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# FACTORIZATION METHOD FOR FINITE FINE STRUCTURES

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**Abstract**—This paper deals with the development of the Wiener-Hopf method for solving the diffraction of waves at fine strip-slotted structures. The classical problem for diffraction of plane wave at a strip is an important canonical problem. The boundary value problem is consecutively solved by a reduction to a system of singular boundary integral equations, and then to a system of Fredholm integral equations of the second kind, which effectively is solved by one of three presented methods: A reduction to a system of the linear algebraic equations with the help of the etalon integral and the saddle point method numerical discretization based on Gauss quadrature formulas the method of successive approximations. The solution to the problem in the first method contains a denominator that takes into account the resonance process. Moreover, the precision of the main contribution of the short-wave asymptotic solution is ensured down to the quasi-stationary limit. The paper presents also comparisons of with earlier known results.

## 1. INTRODUCTION

To present time, there are in essence only two rigorous analytical methods for solving diffraction problems: the Wiener-Hopf (WH) [1] and the Riemann-Hilbert method [2]. The WH method is also known as the factorization method. Both methods give solutions in closed form for plane and cylindrical semi-infinite wave guides. The rigorous formulation of a boundary value problem for semi-infinite structure is equivalent to a corresponding Riemann-Hilbert boundary value problem. These methods give solutions to canonical problems that can be used for checking approximate methods intended for more complex geometries. It is therefore of high interest to find a way of using the

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WH method for finite structures. The key problem for plane finite structures is diffraction of electromagnetic waves from a strip or slot. Generally it concerns a classical problem that has a rigorous solution in the form of a series of elliptical cylinder functions found by the method of a separation of variables. However, this series is inconvenient in the short-wavelength range, because of bad convergence properties [3].

Many works are devoted to an asymptotic solution of diffraction problems on a strip (slot). Some first terms of asymptotic expansion for a diffraction field on parameter $1/kl$ are obtained on the basis of integral equations in works of Millar [4], Westpfahl [5], Lüneburg [6], Kieburts [7], Stockel [8], $k$ — wave number, $2l$ — a width of a strip (slot). Using asymptotic formulas for an elliptical cylinder function, Hansen [9] has selected the expressions from a strict solution of a boundary problem in an elliptic frame corresponding to primary boundary waves of Sommerfeld. The approximated solution of an integral equation of the second kind for a current on the plane screen with a slot is given by Grinberg [10, 11]. The asymptotic solution to within terms of the order $(kl)^{5/2}$ of the key equation for a current on a strip is received by him [12]. The further research of this equation is carried out by Kuritsin [13], Popov [14]. The substantiation and an improvement of the approximated expressions for the scattered field is given by means of a method of integral equations in works Khaskind, Weinstein [1, 15], Fialkovski [16], Nefyodov [17], Borovikov [18] and Popichenko [19]. The obtained formulas are valid at arbitrary angle of incidences and observations. The fullest asymptotic research of strip problem is given in works of [20, 21]. Other methods like Kobayashi potential method [22], Maliuzhinet's techniques [23] may be used to solve the problem.

The factorization of functions by the WH method corresponds essentially to a splitting of the Fourier transform into the sum of two one-sided Fourier integrals along the negative and positive semi-axes of the coordinate. To put it briefly, the coordinate semi-axis is mapped on a complex half-plane, where the factorized functions are determined by the WH method. To split the Fourier integral in more than two integrals, present in diffraction problems for finite structures, introduces difficulties in combination with the WH method. In [24], the diffraction problem for a strip is considered by the WH method and reduced to system of Fredholm integral equations of the second kind. Unlike the results of D. S. Jones in this paper, the exact solution in the current paper is reduced to the form of series and an asymptotic solution containing a resonant denominator. An important property of the WH method is that, it keeps simplicity and physical transparency in the results.

## 2. STATEMENT OF PROBLEM. REDUCING OF PROBLEM TO SYSTEM OF INTEGRAL EQUATION

Let the plane wave impinges on ideally conducting strip $|z| \leq a$, $y = 0$, $-\infty < x < \infty$:

$$\begin{aligned}
&E_x^o = -E_0 \exp\big(ik(y\sin\vartheta_0 + z\cos\vartheta_0)\big),\\
&H_y^o = E_x^o\sqrt{\varepsilon/\mu}\cos\vartheta_0, H_z^o = -E_x^o\sqrt{\varepsilon/\mu}\sin\vartheta_0,\\
&H_x^o = 0,\ E_y^o = E_z^o = 0,\ k = \omega/c,\ E_0 = const.
\end{aligned} \tag{1}$$

The direction of propagation of the incident wave is orthogonal to the $x$ axis and makes an angle $\vartheta_0$ with the $z$ axis (Fig. 1). Such a polarization of the incident plane wave is denoted magnetic. Further, the harmonic time factor $\exp(-i\omega t)$ is everywhere omitted.

With no $x$ dependence present the electromagnetic field

$$E_x = ikcA_x, \quad H_y = \frac{1}{\mu}\frac{\partial}{\partial z}A_x, \quad H_z = -\frac{1}{\mu}\frac{\partial}{\partial y}A_x, \tag{2}$$

is expressed by means of convolution of the single component $A_x$ of the vector potential with the fundamental solution $\psi$ of the wave equation:

$$A_x(y,z) = -\mu\psi * J_x(z), \quad \psi = -\frac{i}{4}\mathrm{H}_0^{(1)}(k\sqrt{y^2+z^2}). \tag{3}$$

Here, $*$ is the symbol of convolution in $z$, $\mathrm{H}_0^{(1)}(x)$ is the Hankel function of the first kind, and $J_x(z)$ is the surface current density of the strip.

We can present the convolution through the Fourier transforms of the surface density and the fundamental solution

$$A_x(y,z) = \frac{i\mu}{4\pi}\int_{-\infty}^{\infty}\frac{1}{v}\exp\{i(wz + v|y|)\}F(w)dw, \tag{4}$$

where $v = \sqrt{k^2 - w^2}$. Indeed, the inverse Fourier transform follows from the boundary condition for the jump of the magnetic field on the strip

$$J_x = H_z(+0,z) - H_z(-0,z),$$

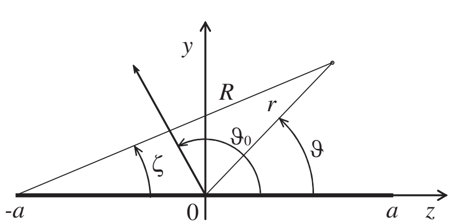


**Figure 1.** Diffraction of plane wave at a strip.

as well as (2) and (4):

$$J_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iwz)F(w)dw. \tag{5}$$

From the boundary condition for the electric field on the strip

$$E_x + E_x^o = 0 \quad \text{at} \quad |z| \le a \quad (y = 0, -\infty < x < \infty),$$

and (2) and (4) we obtain an integral equation

$$\int_{-\infty}^{\infty} \exp(iwz)\frac{1}{v}F(w)dw + A_0 \exp(ihz) = 0 \quad \text{at} \quad |z| \le a, \tag{6}$$

where $A_0 = 4\pi E_0/(\omega\mu)$ and $h = k\cos\vartheta_0$. For concreteness the value $h$ is fixed for example in the lower $w$ half-plane (LHP).

We have the following integral equation from the continuity condition of the magnetic field ($H_z$) on the continuation of the strip:

$$\int_{-\infty}^{\infty} \exp(iwz)F(w)dw = 0 \quad \text{at} \quad a < |z|, \tag{7}$$

which follows immediately from (5) expressing absence of currents on the prolongation of the strip.

Thus, for the solution of a boundary value problem it is necessary to find $F(w)$ that satisfies the system of integral Equations (6) and (7). It is necessary to note, that the solution of a boundary value problem should satisfy an additional edge condition (Meixner condition), i.e., the magnetic field, for example, should increase asymptotically as $H \sim \rho^{-1/2}$ near the knife edge, when the distance $\rho \to 0$.

## 3. THE SOLUTION OF THE BOUNDARY VALUE PROBLEM WITH THE WIENER-HOPF METHOD

Let $k$ have a small positive imaginary part that will vanish in the final formulas. Taking into account that the edges of a strip are secondary sources of waves, the Fourier-component of the current density is written as a sum from two analytical sources:

$$F(w) = F_1 + F_2 \tag{8}$$

where

$$\begin{aligned} F_2(w) &= \sqrt{k-w}\left(A_2(w) + B^+(w)\right)\exp(iwa), \\ F_1(w) &= \sqrt{k+w}\left(A_1(w) + B^-(w)\right)\exp(-iwa). \end{aligned}$$

The fields from the analytical sources must satisfy Meixner's condition, i.e., behaving at infinity as $w^{-1/2}$.

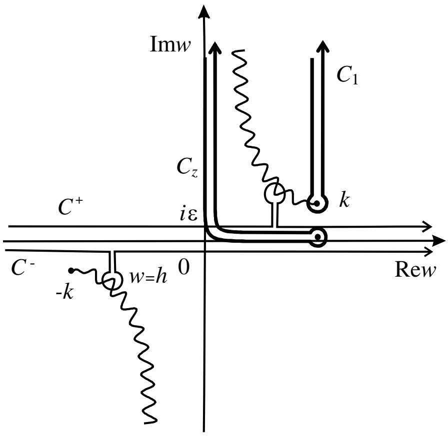


**Figure 2.** A plane of a complex variable $w$ (or $u$).

The functions $F_1$, $F_2$ absorb completely all incident waves and reradiate them in all directions according to the boundary conditions. The terms $F_1$ and $F_2$ are constructed by the WH method such that $A_1$ and $A_2$ correspond to plane wave amplitudes; $B^+$ and $B^-$ correspond to the amplitudes of the reflected waves from the strip edges. Furthermore, the following cancelations appear: $A_1$ cancels the incident plane wave at the left of the first edge ($z \leq a$), $A_2$ cancels the plane wave of $A_1$ at the left of the second edge ($z \leq -a$), $B^+$ cancels the incident waves from the first edge ($B^-$) in the field $z < -a$, $B^-$ cancels the waves from the second edge ($B^+$) in the field $z > a$.

From this follows, that $A_1$ and $A_2$ should be analytical functions on the entire complex $w$ plane except for a simple pole at $w = h$. As the singular points in the upper half plane (UHP) correspond to traveling waves to the right along the $z$ axis, $B^-$ should be analytical in the LHP, and $B^+$ in the UHP. Thus,

$$B^+(w) = \frac{1}{2\pi i}\int_{C^-}\frac{\psi_1(u)}{u-w}du, \quad B^-(w) = -\frac{1}{2\pi i}\int_{C^+}\frac{\psi_2(u)}{u-w}du,$$

where $\psi_1$ and $\psi_2$ are some analytical functions in the band $|\mathrm{Im}u| < \mathrm{Im}k$, $0 < \varepsilon < \mathrm{Im}k$, $C^-$ and $C^+$ are integration contours (IC) laying parallel at a distance $\mp i\varepsilon$ from the real axis and consisting of an infinitely narrow loop enveloping a point $w = \pm h$ from below or from above (Fig. 2).

Let us now consider the system of integral Equations (6) and (7). Substituting the expression (8) for (6), the integral is calculated using Jordan's lemma. Here it is necessary to close the IC for the integrand $F_1$ by a semicircle of infinite radius in the LHP, and for $F_2$ in the UHP.

The integrands are analytical inside the closed IC except for a single simple pole at the point $w = h$. Calculating the integral in (6) by means of a residue, we obtain

$$A_1(w) = \frac{A_o}{2\pi i} \frac{\sqrt{k-h}}{w-h} \exp(iha). \tag{9}$$

Similarly it is convenient to close the IC in (7) in the $w$ LHP when $z < -a$. As the integrands are analytical in the band $|\mathrm{Im} w| < \mathrm{Im} k$, the integration along the real axis in (7) is equivalent to the contour $C^-$ and the additional infinitesimal circle round the point $w = h$. Therefore, allowing for the formula (9) we compensate for the pole at this point and find

$$A_2(w) = -\frac{A_o}{2\pi i} \frac{\sqrt{k+h}}{w-h} \exp(-iha). \tag{10}$$

(Note that (9) and (10) are valid when the plane wave is incident from the left.) Note also that the integrand $F_2$ in (7) has a simple pole at the point $w = h$. Therefore, the integration path for $B^+$ passes in the $u$ LHP. Hence, by changing the order of integration and calculating the integral along the real $w$ axis with help of the theory of residues, (7) is reduced to:

$$\begin{aligned} &\int_{C^-} \exp(iuz)\Big(\sqrt{k+u}\big(A_1(u) + B^-(u)\big)\exp(-iua) \\ &+\sqrt{k-u}\psi_1(u)\exp(iua)\Big)du = 0, \quad z < -a. \end{aligned}$$

Here, the first term in the integrand has a branch point $u = -k$ in the LHP. To eliminate the singularity the integrand is equated to zero, also as well as poles. Using this solution for $\psi_1$, the functional relation

$$B^+(w) = -\frac{1}{2\pi i} \int_{C^-} \frac{\exp(-i2au)}{u-w} \sqrt{\frac{k+u}{k-u}} \big(A_1(u) + B^-(u)\big)\, du \tag{11}$$

is achieved. Analogously, the functional relation

$$B^-(w) = \frac{1}{2\pi i} \int_{C^+} \frac{\exp(i2au)}{u-w} \sqrt{\frac{k-u}{k+u}} \big(A_2(u) + B^+(u)\big)\, du, \tag{12}$$

is obtained. By means of the replacement $u \to -u$, (11) is represented as

$$B^+(w) = \frac{1}{2\pi i} \int_{C^+} \frac{\exp(i2au)}{u+w} \sqrt{\frac{k-u}{k+u}} \big(A_1(-u) + B^-(-u)\big)\, du. \tag{13}$$

Note that the Equations (12) and (13) is a system of integral equations. By introducing the integral operator

$$\mathbf{I}(w,u) = \frac{1}{2\pi i} \int_{C^+} du \frac{\exp(i2au)}{u-w} \sqrt{\frac{k-u}{k+u}},$$

the system of Equations (12) and (13) may be represented compactly as

$$\begin{aligned} B^{+}(w) &= \mathbf{I}(-w,u)\left(A_1(-u)+B^{-}(-u)\right), \\ B^{-}(w) &= \mathbf{I}(w,u)(A_2(u)+B^{+}(u)). \end{aligned} \tag{14}$$

The simple integral $I(w)=\mathbf{I}\cdot 1$ will be used and is represented in terms of special functions (see appendix A).

Thus, the boundary value problem is reduced to solving the system (14). It may be readily checked by calculating the integrals (6) and (7), using the theory of residues and Jordan's lemma, that the constructed solutions for $F_1(w)$ and $F_2(w)$ imply that $A_x$ in (4) meet the boundary conditions (6) and (7). Below we will present three methods for the solution of (14).

### 3.1. 1st Method

It is easy to obtain the solution of (14) in the form of successive approximations:

$$\begin{aligned} B^{+}(w) &= \sum_{n=1}^{\infty}\mathbf{I}^{2n-1}(-w,u)A_1(-u)+\sum_{n=1}^{\infty}\mathbf{I}^{2n}(-w,u)A_2(u), \\ B^{-}(w) &= \sum_{n=1}^{\infty}\mathbf{I}^{2n-1}(w,u)A_2(u)+\sum_{n=1}^{\infty}\mathbf{I}^{2n}(w,u)A_1(-u). \end{aligned} \tag{15}$$

Here, the product $\mathbf{I}^k(w,w_0)$ of integral operators is denoted by

$$\mathbf{I}^k(w,w_0) \;=\; \mathbf{I}(w,w_{k-1})\prod_{m=k-1}^{1}\mathbf{I}(-w_m,w_{m-1})$$

$$\left(\prod_{m=k-1}^{1}\mathbf{I}(-w_m,w_{m-1}) \;\equiv\; \mathbf{I}(-w_{k-1},w_{k-2})\ldots\mathbf{I}(-w_1,w_0),\right.$$

$$\left.\mathbf{I}^1(w,w_0) \;\equiv\; \mathbf{I}(w,w_0),\; k\geq 1\right).$$

The exact solution of the boundary value problem is obtained in the form of multiple reflections of waves from the edges of the strip by substituting (15) in (8) also allowing for (9) and (10) as well as (4) and (2):

$$E_x=E_x^1+E_x^2,$$

where

$$\begin{aligned} E_x^1 &= E_0\big(\mathbf{J}_1(h,w)+\mathbf{J}(-h,w_1)\mathbf{J}(w_1,w)+\ldots\big)e^{i(-wz+v|y|)}, \\ E_x^2 &= E_0\big(\mathbf{J}_1(-h,w)+\mathbf{J}(h,w_1)\mathbf{J}(w_1,w)+\ldots\big)e^{i(wz+v|y|} \end{aligned}$$

or

$$
\begin{aligned}
E_x^1 &= E_0 \sum_{n=1}^{\infty} E_n^1, \quad E_x^2 = E_0 \sum_{n=1}^{\infty} E_n^2, \qquad (16)\\
E_n^1 &= \mathbf{J}\big((-1)^{n-1}h, w_1\big) \prod_{k=1}^{n-2} \mathbf{J}(w_k, w_{k+1}) \mathbf{J}(w_{n-1}, w) e^{i(-wz+v|y|)},\\
E_n^2 &= \mathbf{J}\big((-1)^{n}h, w_1\big) \prod_{k=1}^{n-2} \mathbf{J}(w_k, w_{k+1}) \mathbf{J}(w_{n-1}, w) e^{i(wz+v|y|)}\\
&\qquad (n = 2, 3, \ldots).
\end{aligned}
$$

In (16), the strip diffraction operators

$$\mathbf{J}(w, u) = \frac{\sqrt{k-w}}{2\pi i} \exp(iaw) \int_{C_1} du \frac{\exp(iau)}{\sqrt{k+u}(u+w)},$$

and

$$\mathbf{J}_1(w, u) = \frac{\sqrt{k-w}}{2\pi i} \exp(iaw) \int_{-\infty}^{\infty} du \frac{\exp(iau)}{\sqrt{k+u}(u+w)}.$$

Note that the integration contour $C^+$, present in the integral representation of the operator $\mathbf{I}$, as well as the real line can be deformed to the contour $C_1$ without any intersection of poles, except the first term of the series (Fig. 2). By the coordinate transformation (Fig. 1):

$$z = r\cos\vartheta,\ y = r\sin\vartheta \quad (0 < y,\ 0 < \vartheta < \pi)$$

and the substitution

$$w = k\sin\tau,\ v = k\cos\tau,\ (\mathrm{Im}k = 0)$$

of the integration variable, the diffraction operator $\mathbf{J}(k\sin\tau, k\sin\alpha) \equiv \mathbf{J}(\tau, \alpha)$

$$
\begin{aligned}
\mathbf{J}(\tau, \alpha) &= \frac{1}{4\pi i} \int_S d\alpha g(\tau, \alpha), \quad \mathbf{J}_1(\tau, \alpha) = \frac{1}{4\pi i} \int_{S_z} d\alpha g(\tau, \alpha), \qquad (17)\\
g(\varphi, \tau) &= \left(\sin^{-1}\frac{\varphi+\tau}{2} - \cos^{-1}\frac{\varphi-\tau}{2}\right) \exp\big(ika(\sin\varphi + \sin\tau)\big),
\end{aligned}
$$

is represented in polar form, where $S$ and $S_z$ are the Sommerfeld integration contours (Fig. 3).

The term of the series expansion for $E_x^2$ (for the second edge) corresponds to the $n$-th diffraction of the plane wave, i.e., the first term corresponds to diffraction of the plane wave falling on the half-plane ($z = -a$), the second corresponds to the diffraction of the primary cylindrical wave on the second edge, etc.

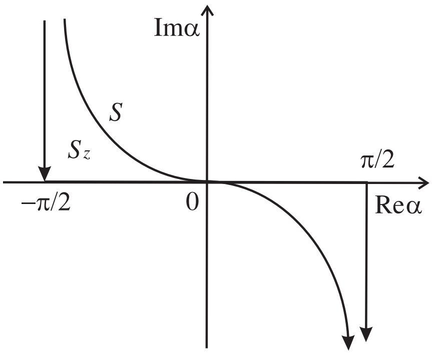


**Figure 3.** Sommerfeld integration contours.

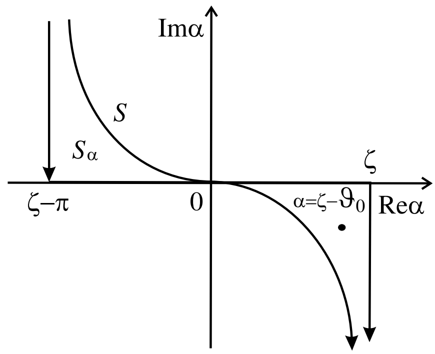


**Figure 4.** Integration contours $S_a$ and $S$.

Let us now investigate the primary diffraction (for example at the second edge):

$$E_1^2 = \mathbf{J}_1(\vartheta_0 - \pi/2, \tau)e^{ikr\sin(\tau+\vartheta)} \quad (0 < \vartheta < \pi).$$

To calculate the integral

$$E_1^2 = \frac{1}{4\pi i}\int_{S_z} e^{ik(r\sin(\tau+\vartheta)+a\sin\tau+a\sin\tau_0)}\left(\frac{1}{\sin\frac{\tau_0+\tau}{2}} - \frac{1}{\cos\frac{\tau_0-\tau}{2}}\right)d\tau,$$

where $\tau_0 = \vartheta_0 - \pi/2$, it is necessary to transfer the origin of coordinates from the center of the strip to the second edge:

$$R\sin\zeta = r\sin\vartheta, \quad R\cos\zeta = a + r\cos\vartheta. \tag{18}$$

Introducing a new integration variable $\alpha = \zeta + \tau - \pi/2$, with integration along the contour $S_\alpha$ (Fig. 3), also excluding the variables $r$, $\vartheta$, the integral is easily transformed to the form:

$$E_1^2 = \frac{\exp(-ika\cos\vartheta_0)}{4\pi i}\left(\int_{S_\alpha}\frac{\exp(ikR\cos\alpha)}{\sin\left((\alpha-\zeta+\vartheta_0)/2\right)}d\alpha + \int_{S_\alpha}\frac{\exp(ikR\cos\alpha)}{\sin\left((\alpha-\zeta-\vartheta_0)/2\right)}d\alpha\right).$$

It is convenient to introduce the special function

$$\Xi(R,\omega) = \frac{1}{4\pi i}\int_S \frac{\exp\left(ikR(\cos\alpha - \cos\omega)\right)}{\sin\left((\alpha+\omega)/2\right)}d\alpha, \tag{19}$$

which can be represented in the form [1]:

$$\Xi(R,\omega) = \frac{\exp(-i\pi/4)}{\sqrt{2\pi}}\int_{\infty\sin(\omega/2)}^{2\sqrt{kR}\sin(\omega/2)} \exp\left(it^2/2\right)dt, \tag{20}$$

which is a Fresnel integral.

So, the exact expression of primary diffraction is obtained (at $y > 0$) by deforming the path of integration $S_\alpha$ into the contour $S$ (Fig. 3), taking account of a residue in the point $\alpha = \zeta - \vartheta_0$ if the contour deformation captures a pole ($\zeta - \vartheta_0 > 0$):

$$\begin{aligned} E_1^2 &= e^{ikr\cos(\vartheta-\vartheta_0)}\Xi(R, \vartheta_0 - \zeta) - e^{ikr\cos(\vartheta+\vartheta_0)}\Xi(R, \vartheta_0 + \zeta) + E_0^2, \\ E_0^2 &= -\theta(\zeta - \vartheta_0)e^{ikr\cos(\vartheta-\vartheta_0)}. \end{aligned} \quad (21)$$

Here, $R = \sqrt{r^2 + a^2 + 2ar\cos\vartheta}$, $\zeta = \arcsin(r\sin\vartheta/R)$ and $\theta(\zeta)$ is the Heaviside function.

Although (21) is derived for positive $y$, it is applicable for any $y$ if

$$0 < \vartheta < 2\pi$$

using

$$E_0^2 = -\theta(\pi - \vartheta_0 - |\zeta - \pi|)\exp\big(ikr\cos(\pi - \vartheta_0 - |\vartheta - \pi|)\big). \quad (22)$$

Indeed, $\cos(\vartheta - \vartheta_0)$ passes to $\cos(\vartheta + \vartheta_0)$ and vice versa when replacing $y$ with $-y$ (i.e., replacing $\vartheta$ with $2\pi - \vartheta$ and $\zeta$ with $2\pi - \zeta$), $\sin\big((\vartheta_0 - \zeta)/2\big)$ with $-\sin\big((\vartheta_0 + \zeta)/2\big)$ and vice versa. Hence, since the first and the second terms in (21) change places, (21) remains in force. Note also that the Heaviside function in (21) is responsible for shielding and reflecting the plane wave. One should take into account the plane waves also from the first edge. Obviously, the angle $\vartheta_0$ in (22) (Fig. 1) corresponds to the geometrical boundary of a shadow from the strip edge.

Instead of (18), we have approximately

$$R \cong r + a\cos\vartheta, \quad \zeta \cong \vartheta - \frac{a}{r}\sin\vartheta, \quad (23)$$

for the intermediate wave zone ($r > a$) and

$$R \sim r, \quad \zeta \sim \vartheta.$$

for far-zone ($r \gg a$ or $a/r \to 0$). The asymptotic formula for primary diffraction

$$\mathbf{J}(\vartheta_0 - \pi/2, \tau)\exp\big(ikr\sin(\tau + \vartheta)\big) \cong Ag\,(\vartheta_0 - \pi/2, \vartheta + \pi/2) \quad (24)$$

follows from (21) with help of expressions for $R$, $\zeta$ in (23) and the asymptotic expression

$$\Xi(r, \omega) \cong -\frac{1}{2\sqrt{2\pi kr}}\frac{\exp\big(ikr(1 - \cos\omega) + i\pi/4\big)}{\sin(\omega/2)},$$

valid for large $r$. The solution of the diffraction problem for an incident plane wave on a half-plane [1] is

$$E_1^2 \approx -\frac{\cos(\vartheta_0/2)\sin(\vartheta/2)}{\cos\vartheta - \cos\vartheta_0}\sqrt{\frac{2}{\pi kr}}\exp(ikr + i\pi/4),$$

in the far-zone. Now let us consider the secondary diffraction in (16)

$$E_2^2 = \mathbf{J}(\vartheta_0 + \pi/2, \alpha)\mathbf{J}(\alpha, \tau)\exp\big(ikr\sin(\tau+\vartheta)\big), \tag{25}$$

where the diffraction operator $\mathbf{J}$ now acts on the cylindrical wave reflected from the first edge. The asymptotic formula (24) may be used for the estimate

$$E_2^2 \cong A\mathbf{J}(\vartheta_0 + \pi/2, \alpha)g(\alpha, \vartheta + \pi/2).$$

Introducing a variable $\tau = \alpha - \pi/2$, using the identity

$$\left(\frac{\cos(\vartheta/2)}{\sin((\alpha-\vartheta)/2)} - \frac{\cos(\varphi/2)}{\sin\big((\alpha-\varphi)/2\big)}\right)\frac{1}{\sin\big((\alpha-\vartheta)/2\big)}\frac{1}{\sin\big((\alpha-\varphi)/2\big)}$$
$$= \frac{1}{\sin\big((\vartheta-\varphi)/2\big)}\frac{1}{\cos(\alpha/2)}$$

and the symmetry of the integrand with respect to $\tau$, the integral in (25) is calculated by means of the special function $\Upsilon(l, \alpha)$ (A4):

$$\begin{aligned} E_2^2/A \;&\cong\; \Upsilon\big(ka, \cos(\vartheta_0+\pi)\big)g(\vartheta+\pi/2, \vartheta_0-\pi/2) \\ &\quad +\Upsilon\big(ka, \cos(\vartheta+\pi)\big)g(\vartheta_0+\pi/2, \vartheta-\pi/2) \end{aligned} \tag{26}$$

or using the properties $g$ and $\Upsilon$:

$$\begin{aligned} E_2^2/A \;&\cong\; -\Upsilon\big(ka, \cos(\vartheta_0+\pi)\big)g(\vartheta_0+3\pi/2, \vartheta+\pi/2) \\ &\quad -\Upsilon\big(ka, \cos(\vartheta+\pi)\big)g(\vartheta+3\pi/2, \vartheta_0+\pi/2). \end{aligned}$$

For future reference the useful integral relation

$$\begin{aligned} \mathbf{J}(\alpha, \beta)g(\beta, \vartheta+\pi/2) \;&=\; \Upsilon\big(ka, \cos(\alpha+\pi/2)\big)g(\vartheta+\pi/2, \alpha-\pi) \\ &\quad +\Upsilon\big(ka, \cos(\vartheta+\pi)\big)g(\vartheta-\pi/2, \alpha). \end{aligned} \tag{27}$$

is also given. A change of the origin of the angle coordinates in (26)

$$\vartheta \to \vartheta - \pi/2, \quad \vartheta_0 \to \vartheta_0 + \pi/2,$$

gives the result in [19]:

$$\begin{aligned} E_2^2/A \;&=\; \Upsilon\big(ka, \cos(\vartheta_0-\pi/2)\big)g(\vartheta_0, \vartheta) \\ &\quad -\Upsilon\big(ka, \cos(\vartheta+\pi/2)\big)g(\vartheta+\pi, \vartheta_0+\pi), \end{aligned}$$

or, using the notation of [19],

$$E_2^2 = \frac{g_2^2}{\sqrt{kr}}\exp\big(i(ka\sin\vartheta + kr + \pi/4)\big).$$

It is easy to incorporate the subsequent terms of the series (16). For example, the tertiary diffraction may be considered. Using relations (24) and (27), the expression

$$E_3^2 = \mathbf{J}\left(\vartheta_0 - \frac{\pi}{2}, \alpha\right)\mathbf{J}(\alpha, \beta)\mathbf{J}(\beta, \gamma)\exp\big(ikr\sin(\gamma+\vartheta)\big)$$

is represented in the form:

$$E_3^2 \cong A\mathbf{J}(\vartheta_0-\pi/2,\alpha)\Big(\Upsilon\big(ka,\cos(\alpha+\pi/2)\big)g(\vartheta+\pi/2,\alpha-\pi) \\ +\Upsilon\big(ka,\cos(\vartheta+\pi)\big)g(\vartheta-\pi/2,\alpha)\Big). \tag{28}$$

The contribution of the first term can be neglected since $\Upsilon(ka,-1)=0$ is satisfied in the saddle point $\alpha=\pi/2$, to be used when estimating the integral with the saddle point method.

The final estimate of the tertiary diffraction is obtained from (28), using (27):

$$E_3^2 = -A\Upsilon\big(ka,\cos(\vartheta+\pi)\big)\big(\Upsilon(ka,\cos\vartheta_0)g(\vartheta_0+\pi/2,\vartheta-\pi/2) \\ +\Upsilon(ka,\cos\vartheta)g(\vartheta+\pi/2,\vartheta_0-\pi/2)\big).$$

### 3.2. 2nd Method

By a deformation of the integration contour $C^+$ up to edges of the cut $C_z$ (Fig. 2), the system (14) is presented in the form:

$$B^+(w) = \frac{1}{\pi i}\int_0^1 \sqrt{1-x}\left(\frac{\exp(i2kax)B^-(-kx)}{(x+w/k)\sqrt{1+x}}\right)dx \\ +\frac{1}{\pi}\int_0^\infty \exp(-x)\left(\sqrt{\frac{x+i2ka}{x-i2ka}}\frac{B^-(-ix/2a)}{(x-i2aw)}\right)dx+B_0^+,$$

and

$$B^-(w) = \frac{1}{\pi i}\int_0^1 \sqrt{1-x}\left(\frac{\exp(i2kax)B^+(kx)}{(x-w/k)\sqrt{1+x}}\right)dx \\ +\frac{1}{\pi}\int_0^\infty \exp(-x)\left(\sqrt{\frac{x+i2ka}{x-i2ka}}\frac{B^+(ix/2a)}{(x+i2aw)}\right)dx+B_0^-,$$

where

$$B_0^+=\mathbf{I}(-w,u)A_1(-u),\quad B_0^-=\mathbf{I}(w,u)A_2(u).$$

Here, the improper integrals are converging fast. Therefore, it is convenient for their numerical generation to use Gauss' quadrature formulas with weights [25]. In this case, the boundary value problem is reduced to the solution of a system of linear algebraic equations.

### 3.3. 3rd Method

The short-wave asymptotic behavior is achieved by means of the etalon integral

$$I(w)\equiv \mathbf{I}(w,u)\cdot 1,$$

using the stationary phase method [26] (see (A5)). Here, the integration path in (14) is deformed up to the edge of the cut $C_1$, (Fig. 2) to get the contour of steepest descent that is a line parallel to imaginary axis upwards from the branch point. The result is

$$\begin{aligned} B^+(w) &\cong \mathbf{I}(-w,u)A_1(-u) + B^-(-k)I(-w), \\ B^-(w) &\cong \mathbf{I}(w,u)A_2(u) + B^+(k)I(w). \end{aligned} \tag{29}$$

The functions in (29) are found by solving the following system of linear algebraic equations:

$$B^+(k) = \big(1 - I^2(-k)\big)^{-1} \big(\mathbf{I}(-k,u)A_1(-u) + I(-k)\mathbf{I}(-k,u)A_2(u)\big), \tag{30}$$

$$B^-(-k) = \big(1 - I^2(-k)\big)^{-1} \big(\mathbf{I}(-k,u)A_2(u) + I(-k)\mathbf{I}(-k,u)A_1(-u)\big), \tag{31}$$

$$\mathbf{I}(-k,u)A_1(-u) = \big(I(-k) - I(-h)\big)A_1(k),$$

$$\mathbf{I}(-k,u)A_2(u) = \big(I(-k) - I(h)\big)A_2(-k).$$

Thus, the above-stated expressions give the dominant contribution to the solution of (14).

In order to take a account of the corrections of higher order it is necessary to expand the required functions $B^+$ and $B^-$ in (14) in a Taylor series in the neighborhood of the point $u = k$ with the result:

$$\mathbf{I}(-w,u)B^-(-u) = e^{i2ak} \sum_{n=0}^{N} \frac{(i)^n}{n!} B^{-(n)}(-k) \frac{\partial^n}{\partial(2a)^n} \big(e^{-i2ak} I(-w)\big), \tag{32}$$

$$\mathbf{I}(w,u)B^+(u) = e^{i2ak} \sum_{n=0}^{N} \frac{(-i)^n}{n!} B^{+(n)}(k) \frac{\partial^n}{\partial(2a)^n} \big(e - i2akI(w)\big). \tag{33}$$

A substitution of (32), (33) in the system (14), taking the derivatives in the branch points will yield $2N$ algebraic equations for the functions $B^+(k)$, $B^-(-k)$ and their derivatives.

Substituting the solution (29) in (8), (4) and (2) and using the result from (18), we get the dominating contribution to the electric field by the saddle point method and an etalon integral in (A5):

$$\begin{aligned} E_x &= E_x^1 + E_x^2, \\ E_x^1 &\simeq E_0\Big(E_1^1 + E_2^1 - \frac{\pi}{A_0}\sqrt{2k}\sin\frac{\vartheta}{2} I(k\cos\vartheta)\mathrm{H}_0^{(1)}(kr)B^+(k)\Big), \\ E_x^2 &\simeq E_0\Big(E_1^2 + E_2^2 - \frac{\pi}{A_0}\sqrt{2k}\sin\frac{\vartheta}{2} I(-k\cos\vartheta)\mathrm{H}_0^{(1)}(kr)B^-(-k)\Big). \end{aligned}$$

It is interesting to observe that the precision of the dominating contribution in the third method turns out to be not less than the precision of the solution of the tertiary diffraction achieved by the method of successive approximations in the first method.

## 4. RESONANCE

Below, we will consider a resonance on the strip that follows immediately from the solution (29). Zeroing a denominator in (30) and (31) gives a characteristic equation in the first approximation:

$$\det_1 = 1 - I^2(-k) = 0, \tag{34}$$

which determines the complex resonance frequencies of a strip, i.e., frequencies of self oscillations in absence of external incident waves. By means of the formula

$$\int_\infty^z \exp(it)\mathrm{H}_0^{(1)}(t)dt = z\exp(iz)\left(\mathrm{H}_0^{(1)}(z) - i\mathrm{H}_1^{(1)}(z)\right),$$

we obtain from (A5),

$$I(-k) = \frac{1}{2i}\mathrm{H}_0^{(1)}(2ka) - 2ak\left(\mathrm{H}_0^{(1)}(2ka) - i\mathrm{H}_1^{(1)}(2ka)\right),$$

to be used in (34). Using the asymptotic form of the Hankel functions ($ka \to \infty$) in (34) we get the known result [1]:

$$\exp\left(i(2ka - 3\pi/4)\right) = \pm 2\sqrt{\pi ka}.$$

Now, we will consider the characteristic equation in the second approximation for the estimation of the precision of the basic contribution of the integration by the saddle point method taking into account only the first derivatives in (32), (33) ($N = 1$):

$$\begin{aligned}\mathbf{I}(-w,u)B^-(-u) &\simeq B^-(-k)I(-w) + B^{-(1)}(-k)\big(kI(-w) + iI_a(-w)\big),\\ \mathbf{I}(w,u)B^+(u) &\simeq B^+(k)I(w) - B^{+(1)}(k)\big(kI(-w) + iI_a(-w)\big),\end{aligned}$$

where the following notation for the derivatives in the specified (indicated) points with respect to $w$ and the parameter $b = 2a$ is introduced:

$$\begin{aligned}I_a(-w) &\equiv \frac{\partial}{\partial b}I(-w), \quad B^{-(1)}(-k) \equiv \frac{\partial}{\partial w}B^-(w)\big|_{w=-k},\\ I_a &\equiv I_a(-k), \qquad\quad B^{+(1)}(k) \equiv \frac{\partial}{\partial w}B^+(w)\big|_{w=k}.\end{aligned}$$

Substituting these expressions in (14), we find the matrix

$$\begin{pmatrix} 1 & 0 & -I(-k) & -kI(-k) - iI_a \\ 0 & 1 & -I_w & -kI_w - iI_{wa} \\ -I(-k) & kI(-k) + iI_a & 1 & 0 \\ -I_w & kI_w + iI_{wa} & 0 & 1 \end{pmatrix}$$

for the system of algebraic equations, where the following notation is introduced:

$$I_w \equiv \frac{\partial}{\partial w} I(w)\big|_{w=-k}, \quad I_{wa} \equiv \frac{\partial}{\partial b} I_w.$$

We get the characteristic equation in the second approximation by equating the determinant of the matrix to zero:

$$\begin{aligned}
\det{}_2 \;=\; & 1 - \frac{H_0^4}{16} + \frac{ka}{12}\left(8H_0H_1\left(1+H_0^2\right) + iH_0^2\left(24+9H_0^2+H_1^2\right)\right) \\
& + \frac{(ka)^2}{36}\left(H_1^2\left(H_1^2-112\right) + 2H_0^2\left(312-47H_1^2\right) + 129H_0^4\right. \\
& \left. - i16H_0H_1\left(H_1^2+15H_0^2+42\right)\right) + \frac{8(ka)^3}{9}(H_1+iH_0) \\
& \left(H_0(7H_1^2-9H_0^2-48) + iH_1\left(H_1^2+17H_0^2+32\right)\right) \\
& - \frac{64(ka)^4}{9}(H_1+iH_0)^2\left((H_1+iH_0)^2-4\right) = 0, \qquad (35)
\end{aligned}$$

where for brevity the values of the Hankel functions are designated as

$$H_0 \equiv \mathrm{H}_0^{(1)}(2ak), \quad H_1 \equiv \mathrm{H}_1^{(1)}(2ak).$$

The following values of the derivatives of $I$ have been used in the characteristic equation:

$$\begin{aligned}
I_w(-k) &= a(H_0 - \frac{i}{3}H_1) - \frac{8}{3}ika^2(H_0 - iH_1), \\
I_a &= -\frac{1}{2}kH_0 + \frac{i}{2}kH_1 - ikI(-k), \\
I_{wa} &= \frac{1}{2}H_0 - 3ak(iH_0 + \frac{7}{9}H_1) - \frac{8}{3}(ak)^2(H_0 - iH_1).
\end{aligned}$$

The variation of $\det_1$ and $\det_2$ with real $k$ is presented in Fig. 5 using the basic contribution of the solution (14) with the saddle point method.

The behavior of the real part of the characteristic function for real values of $k$ is maintained if $2ak$ is larger than about 0.04 in the first and in the second approach (Fig. 5). The behavior of the imaginary part of the characteristic functions for real values $k$ are basically identical both in the first and in the second approximation.

Hence, on solving similar diffraction problems with the saddle point method and an etalon integral, the basic contribution of the integration should be restricted to the frequency band $2ka > 0.4$.

Note that the characteristic Equations (34) and (35) have no real roots. Therefore the strip eigenfrequencies are complex.

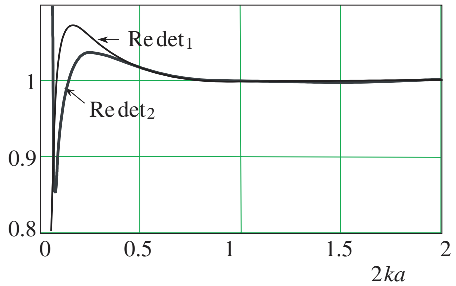


**Figure 5.** Real part of $\det_1$, $\det_2$ ($\mathrm{Im}ka = 0$).

**Table 1.** Resonance wave lengths for the strip and their asymptotic values.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $2a/\lambda$ | 0.23 | 0.67 | 1.15 | 1.64 | 2.14 | 2.64 | 3.13 |
| $2a/\lambda_0$ | 0.12 | 0.62 | 1.12 | 1.62 | 2.12 | 2.62 | 3.12 |

The imaginary part of the roots of the characteristic Equation (34) approach asymptotically the value

$$\mathrm{Im}(2ak) = -2.305 - 1.5 \ln \mathrm{Re}(2ak) \tag{36}$$

at $\mathrm{Re}k \gg 1$, which is conveniently derived by the asymptotical formula for the Hankel functions:

$$\mathrm{H}_n^{(1)}(\xi) \simeq \exp(-\mathrm{Im}\xi)\mathrm{H}_n^{(1)}(\mathrm{Re}\xi).$$

The real parts of the roots of (34) have the asymptotic form:

$$\mathrm{Re}(2ak) \simeq \pi(n + 1/4) \qquad (n = 1, 2, \ldots). \tag{37}$$

The obtained asymptotic formulas coincide with the result of [17]:

$$2ak \simeq (n + 1/4)\pi - i1.5 \ln\left(2\sqrt[3]{4\pi}(n + 1/4)\pi\right).$$

Numerical and asymptotic values of the resonance wave lengths of the strip are presented in Table 1, where $\lambda_0$ are the asymptotic values of $\lambda$ given in (37):

$$2a/\lambda_0 = (n - 1)/2 + 1/8.$$

## 5. CONCLUSION

The WH method is developed for plane structures, e.g., the diffraction problem for the strip. The exact solution of the boundary value problem is obtained in the form of an infinite series by means of a diffraction integral operator for the strip. The short-wave asymptotic solution, which contains a resonant denominator is also presented. The diffraction fields are found for each solution. The characteristic equation and the numerical values of the complex resonance frequencies are obtained. The results of the paper are also compared to earlier known asymptotic solutions obtained by the edge wave method and the method of self-consistent fields. Numerical calculations have shown a high precision of the integration by means of the saddle point method and the etalon integral. Therefore it is possible to use the basic contribution of the short-wave asymptotic solution for frequencies above the quasi stationary limit. The developed WH-method can be applied to other structures than the strip and can also be extended to a cylindrical geometry without any essential changes. An important property is that the mathematical apparatus is maintained for analogous problems with excitation with electromagnetic waves and diffraction of charged particles.

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## APPENDIX A. ETALON INTEGRAL, SPECIAL FUNCTION $\Upsilon$

An important ingredient in the current paper is to calculate the etalon integral to obtain the asymptotic solutions of the system of integral Equations (6) and (7):

$$I(w) = \frac{1}{2\pi i}\int_{C^+}\frac{\exp(i2au)}{u-w}\sqrt{\frac{k-u}{k+u}}du,$$

which can be presented in the form of the sum of two integrals as:

$$I(w) = -\frac{1}{2\pi i}\int_{C^+}\frac{\exp(i2au)}{\sqrt{k^2-u^2}}du + \frac{k-w}{2\pi i}\int_{C^+}\frac{\exp(i2au)}{(u-w)\sqrt{k^2-u^2}}du,$$

where $C^+$ is the integration contour that is parallel to the real axis in the upper $u$ half plane (UP) with an additional infinite narrow loop enveloping the point $u = -h$ from above (Fig. 2).

We use the Sommerfeld integral [27] to calculate the first integral:

$$\frac{1}{\pi}\int_S \exp(ix\cos\alpha)d\alpha = \mathrm{H}_0^{(1)}(x), \tag{A1}$$

where $S$ is the integration contour passing from top to down in the second and fourth quadrants through the origin of the coordinates (Fig. 4).

Note that the Sommerfeld contour $S_\alpha$ in the complex $\alpha$ plane (Fig. 3) corresponds to the integration contour $S_z$ in the $w$ complex plane, which passes along the edges of the cut of the function $\sqrt{k-w}$ at $0 < \mathrm{Im}k \to 0$ (Fig. 2).

We calculate the first integral with the help of (A1) by deforming the contour $C^+$ up to $C_1$ and introducing the new integration variable $u = k\cos\alpha$, which maps one-to-one and conformally the half-plane $\mathrm{Im}u > 0$ on the half strip $0 < \mathrm{Re}\alpha < \pi$, $\mathrm{Im}\alpha < 0$:

$$-\frac{1}{2\pi i}\int_{C_1}\frac{\exp(i2au)}{\sqrt{k^2-u^2}}du = \frac{1}{2i}\mathrm{H}_0^{(1)}(2ak). \tag{A2}$$

To get the second integral, we determine the special function as:

$$\Upsilon(x,\cos\beta) = \sin\beta\int_\infty^x \mathrm{H}_0^{(1)}(2t)\exp(-2it\cos\beta)dt. \tag{A3}$$

Multiplying both parts of (A2) with $\exp(i2bw)$, substituting $a$ by $b$ and integrating in $b$ from infinity to $a$ with $\mathrm{Im}u > 0$, we get the function as the contour integral [28]

$$\Upsilon(ka, w/k) = -\frac{\sqrt{k^2-w^2}}{2\pi i}e^{-i2aw}\int_{C_1}\frac{e^{i2au}}{(u-w)\sqrt{k^2-u^2}}du. \tag{A4}$$

With the help of (A4) and (A2) we finally obtain the etalon integral [29]:

$$I(w) = \frac{1}{2i}H_0^{(1)}(2ka) - \sqrt{\frac{k-w}{k+w}}e^{i2aw}\Upsilon(ka, w/k). \tag{A5}$$

The following asymptotic formula is useful for numerical calculations of the special function $\Upsilon$ [19]:

$$\begin{aligned}\Upsilon(ka,\cos\beta) \;=\; & e^{-i\pi/4}\frac{2}{\sqrt{\pi}}\mathrm{sgn}\left(\sin\frac{\beta}{2}\right)\int_\chi^\infty e^{i\mu^2}d\mu \\ & -\frac{e^{i\pi/4}}{2\sqrt{\pi ka}}e^{i4ka\sin^2(\beta/2)}\sum_{n=0}^\infty(-i2ak)^{-n}D_n\mathrm{ctg}\frac{\beta-2\pi}{4}.\end{aligned}$$

The following properties of the special function is useful:

$$\Upsilon(ka,\cos\beta) = \Upsilon(ka,\cos(\beta\pm 2\pi)),\ \Upsilon(ka,1) = -1,\ \Upsilon(ka,-1) = 0.$$

When estimating the integrals with the saddle point method (or with the stationary phase method) and the etalon integral $I(w)$ it is assumed that the integration contour is the line of fastest descent and also that the required integrands have no singularities near the saddle-point. Such a contour on the complex $\alpha$ plane is the path $S$ (Fig. 4). It corresponds to the integration contour $C_1$ on the $w$ complex plane which passes along the edges of the cut of the function $v = \sqrt{k - w}$ and is taken vertically from the point $k$ (Fig. 2).

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