

In the case of magnetic materials, the standard formulation establishes that an induced magnetic moment is proportional to $\mathbf{B}$, and this $\mathbf{B}$ produces a magnetic force ($\mathbf{v} \times \mathbf{B}$) which can modify the current structure that, in turn, modifies the magnetic polarization. If we consider the most simple case of a single electron moving along a circular path with a constant speed, the presence of a uniform steady field $\mathbf{B}$ existing in the region, will produce a magnetic force, which generates a change in the angular velocity, and a consequent change in the magnetic moment, proportional to the force. When we make an extension to the dyons we must consider the part of the Lorentz force proportional to the velocity,

$$\mathbf{F}_v = \mathbf{v} \times \left(q\mathbf{B} - \frac{\kappa}{c} g\mathbf{E}\right)$$

and the kinetic polarization will suffer a change whose expression is

$$\begin{pmatrix} \mathbf{M}_e \\ \mathbf{M}_m \end{pmatrix} = n\alpha' \begin{pmatrix} q \\ g \end{pmatrix} \left(q\mathbf{B} - \frac{\kappa}{c} g\mathbf{E} \right), \tag{43}$$

where α' is a response function which depends on the characteristic features of material, since this answer depends on the speed, we call it as kinetic polarizability of the material. Following a similar procedure to that used in the determination of the split-charge polarization, we have

$$\begin{pmatrix} \mathbf{M}_e \\ \kappa\mathbf{M}_m \end{pmatrix} = \frac{n\alpha'}{c} \begin{pmatrix} q \\ \kappa g \end{pmatrix} (-\kappa g, q) \begin{pmatrix} \mathbf{E} \\ c\mathbf{B} \end{pmatrix}. \tag{44}$$

This latter expression can be given in function of the kinetic susceptibility, χ_k, which is defined by $n|Q|^2\alpha'\mu_0 \equiv \chi_k/(1+\chi_k) \equiv \xi$,

$$\mathbf{M} = \frac{\xi}{\mu_0 c} \begin{pmatrix} -\frac{1}{2}\sin 2\zeta & \cos^2\zeta \\ -\sin^2\zeta & -\frac{1}{2}\sin 2\zeta \end{pmatrix} \mathbf{G}, \tag{45}$$

this relationship between vectors **M** and **G** also reduced, obviously, to the expression of standard electromagnetism when there is no magnetic monopoles, and then, the so-called kinetic susceptibility is the magnetic susceptibility.

Consequently, we can define the permeability matrix, $\overleftrightarrow{\mu}$, in a similar way to that which allowed us to define the permittivity matrix,

$$\begin{aligned} \mathbf{h} &= \frac{1}{\mu_0 c}\mathbf{G} - \Omega\mathbf{M} \equiv \frac{1}{c} \overleftrightarrow{\mu}^{-1} \mathbf{G} \\ &= \frac{1}{\mu_0 c} \left[\mathbb{1} - \xi\Omega \begin{pmatrix} -\frac{1}{2}\sin 2\zeta & \cos^2\zeta \\ -\sin^2\zeta & \frac{1}{2}\sin 2\zeta \end{pmatrix} \right] \mathbf{G}. \end{aligned} \tag{46}$$

By inversion of the matrix

$$\begin{aligned} \overleftrightarrow{\mu} &= \mu_0 \left[\mathbb{1} - \xi \begin{pmatrix} \sin^2\zeta & -\frac{1}{2}\sin 2\zeta \\ -\frac{1}{2}\sin 2\zeta & \cos^2\zeta \end{pmatrix} \right]^{-1} \\ &= \frac{\mu_0}{1-\xi} \left(\mathbb{1} - \xi\Theta \right), \end{aligned} \tag{47}$$

and therefore the corresponding relationship between the **G** and **h**-vectors

$$\mathbf{G} = c \overleftrightarrow{\mu} \mathbf{h}. \tag{48}$$

With this expression and from Eq. (45), the kinetic polarization is given by

$$\mathbf{M} = \chi_k \begin{pmatrix} -\frac{1}{2}\sin 2\zeta & \cos^2\zeta \\ -\sin^2\zeta & \frac{1}{2}\sin 2\zeta \end{pmatrix} \mathbf{h}. \tag{49}$$

In the case of absence of magnetic monopoles, becomes

$$\overleftrightarrow{\mu} = \mu_0 \begin{pmatrix} 1 & 0 \\ 0 & 1+\chi_k \end{pmatrix}, \quad \mathbf{M}_e = \chi_k \mathbf{H}, \tag{50}$$

which is the corresponding expression of the standard classical electromagnetism, with $\chi_k = \chi_m$ being the magnetic susceptibility.

Similar to the case of the split-charge susceptibility, for materials and systems with several kinds of particles of the same class one needs to modify only the value of χ_k, and if one considers several class of particles, the expression $\mathbf{G} = c \overleftrightarrow{\mu} \mathbf{h}$, continues being valid, but

$$\begin{aligned} \overleftrightarrow{\mu} &= \mu_0 \begin{pmatrix} 1-\sum_p \xi_p \sin^2 \zeta_p & \frac{1}{2}\sum_p \xi_p \sin 2\zeta_p \\ \frac{1}{2}\sum_p \xi_p \sin 2\zeta_p & 1-\sum_p \xi_p \cos^2 \zeta_p \end{pmatrix}^{-1} \\ &\equiv \mu_0 \begin{pmatrix} 1-\alpha & \beta \\ \beta & 1-\gamma \end{pmatrix}^{-1}. \end{aligned} \tag{51}$$

We can simplify the notation by means the following variable changes:

$$\begin{aligned} \alpha &\equiv \sum_p \xi_p \sin^2 \zeta, \\ \beta &\equiv \frac{1}{2}\sum_p \xi_p \sin 2\zeta_p, \\ \gamma &\equiv \sum_p \xi_p \cos^2 \zeta_p \ ; \end{aligned} \tag{52}$$

and with these changes of notation we have that $\overleftrightarrow{\mu}$ is,

$$\begin{aligned} \overleftrightarrow{\mu} &= \frac{\mu_0}{(1-\alpha)(1-\gamma)-\beta^2} \begin{pmatrix} 1-\gamma & -\beta \\ -\beta & 1-\alpha \end{pmatrix} \\ &= \frac{\mu_0}{(1-\alpha)(1-\gamma)-\beta^2} \left(\mathbb{1} - \sum_p \xi_p \Theta_p \right). \end{aligned} \tag{53}$$

The permeability can be defined by:

$$\mu \equiv \frac{\det \left| \overleftrightarrow{\mu} \right|}{\mu_0}, \tag{54}$$

which, for one class of dyons, becomes

$$\mu = \frac{\mu_0}{1-\xi} = \mu_0 (1+\chi_k). \tag{55}$$

Here, we can also observe that this expression is invariant by a duality transformation.

4. EM PROPAGATION IN MATTER WITH DYONS

The response functions of the materials, $\overleftrightarrow{\sigma}$, $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$ before the presence of the electromagnetic **G**-field and the corresponding relationships among the different resulting vectors, given in Eqs. (30), (37) and (48), allows us to determine the characteristic features of the electromagnetic wave propagation within this extended theory. This analysis should serve as a guide for understanding the classical electro-magneto-optical properties of the systems in which the electronic charge is substituted by the dyonic charge, i.e., those systems in which there is presence of magnetic monopoles.

In a first step, we consider the simplest case with a plane wave in a linear and uniform media in space regions with absence of charges. With these conditions, the divergence of the **d** field is,

$$0 = \boldsymbol{\nabla}\cdot\mathbf{d} = \boldsymbol{\nabla}\cdot \overleftrightarrow{\epsilon}\ \mathbf{G} = c\ \overleftrightarrow{\epsilon}\overleftrightarrow{\mu}\ \boldsymbol{\nabla}\cdot\mathbf{h} \equiv c\ \overleftrightarrow{\epsilon}\overleftrightarrow{\mu}\ \boldsymbol{\nabla}\cdot\begin{pmatrix}\mathbf{f}\\ \mathbf{b}\end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\mathbf{t})}, \quad (56)$$

where the vectors **f** and **b** are the initial electric and magnetic components (flavours) of **h**. In the simplified case of a monochromatic plane wave, it is satisfied the following expression

$$\overleftrightarrow{\epsilon}\overleftrightarrow{\mu}\begin{pmatrix}\mathbf{k}\cdot\mathbf{f}\\ \mathbf{k}\cdot\mathbf{b}\end{pmatrix} = 0. \quad (57)$$

If $\det|\ \overleftrightarrow{\epsilon}\overleftrightarrow{\mu}\ | \neq 0$, then, we have the restricted conditions on the propagation vector **k**:

$$\mathbf{k}\cdot\mathbf{f} = \mathbf{k}\cdot\mathbf{b} = 0. \quad (58)$$

These are the transversality relationships of **f** and **b**-fields.

In order to determine the **h**-vector versus conductivity of Eq. (33), permittivity of Eq. (40) and permeability of Eq. (53), we can use the equation of the curl of **h**:

$$\boldsymbol{\nabla}\times\mathbf{h} = c\,\Omega\left(\overleftrightarrow{\sigma} + \overleftrightarrow{\epsilon}\ \frac{\partial}{\partial t}\right)\overleftrightarrow{\mu}\ \mathbf{h}, \quad (59)$$

and for a monochromatic plane wave, this expression takes the following algebraic form,

$$\mathbf{k}\times\begin{pmatrix}\mathbf{f}\\ \mathbf{b}\end{pmatrix} = -\begin{pmatrix}S & -R\\ U & -T\end{pmatrix}\begin{pmatrix}\mathbf{f}\\ \mathbf{b}\end{pmatrix}, \quad (60)$$

where the definition of the *propagation matrix* is

$$\begin{pmatrix}U & -T\\ -S & R\end{pmatrix} = c\left(\omega\ \overleftrightarrow{\epsilon} + i\ \overleftrightarrow{\sigma}\right)\overleftrightarrow{\mu}\ . \quad (61)$$

The specific development of the matrix elements U, T, S and R of this propagation matrix is given in Appendix A. The cross product can be written in matrix form as

$$\mathbf{k}\times\mathbf{f}=\begin{pmatrix}0 & -k_z & k_y\\ k_z & 0 & -k_x\\ -k_y & k_x & 0\end{pmatrix}\begin{pmatrix}f_x\\ f_y\\ f_z\end{pmatrix}. \tag{62}$$

The vector $\mathbf{k}$ can be complex if the conductivity is different from zero and the simplest case is to consider that both real part and imaginary part are parallel, i.e., the attenuation and the propagation have the same direction. Then, in this case, we can choose a coordinate system in such a way that

$$\mathbf{k}\equiv k\mathbf{e}_x, \tag{63}$$

then, due to the transversality, Eq. (58),

$$\mathbf{f}\equiv f_y\mathbf{e}_y+f_z\mathbf{e}_z,\qquad \mathbf{b}\equiv b_y\mathbf{e}_y+b_z\mathbf{e}_z, \tag{64}$$

and a consequence Eq. (60) can be written as follows

$$0=\begin{bmatrix}\begin{pmatrix}S & 0 & 0\\ 0 & S & -k\\ 0 & k & S\end{pmatrix} & -R\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ \\ U\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} & \begin{pmatrix}-T & 0 & 0\\ 0 & -T & -k\\ 0 & k & -T\end{pmatrix}\end{bmatrix}\begin{pmatrix}0\\ f_y\\ f_z\\ \\ 0\\ b_y\\ b_z\end{pmatrix}, \tag{65}$$

this latter expression can be reduced and the result is the following equation

$$0=\begin{pmatrix}S & -k & -R & 0\\ k & S & 0 & -R\\ U & 0 & -T & -k\\ 0 & U & k & -T\end{pmatrix}\begin{pmatrix}f_y\\ f_z\\ b_y\\ b_z\end{pmatrix}\equiv\begin{pmatrix}\mathbf{A} & -R\mathbb{1}\\ U\mathbb{1} & \mathbf{B}\end{pmatrix}\begin{pmatrix}\mathbf{f}\\ \mathbf{b}\end{pmatrix}, \tag{66}$$

where we have made the following notation changes

$$\mathbf{A}\equiv\begin{pmatrix}S & -k\\ k & S\end{pmatrix},\quad \mathbf{B}\equiv\begin{pmatrix}-T & -k\\ k & -T\end{pmatrix}. \tag{67}$$

Eq. (66) is a eigenvalue-eigenvector equation which should have a non trivial solution if and only if

$$0=\det|\mathbf{AB}+RU\mathbb{1}|=\det\begin{vmatrix}-ST-k^2+RU & k(T-S)\\ -k(T-S) & -ST-k^2+RU\end{vmatrix}. \tag{68}$$

Here, k is the magnitude of the complex wave vector $k\mathbf{e}_x$, which is the variable of Eq. (68). The condition that should accomplish k is:

$$\left(-ST - k^2 + RU\right)^2 = -k^2(S-T)^2, \tag{69}$$

namely

$$k^2 \pm ik(S-T) - RU + ST = 0. \tag{70}$$

Obtained the values of $\mathbf{k}$ vector from Eqs. (63) and (69), we have to obtain the eigenvectors whose components are $\mathbf{f}$ and $\mathbf{b}$; this is carried out by means of the following matrix expressions.

$$\mathbf{Af} - R\mathbf{b} = 0, \qquad U\mathbf{f} + \mathbf{Bb} = 0. \tag{71}$$

These two equations after simple mathematical manipulations are converted in:

$$\mathbf{BAf} + UR\mathbf{f} = 0, \qquad \mathbf{ABb} + UR\mathbf{b} = 0, \tag{72}$$

which will have a different solution of the trivial if the determinantal Eq. (68) is satisfied.

The solutions of Eq. (72) can be written in compact form as:

$$\left(k^2 - RU + ST\right)\begin{pmatrix} f_y \\ b_y \end{pmatrix} - k(T-S)\begin{pmatrix} f_z \\ b_z \end{pmatrix} = 0, \tag{73}$$

or also

$$k(T-S)\begin{pmatrix} f_y \\ b_y \end{pmatrix} + \left(k^2 - RU + ST\right)\begin{pmatrix} f_z \\ b_z \end{pmatrix} = 0, \tag{74}$$

which are coherent with the condition of vanishing the determinant, Eq. (69). Having in mind Eq. (69), one can deduce that if $S = T$ the latter equations do not give information since both Eq. (73) and Eq. (74) are always $0 = 0$.

If $f_z = 0$ ($f_y = 0$), one must have $f_y = 0$ ($f_z = 0$) or $S = T$ and $k^2 - RU + ST = 0$. Also, if $b_z = 0$ ($b_y = 0$), one must have $b_y = 0$ ($b_z = 0$) or $S = T$ and $k^2 - RU + ST = 0$. In other words, if $S \neq T$ and a component of either $\mathbf{f}$ or $\mathbf{b}$ vanishes, then the electromagnetic field $\mathbf{h}$ can not be propagated, i.e., *there is not electromagnetic wave with linear polarization, within the material, if the propagation matrix is non-symmetric.*

4.1. Non-symmetric Propagation Matrix, $S \neq T$

The condition for obtaining the characteristic features of the electromagnetic propagation in matter with dyons, in general, and magnetic monopoles, as a class of dyons, can be deduced from

Eq. (66) which is constituted as the golden rule for the electromagnetic propagation in linear and isotropic materials. The basic properties of these materials are reflected in the response tensors $\overleftrightarrow{\sigma}$, $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$. In this golden rule, there are two different cases, which represent several physical classifications of materials, according to the balance of the components of the three linear response tensors which are joined in the propagation matrix of Eq. (61). If $S = T$, the Eq. (72) do not provide information, but if $S \neq T$, as the determinant of Eq. (68) has to be zero [Eq. (69)], then

$$\begin{pmatrix} f_y \\ b_y \end{pmatrix} = \frac{k(T-S)}{k^2 - RU + ST} \begin{pmatrix} f_z \\ b_z \end{pmatrix} = \pm i \begin{pmatrix} f_z \\ b_z \end{pmatrix}, \tag{75}$$

and therefore a solution with plane polarization is impossible. In this case, a solution of Eq. (66) can be expressed in the following way:

$$0 = \begin{pmatrix} S & -k & -R & 0 \\ k & S & 0 & -R \\ U & 0 & -T & -k \\ 0 & U & k & -T \end{pmatrix} \begin{pmatrix} \pm i f_z \\ f_z \\ \pm i b_z \\ b_z \end{pmatrix}. \tag{76}$$

This equation can be reduced to a two-dimensional matrix expression

$$0 = \begin{pmatrix} \pm ik + S & -R \\ U & \pm ik - T \end{pmatrix} \begin{pmatrix} f_z \\ b_z \end{pmatrix}, \tag{77}$$

and a nontrivial solution can be obtained if the determinant vanishes

$$0 = -k^2 \pm ik(S-T) - ST + RU, \tag{78}$$

which allows us to determine k that is the same k that above given in Eq. (70). The solution for the components of the vectors f_z and b_z are

$$b_z = \frac{\pm ik + S}{R} f_z = -\frac{U}{\pm ik - T} f_z. \tag{79}$$

And as a consequence, we have determined the **h**-field which can be written

$$\mathbf{h} = \begin{pmatrix} 1 \\ \dfrac{\pm ik + S}{R} \end{pmatrix} (\pm i\mathbf{e}_y + \mathbf{e}_z)\, \psi\, e^{i(kx - \omega t)}, \tag{80}$$

where we have named $\psi \equiv f_z$. With the determination of vector **h**, we can obtain **M** with Eq. (49), **G** with Eq. (48), and with this latter we can also obtain **d** with Eq. (37), and all electromagnetic information is then determined.

4.2. Symmetric Propagation Matrix, $S = T$

In this another case of $S = T$, the golden rule, Eq. (66), can be written by means the following expression:

$$0 = \begin{pmatrix} -k & U & -T & 0 \\ -R & k & 0 & S \\ -T & 0 & k & U \\ 0 & S & -R & -k \end{pmatrix} \begin{pmatrix} b_z \\ f_y \\ b_y \\ f_z \end{pmatrix}. \tag{81}$$

If case $S = T = 0$

$$0 = \begin{pmatrix} -k & U & 0 & 0 \\ -R & k & 0 & 0 \\ 0 & 0 & k & U \\ 0 & 0 & -R & -k \end{pmatrix} \begin{pmatrix} b_z \\ f_y \\ b_y \\ f_z \end{pmatrix}, \tag{82}$$

in this case, *the values of* (b_z, f_y) *are independent of* (b_y, f_z), *and therefore any plane wave polarization is possible.*

Now, we analyze the it solution when $S = T \neq 0$. We can assume, by rotating the coordinate system, that $f_z = 0$, then,

$$0 = \begin{pmatrix} -k & U & -T \\ -R & k & 0 \\ -T & 0 & k \\ 0 & S & -R \end{pmatrix} \begin{pmatrix} b_z \\ f_y \\ b_y \end{pmatrix} \tag{83}$$

and their solutions are

$$b_y = \frac{S}{R} f_y, \qquad b_z = \frac{k}{R} f_y, \qquad b_y = \frac{T}{k} b_z = \frac{T}{R} f_y. \tag{84}$$

The first is only compatible with the third when $S = T$, which is the starting assumption, and then the field $\mathbf{h}$ can be written by means of the following expression

$$\mathbf{h} = \begin{pmatrix} \mathbf{e}_y \\ \dfrac{S}{R}\,\mathbf{e}_y + \dfrac{k}{R}\,\mathbf{e}_z \end{pmatrix} \psi\, e^{i(kx-\omega t)}, \tag{85}$$

where $\psi \equiv f_y$.

In the case that we consider $b_y = 0$, Eq. (81) can be written as

$$0 = \begin{pmatrix} -k & U & 0 \\ -R & k & S \\ -T & 0 & U \\ 0 & S & -k \end{pmatrix} \begin{pmatrix} b_z \\ f_y \\ f_z \end{pmatrix} \tag{86}$$

and

$$f_z = \frac{T}{U}\, b_z, \quad f_y = \frac{k}{U}\, b_z, \quad f_z = \frac{S}{k}\, f_y = \frac{S}{U}\, b_z, \tag{87}$$

where the first is compatible with the third for $S = T$, and the field $\mathbf{h}$ is

$$\mathbf{h} = \begin{pmatrix} \frac{k}{U}\,\mathbf{e}_y + \frac{S}{U}\,\mathbf{e}_z \\ \mathbf{e}_z \end{pmatrix} \psi\, e^{i(kx-\omega t)}, \tag{88}$$

with $\psi \equiv b_z$.

In both cases, f_y or b_z can be linearly polarized if one conveniently chooses the time origin. That is, the amplitude is real and only in one direction, however, $\mathbf{f}$ *and* $\mathbf{b}$ *are not perpendicular. On the other hand, in the first case $b_y/b_z = S/k$ and in the second $f_z/f_y = S/k$, and then we conclude that the other field will not always be linear polarized.*

5. SOME SPECIAL CASES REGARDING DYONS

If there are no magnetic monopoles the propagation matrix, Eq. (61), becomes:

$$\begin{aligned} \begin{pmatrix} U & -T \\ -S & R \end{pmatrix} &= c\left(\omega \overleftrightarrow{\epsilon} + i \overleftrightarrow{\sigma}\right) \overleftrightarrow{\mu} \\ &= c\left[\omega\varepsilon_0 \begin{pmatrix} 1+\chi_e & 0 \\ 0 & 1 \end{pmatrix} + i\sigma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \mu_0 \begin{pmatrix} 1 & 0 \\ 0 & 1+\chi_m \end{pmatrix} \\ &= c\mu_0 \begin{pmatrix} \omega\varepsilon + i\sigma & 0 \\ 0 & \omega\varepsilon_0(1+\chi_m) \end{pmatrix}, \end{aligned} \tag{89}$$

and then from (70)

$$k^2 = RU = \mu(\omega\varepsilon + i\sigma)\omega = \varepsilon\mu\omega^2 + i\sigma\mu\omega, \tag{90}$$

which is as expected the standard expression in a conductor.

If in a medium with electric charges, we add magnetic monopoles and one does not consider kinetic susceptibility, then the propagation matrix is:

$$\begin{aligned} \begin{pmatrix} U & -T \\ -S & R \end{pmatrix} &= c\left[\omega\varepsilon_0 \begin{pmatrix} 1+\chi_e & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} \sigma & 0 \\ 0 & \sigma_m \end{pmatrix}\right] \mu_0 \\ &= c\mu_0 \begin{pmatrix} \omega\varepsilon + i\sigma & 0 \\ 0 & \omega\varepsilon_0 + i\sigma_m \end{pmatrix}, \end{aligned} \tag{91}$$

where σ_m is the monopole conductivity. Then

$$k^2 = RU = \varepsilon\mu_0\omega^2 - \frac{\mu_0}{\varepsilon_0}\,\sigma\sigma_m + i\mu_0\omega\left(\sigma + \frac{\varepsilon\sigma_m}{\varepsilon_0}\right), \tag{92}$$

that is compatible with Eq. (A10). This situation would be likened to that which should be applied to the case of the spin-ice materials [11, 15]. One can express k^2 as

$$k^2 = \kappa k_0^2 - \delta + i\Gamma\omega \equiv (k_r + ik_i)^2 \tag{93}$$

where $\kappa = \varepsilon/\varepsilon_0$, $k_0^2 = \varepsilon_0\mu_0\omega^2$, $\delta = \frac{\mu_0}{\varepsilon_0}\sigma\sigma_m$ and $\Gamma = \mu_0(\sigma + \kappa\sigma_m)$. Then k can be calculated

$$\begin{aligned} k_r^2 &= \frac{\kappa k_0^2 - \delta}{2}\left[1 \pm \sqrt{1 + \left(\frac{\Gamma\omega}{\kappa k_0^2 - \delta}\right)^2}\,\right], \\ k_i^2 &= \frac{\kappa k_0^2 - \delta}{2}\left[-1 \pm \sqrt{1 + \left(\frac{\Gamma\omega}{\kappa k_0^2 - \delta}\right)^2}\,\right], \end{aligned} \tag{94}$$

where the only sign considered is that which $k_r^2,\ k_i^2 > 0$. In a good conductor, $\varepsilon_0\omega/\sigma,\ \varepsilon_0\omega/\sigma_m \ll 1$, one can simplify and

$$k_r \simeq \frac{1}{2}\frac{\sigma + \sigma_m}{\sigma\sigma_m}\sqrt{\varepsilon_0\mu_0}\,\omega, \quad k_i \simeq \sqrt{\frac{\mu_0}{\varepsilon_0}\,\sigma\sigma_m}, \tag{95}$$

which differs considerably from the standard expressions for a good conductor without monopoles. For example, the penetration depth ($1/k_i$) does not depend on the frequency. These expressions may be useful in order to experimentally detect the monopoles in spin-ices.

6. NON-CONDUCTIVE MATERIAL

The propagation in materials in which all dyons are motionless (i.e., in non-conductive media) can be analyzed by considering $\sigma = 0$ in Eq. (61). Then,

$$\begin{pmatrix} U & -T \\ -S & R \end{pmatrix} = c\omega \overset{\leftrightarrow}{\epsilon}\overset{\leftrightarrow}{\mu} \equiv \omega \begin{pmatrix} U' & -T' \\ -S' & R' \end{pmatrix}, \tag{96}$$

and for a monochromatic plane wave, the general condition of Eq. (60) can be written

$$\mathbf{k} \times \begin{pmatrix} \mathbf{f} \\ \mathbf{b} \end{pmatrix} = -\omega \begin{pmatrix} S' & -R' \\ U' & -T' \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{b} \end{pmatrix}, \tag{97}$$

therefore, for a non-conductive medium, one must make the substitutions $R, S, T, U \rightarrow \omega(R', S', T', U')$, where R', S', T', U' are real, and if we also impose that the attenuation and propagation are in the same direction, then we can choose a coordinate system in such a way that

$$\mathbf{k} \equiv k\mathbf{e}_x. \tag{98}$$

We can carry out the same analysis that in the case of Eq. (69), and then, the corresponding result gives the value of the magnitude of the k-wave vector with respect to $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$ that is this case is:

$$k^2 \pm ik\omega(S' - T') - \omega^2(R'U' - S'T') = 0, \tag{99}$$

if $S' = T'$, then k can be real, and its value is

$$k = \pm\omega\sqrt{R'U' - S'^2}, \tag{100}$$

and if $S' \neq T'$, then k, solution of (99) is a complex number.

In order to determine the **h**-vector, in a similar way to the case of the conductor media, we distinguish two situations: $S' \neq T'$ (non-symmetric propagation matrix) and $S' = T'$ (symmetric propagation matrix). In the first case,

$$f_y = \pm i f_z, \qquad b_z = \frac{\pm ik + \omega S'}{\omega R'}\, f_z, \tag{101}$$

and as a consequent, a solution for the field **h** is

$$\mathbf{h} = \begin{pmatrix} 1 \\ \dfrac{\pm ik + \omega S'}{\omega R'} \end{pmatrix} (\pm i\mathbf{e}_y + \mathbf{e}_z)\psi\, e^{i(kx-\omega t)}, \tag{102}$$

where $\psi \equiv f_z$.

In the situation where $S' = T'$, one can find a solution so that $f_z = 0$, and then,

$$b_y = \frac{S'}{R'}\, f_y, \qquad b_z = \frac{k}{\omega R'}\, f_y \tag{103}$$

and the field **h** is

$$\mathbf{h} = \begin{pmatrix} \mathbf{e}_y \\ \dfrac{S'}{R'}\,\mathbf{e}_y + \dfrac{k}{\omega R'}\,\mathbf{e}_z \end{pmatrix} \psi\, e^{i(kx-\omega t)}, \tag{104}$$

where $\psi \equiv f_y$.

It is possible, choosing the time origin, that f_y was linear polarized. That is, the amplitude is real and only in one direction. But **f** *and* **b** *are not perpendicular, since $b_y \neq 0$, and $b_y/b_z = \omega S'/k$. As k is real then the* **b***-field is also linearly polarized.* Remember that this did not happen in conducting media, where only one of the fields could have linear polarization.

7. SUMMARY, CONCLUDING REMARKS AND PERSPECTIVES

The experimental appearance of composite entities whose behaviour is similar to that of the magnetic monopoles is the main cause for establishing a new electrodynamics within the solid, in which there are magnetic fields whose divergence is different from zero. This implies that the four "Maxwell" equations, whose formulation in the empty space (the vacuum) are already well known [19, 21–24], have been reconstructed in the case of existence of matter with electric and magnetic charges. In order to make the most possible generalized theory that leads to these equations, we have considered the dyon idea as a particle which can present electric and magnetic charge. With the acceptation of this idea, the charge representation can be set in a complex plane in such a way that the electric charge is located in the real axis (on the left of the origin the electronic charge and on the right of the origin the protonic charge) and the magnetic charge is represented in the imaginary axis, where over and under the origin the positive and negative magnetic charges are located.

However, we want to emphasize that the magnetic charge found at the present time is an effective concept which does not completely contain the properties of an elementary particle, but the phenomenology created by these objects within the solid state presents similar characteristic features as those attributed to the magnetic monopoles. It must be remembered that while the electric charge is quantized, the magnetic charge in the spin-ice structures is not, in contrast, of the monopoles of Dirac. Therefore, the dyonic complex plane where the real axis is a one-dimensional lattice and the magnetic charge can completely fulfill the imaginary axis. Obviously, a necessary condition for the validity of this extended theory is that it must be able to obtain all the results of the standard electrodynamics when the application of the generalized theory is carried out considering only dyons of the real axis.

In addition, when the physical activity is exclusively produced by magnetic charges such as it occurs in some recent experiments of magnetic conductance [11] (without electric charges), the phase of the charges in all equations should be 1/2 (i.e., $\zeta = \pi/2$) in this particular case.

The experiment of Bramwell et al. [11] measures the conductance of an electrolyte constituted of molecular compounds dissociated in a solvent. This experiment is explained in the light of the Onsager theory of the Wien effect [17]. This theory is a mixture of the thermodynamical kinetic theory and classical electrodynamics. These measurements prove that the movement of magnetic monopoles can be analyzed via the substitution in Onsager's theory of the Wien effect of the electric universal constants, q and ε_0 by the magnetic charge g and μ_0 and using the same formalized equations. In our theory, there exist symmetry between magnetic and electric field equations, therefore, also the phenomenologies are inter-changeable before changes of the magnetic and electric universal constants when there are either exclusively electric charges under the only electric field action or magnetic charges under the only magnetic field action. Therefore, there is a plausible and foreseeable coherence between our theory and the experiments of Bramwell. However, the experimental data, available at the present time, are almost reduced to these Bramwell experiments and those coming from the neutron diffraction [12, 15] whose technical and physical ingredients proceed from quantum mechanics and therefore are outside the classical electrodynamics.

The true novelty of the study of this paper is the physical response of a material when electric and magnetic fields simultaneously exert their actions over the electric and magnetic charges; this is the most general case and it is analyzed in Sections 4, 5 and 6. Therefore, the main objective of this paper is to make a systematization of the classical electromagnetic wave propagation. This issue can suggest different macroscopic and relatively simple experiments which can explain the optical conductivity, reflectivity, polarization conditions in the electromagnetic incidence on separation surfaces and limit angle experiments. These experiments can be explained and analyzed by means of the equations deduced in the last sections of this paper, since, we have found several relationships between the magnetic and electric fields and their polarizations with the properties of the materials determined by the response functions.

The first step in our analysis is the study of the linear responses before the electromagnetic **G**-field in the the dyonic systems. In this case, there are three different responses: the movement of the dyonic charges versus the action of the field, which constitutes conductivity; the changes in the magnitude and orientation of the splitting of the gravity center of the positive and negative dyonic charge versus the **G**-field, which is the so-called split-charge susceptibility, and the changes of the magnitude and orientation of the "magnetic" dipoles moments corresponding to the dyonic charges also before the presence of the **G**-field that we name the kinetic susceptibility. These three response functions, given in Eqs. (33), (40) and (53), respectively, allow us to completely formulate the space-time evolution laws of the **d** and **h**-fields [Eqs. (22) and (26)] within the dyonic matter, as a function of the conductivity, permittivity and permeability, Eq. (59).

This analysis is the starting point for analyzing the propagation of a monochromatic wave in a dyonic system with several classes of dyons. The divergence of the **d**-field allows us to obtain the transversality (in the point where the ϱ-density of particles is null) of the **f** and **b**-fields which are the electric and magnetic flavours, respectively, of the "magnetic intensity" **h**-field. The curl of plane wave **h**-field allows us to obtain Eq. (60), which is fundamental in order to make the development of behaviours of the EM fields within matter. From this point in the paper, the analysis is centered in obtaining the corresponding **k**-vector and the polarization of the **f** and **b** vectors (the six components of the **h**-vector), in all possible different situations. This is analyzed in both conductors and general systems with $\overleftrightarrow{\sigma} \neq 0$, and $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$ different from unit. The concepts of the extended conductivity, split-charge susceptibility and kinetic susceptibility are basic in order to determine the optical properties which are founded in the characteristic features of the propagation of **h**-vector with different values of the electric and magnetic charges. The combination of these three response tensors in Eq. (59) allows to define the so-called propagation matrix [Eq. (61)] so that the different polarizations of **h**-vector depend on its matrix element values and the non diagonal matrix elements S and T are decisive for obtaining the different solutions of each physical situation. On the other hand, we have considered the **k**-vector (that in the conductive systems is complex) in such a way that its attenuation part and its propagation part have the same direction. This is a more simplified case, but it is considered for the sake of clarity, since the consideration of different direction for each part of the **k**-vector implies mathematical difficulties that in this first analysis could diminish its understanding. The different direction for each part of this complex wave vector yields interesting consequences in the propagation in semiinfinite media with separation surfaces with different response functions.

The relative value between the non-diagonal matrix elements of the propagation matrix set the different polarizations and wave vector magnitude that are possible in the material. Concerning this fact, we want to highlight that the electromagnetic propagation is impossible when $S \neq T$ and either **f** or **b** are are linearly polarized. The other characteristic features of the **h**-vector are given in the different cases both conductive and non-conductive materials in Sections 5 and 6.

There are many perspectives and futures for this extended classical electrodynamics. The first of them is the consideration of the electromagnetic propagation with separation surfaces which requires the consideration of a more elaborate **k**-vector that allows us to determine the Fresnel relationships in the surface of separation between materials. Another important issue of this extended classical electrodynamics is the determination of the optical conductivity with several class of dyons, i.e., the formulation of an extension of the Lorentz-Drude model. The continuation of this theory outside the Classical Physics can be the formulation of some parts of Solid State Physics within Quantum Mechanics in which the dyons are the active particles which determine the main properties in the material. This requires the formulation of a quantum theory of the solid with dyons. Therefore, the systematized formulation of the extended classical electrodynamics can induce experiments in order to apply the correspondence principle for obtaining likeliness and consistency for the quantum models formulated with the presence of these magnetic charges. On the other hand, the experimental research can be the construction of new materials different from the spin-ices where the generation of monopole-like entities produced by the spin flip actions can coexist with the presence of quasi-free electric charges traveling on the solid.

APPENDIX A. SOME PROPERTIES OF THE PROPAGATION MATRIX

In this Appendix, we develop Eq. (61) in some simple cases in order to give a idea about the dependence of the elements matrix U, T, R and S, as well as the magnitude of the **k**-wave vector in function of the response tensor of materials.

From the definition of Θ

$$\begin{aligned}\Theta\Theta' &= \begin{pmatrix} \cos^2\zeta & \cos\zeta\sin\zeta \\ \cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix}\begin{pmatrix} \cos^2\zeta' & \cos\zeta'\sin\zeta' \\ \cos\zeta'\sin\zeta' & \sin^2\zeta' \end{pmatrix} \\ &= \cos(\zeta-\zeta')\begin{pmatrix} \cos\zeta\cos\zeta' & \cos\zeta\sin\zeta' \\ \sin\zeta\cos\zeta' & \sin\zeta\sin\zeta' \end{pmatrix}, \end{aligned} \tag{A1}$$

then $\Theta\Theta' = 0$ if $\zeta - \zeta' = \pm\frac{\pi}{2}$, and $\Theta^2 = \Theta$.

For a system in which all dyons are of the same class (all have the same value of ζ, regardless $\pm\pi$), then, from Eqs. (33), (40) and (47),

$$\begin{pmatrix} U & -T \\ -S & R \end{pmatrix} = c\left(\omega \overleftrightarrow{\epsilon} + i \overleftrightarrow{\sigma}\right)\overleftrightarrow{\mu}\ . \tag{A2}$$

$$= c\Big[\omega\varepsilon_0(\mathbb{1}+\chi_s\Theta) + i\sigma\Theta\Big]\frac{\mu_0}{1-\xi}(\mathbb{1}-\xi\Theta) \tag{A3}$$

$$= \frac{c\mu_0}{1-\xi}\Big[\omega\varepsilon_0\mathbb{1} + i\sigma(1-\xi)\Theta + \omega\varepsilon_0(\chi_s-\xi-\xi\chi_s)\Theta\Big] \tag{A4}$$

and here, we have used that $\Theta^2 = \Theta$. It is a symmetric matrix and as a consequence, Eq. (70),

$$\begin{aligned} k^2 &= RU - TS \\ &= \frac{\varepsilon_0\mu_0}{(1-\xi)^2}\,\omega^2\left[(1+\chi_s)(1-\xi) + i\frac{\sigma(1-\xi)}{\omega\varepsilon_0}\right] \\ &= \varepsilon\mu\omega^2\left[1 + i\frac{\sigma}{\omega\varepsilon}\right]. \end{aligned} \tag{A5}$$

This value for k is the standard value for a conductor medium without monopoles. This means that the value of k^2 is invariant by a duality transformation.

A.1. Two Classes of Dyons

If the analyzed material has two dyon classes, i.e., there exist charges with different phases, ζ and ζ', then α, β and γ of Eq. (52) are:

$$\begin{aligned} \alpha &\equiv \xi\sin^2\zeta + \xi'\sin^2\zeta', \\ \beta &\equiv \xi\cos\zeta\sin\zeta + \xi'\cos\zeta'\sin\zeta', \\ \gamma &\equiv \xi\cos^2\zeta + \xi'\cos^2\zeta'\ ; \end{aligned} \tag{A6}$$

and consequently, the permeability matrix is:

$$\overleftrightarrow{\mu} = \frac{\mu_0}{1-\xi-\xi'+\xi\xi'\sin^2(\zeta-\zeta')}\left(\mathbb{1} - \xi\Theta - \xi'\Theta'\right). \tag{A7}$$

and the propagation matrix is

$$\begin{aligned} \begin{pmatrix} U & -T \\ -S & R \end{pmatrix} &= c\left(\omega\overleftrightarrow{\epsilon} + i\overleftrightarrow{\sigma}\right)\overleftrightarrow{\mu} \\ &= c\Big[\omega\varepsilon_0(\mathbb{1}+\chi\Theta+\chi'\Theta') + i\sigma\Theta + i\sigma'\Theta'\Big] \\ &\quad\times\frac{\mu_0}{1-\xi-\xi'+\xi\xi'\sin^2(\zeta-\zeta')}\left(\mathbb{1}-\xi\Theta-\xi'\Theta'\right). \end{aligned} \tag{A8}$$

It should be remarked that this matrix is, in general a non-symmetric operator, $(S \neq T)$, because $\Theta\Theta'$ is non-symmetric.

In the case that $\zeta - \zeta' = \pm\pi/2$ (from a duality transformation, there are only electric charges and magnetic monopoles, but there are not dyons with both electric and magnetic charges), then, $\Theta\Theta' = 0$ and $\sin^2(\zeta - \zeta') = 1$, and the propagation matrix is then:

$$\begin{aligned} \begin{pmatrix} U & -T \\ -S & R \end{pmatrix} &= \frac{c\mu_0\omega\varepsilon_0}{(1-\xi)(1-\xi')}\Bigg[\mathbb{1} + (\chi-\xi-\chi\xi)\Theta + i\frac{\sigma(1-\xi)}{\omega\varepsilon_0}\Theta \\ &\quad + (\chi'-\xi'-\chi'\xi')\Theta' + i\frac{\sigma'(1-\xi')}{\omega\varepsilon_0}\Theta'\Bigg], \end{aligned} \tag{A9}$$

that is a symmetric matrix ($S = T$), and consequently

$$\begin{aligned} k^2 &= RU - TS \\ &= \frac{\varepsilon_0(1+\chi)(1+\chi')\mu_0}{(1-\xi)(1-\xi')}\,\omega^2\left[1+i\frac{\sigma}{\omega\varepsilon_0(1+\chi)}\right]\left[1+i\frac{\sigma'}{\omega\varepsilon_0(1+\chi')}\right] \\ &= \varepsilon\mu\,\omega^2\left[1-\frac{\sigma\sigma'}{\omega^2\varepsilon_0\varepsilon}+i\frac{1}{\omega\varepsilon_0}\left(\frac{\sigma}{1+\chi}+\frac{\sigma'}{1+\chi'}\right)\right]. \end{aligned} \tag{A10}$$

This result becomes the standard value of k for a conductor medium, Eq. (A5), when $\sigma' = \xi' = 0$, i.e., when there is only electric charges and the corresponding electric conductivity σ and there are not magnetic monopoles and therefore, their conductivity is null.

REFERENCES

1. Umul, Y. Z., "Improved equivalent source theory," *J. Opt. Soc. Am. A*, Vol. 2, 1798–1804, 2009.
2. Harrington, R. F., *Time-harmonic Electromagnetic Fields*, IEEE-Wiley, 2001.
3. Balanis, C. A., *Advanced Engineering Electromagnetics*, Wiley, 1989.
4. Qi, X.-L., R. Li, J. Zang, and S.-C. Zhang, "Inducing a magnetic monopole with topological surface states," *Science*, Vol. 323, 1184–1187, 2009.
5. Curie, M. P., "Sur la possibilité d'existence de la conductibilité magnétique et du magnétisme libre," *Séances de la Société Française de Physique*, 76–77, 1894.
6. Dirac, P. A. M., "Quantized singularities in the electromagnetic field," *Proc. Roy. Soc.*, Vol. A133, 60, 1931.
7. Dirac, P. A. M., "The theory of magnetic poles," *Phys. Rev.*, Vol. 74, 817–830, 1948.
8. Castelnovo, C., R. Moessner, and S. L. Sondhi, "Magnetic monopoles in spin ice," *Nature*, Vol. 451, 42–45, 2008.
9. Cabrera, B., "First results from a superconductive detector for moving magnetic monopoles," *Phys. Rev. Lett.*, Vol. 48, 1378–1381, 1982.
10. Sondhi, S., "Wien route to monopoles," *Nature*, Vol. 461, 888–889, 2009.
11. Bramwell, S. T., S. R. Giblin, S. Calder, R. Aldus, D. Prabhakaran, and T. Fennell, "Measurement of the charge and current of magnetic monopoles in spin ice," *Nature*, Vol. 461, 956–960, 2009.

12. Fennell, T., P. P. Deen, A. R. Wildes, K. Schmalzl, D. Prabhakaran, A. T. Boothryod, R. J. Aldus, D. F. McMorrow, and S. T. Bramwell, "Magnetic Coulomb phase in the spin ice $Ho_2Ti_2O_7$," *Science*, Vol. 326, 415–417, 2009.
13. Gingrass, M. J., "Observing monopoles in a magnetic analog of ice," *Science*, Vol. 326, 375–376, 2009.
14. Jaubert, L. D. C. and P. C. W. Holsworth, "Signature of magnetic monopole and Dirac string dynamics in spin ice," *Nature Physics*, Vol. 5, 258–261, 2009.
15. Morris, D. J. P., D. A. Tennant, S. A. Grigera, B. Klemke, C. Castelnovo, R. Moessner, C. Czternasty, M. Meissner, K. C. Rule, J.-U. Hoffmann, K. Kiefer, S. Gerischer, D. Slobinsky, and R. S. Perry, "Dirac strings and magnetic monopoles in the spin ice $Dy_2Ti_2O_7$," *Science*, Vol. 326, 411–414, 2009.
16. Ladak, S., D. E. Read, G. K. Perkins, L. F. Cohen, and W. R. Branford, "Direct observation of magnetic monopole defects in an artificial spin-ice system," *Nature Physics*, Vol. 6, 359–363, 2010.
17. Onsager, L., "Deviations from Ohm's law in weak electrolytes," *J. Chem. Phys.*, Vol. 2, 599–615, 1934.
18. Bandyopadhyay, S. and M. Cahay, *Introduction to Spintronics*, CRC Press, 2008.
19. Jackson, J. D., *Classical Electrodynamics*, 3rd edition, John Wiley & Sons, Inc., New York, 1999.
20. Hooft, G., "Magnetic monopoles in unified gauge theories," *Nuclear Physics*, Vol. B79, 276–284, 1974.
21. Costa-Quintana, J. and F. Lopez-Aguilar, "Extended classical electrodynamics with magnetic monopoles," *Far East Journal of Mechanical Engineering and Physics*, Vol. 1, 19–56, 2010.
22. Nowakowski, M. and N. G. Kelkar, "Faraday's law in the presence of magnetic monopoles," *Europhys. Lett.*, Vol. 71, 346–351, 2005.
23. Heras, J. A., "Jefimenko's formulas with magnetic monopoles and the Lienard-Wiechert fields of a dual-charged particle," *Am. J. Phys.*, Vol. 62, 525–531, 1994.
24. Milton, K. A., "Theoretical and experimental status of magnetic monopoles," *Rep. Prog. Phys.*, Vol. 69, 1637–1711, 2006.
25. Umul, Y. Z., "Electric charges that behave as magnetic monopoles," *Progress In Electromagnetic Research Letters*, Vol. 18, 19–28, 2010.
26. Shnir, Y. M., *Magnetic Monopoles*, Springer-Verlag, Berlin, 2005.