

## LOD-LIKE METHOD THAT CHARACTERIZES THE ANALYTICAL SOLUTION

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**Abstract**—A LOD-like method that characterizes the analytical solution is proposed to study the one-dimensional (1-D) chiral media. Through theoretical analysis and numerical simulation, it is found that the proposed scheme is unconditionally stable. This scheme employs the new mesh-dividing method for bi-isotropic media, in which the two sections on the right side of the rearranged curl equations are regarded as two directions and the LOD-like algorithm is used to deal with the equivalent two-dimensional (2-D) problem. In the first substep, the conventional LOD method is used in computation, while for the second substep, the analytical solution is employed instead. By simulating the polarization rotation of a mono-frequency linear polarized wave both in a 1-D homogeneous chiral media and through a chiral slab, the scheme is testified to be unconditionally stable.

### 1. INTRODUCTION

Many unconditionally stable schemes have been applied to the discretization of Maxwell's equations for the purpose of eliminating the Courant-Friedrich-Levy (CFL) stability restraint on the time step value in the finite-difference time-domain method (FDTD). Some common unconditionally stable FDTD methods include the ADI-FDTD based on the alternating direction implicit algorithm [1], the CN-FDTD based on the Crank-Nicolson algorithm [2], and the LOD-FDTD based on the locally one-dimensional algorithm [3]. However, these unconditionally stable algorithms are generally applied to the FDTD methods for isotropic media. As to the FDTD methods for bi-isotropic media [4–10], none of them is unconditionally stable except the one presented in [10]. The new discretization approach proposed in [4] considered the

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characteristics of the constitutive relations for bi-isotropic media [11–16]. Different from the conventional Yee grid, the new sampling method makes the FDTD method for bi-isotropic media became simple. Based on the new mesh-dividing method, an algorithm similar to ADI for 1-D bi-isotropic media was presented in [10], and that was proved to be unconditionally stable through numerical experiments. That algorithm was characteristic of regarding the two sections on the right side of the resulting equations as two directions, thereby making it possible to carry out the alternating direction implicit algorithm, but that scheme is based on the complex field. This paper firstly derives the curl equations for 1-D chiral media based on the time domain under the condition of mono-frequency. By regarding the two sections on the right-hand side of the result equations as two directions and using the mesh-dividing method in [4], the equivalent 2-D problem is solved by using the LOD-like algorithm. In the first substep, the conventional LOD method is used in computation, while for the second substep, the analytical solution is employed instead. By simulating the polarization rotation of a mono-frequency linear polarized wave both in a 1-D homogeneous chiral medium and through a chiral slab, the scheme is proved to be unconditionally stable.

## 2. LOD-LIKE METHOD THAT CHARACTERIZES THE ANALYTICAL SOLUTION

The constitutive relations for bi-isotropic media can be written as

$$\vec{D} = \varepsilon \vec{E} + \xi \vec{H} \quad (1a)$$

$$\vec{B} = \varsigma \vec{E} + \mu \vec{H} \quad (1b)$$

The electromagnetic coupling coefficients  $\xi$ ,  $\varsigma$  are expressed as

$$\xi = (\chi - j\kappa)\sqrt{\mu_0\varepsilon_0}$$

$$\varsigma = (\chi + j\kappa)\sqrt{\mu_0\varepsilon_0}$$

where  $\chi$  is Tellegen parameter,  $\kappa$  is chirality parameter. When  $\chi = 0$  (only this condition is discussed in the paper),  $\kappa \neq 0$  and the source is mono-frequency, assuming a propagating transversal electromagnetic (TEM) wave along the  $z$  axis, one obtains

$$\varepsilon \frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z} - \omega \kappa \sqrt{\mu_0\varepsilon_0} H_x \quad (2a)$$

$$\varepsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \omega \kappa \sqrt{\mu_0\varepsilon_0} H_y \quad (2b)$$

$$\mu \frac{\partial H_x}{\partial t} = -\frac{\partial E_y}{\partial z} + \omega \kappa \sqrt{\mu_0\varepsilon_0} E_y \quad (2c)$$

$$\mu \frac{\partial H_y}{\partial t} = -\frac{\partial E_x}{\partial z} + \omega \kappa \sqrt{\mu_0 \varepsilon_0} E_y \quad (2d)$$

To solve (2a)–(2d), use the LOD method which characterizes the analytical solution.

The first substep  $n \rightarrow n + 1/2$

$$\begin{aligned} & \varepsilon(k) \frac{E_x^{n+1/2}(k) - E_x^n(k)}{\Delta t/2} \\ &= -\frac{1}{\Delta z} \left[ H_y^{n+\frac{1}{2}} \left( k + \frac{1}{2} \right) - H_y^{n+\frac{1}{2}} \left( k - \frac{1}{2} \right) + H_y^n \left( k + \frac{1}{2} \right) - H_y^n \left( k - \frac{1}{2} \right) \right] \end{aligned} \quad (3a)$$

$$\begin{aligned} & \varepsilon \left( k + \frac{1}{2} \right) \frac{E_y^{n+1/2} \left( k + \frac{1}{2} \right) - E_y^n \left( k + \frac{1}{2} \right)}{\Delta t/2} \\ &= \frac{1}{\Delta z} \left[ H_x^{n+\frac{1}{2}}(k+1) - H_x^{n+\frac{1}{2}}(k) + H_x^n(k+1) - H_x^n(k) \right] \end{aligned} \quad (3b)$$

$$\begin{aligned} & \mu(k) \frac{H_x^{n+1/2}(k) - H_x^n(k)}{\Delta t/2} \\ &= \frac{1}{\Delta z} \left[ E_y^{n+\frac{1}{2}} \left( k + \frac{1}{2} \right) - E_y^{n+\frac{1}{2}} \left( k - \frac{1}{2} \right) + E_y^n \left( k + \frac{1}{2} \right) - E_y^n \left( k - \frac{1}{2} \right) \right] \end{aligned} \quad (3c)$$

$$\begin{aligned} & \mu \left( k + \frac{1}{2} \right) \frac{H_y^{n+1/2} \left( k + \frac{1}{2} \right) - H_y^n \left( k + \frac{1}{2} \right)}{\Delta t/2} \\ &= -\frac{1}{\Delta z} \left[ E_x^{n+\frac{1}{2}}(k+1) - E_x^{n+\frac{1}{2}}(k) + E_x^n(k+1) - E_x^n(k) \right] \end{aligned} \quad (3d)$$

Substituting (3d) into (3a) and (3c) into (3b), one obtains a tridiagonal matrix about  $E_x^{n+1/2}$  and  $E_y^{n+1/2}$  respectively, which can be solved by using the conventional chasing method. Then from (3c) and (3d), one can compute the values of  $H_x^{n+1/2}$  and  $H_y^{n+1/2}$ .

The second substep  $n + 1/2 \rightarrow n + 1$  use the analytical solution

$$\varepsilon \frac{\partial E_x}{\partial t} = -\sigma H_x \quad (4a)$$

$$\varepsilon \frac{\partial E_y}{\partial t} = -\sigma H_y \quad (4b)$$

$$\mu \frac{\partial H_x}{\partial t} = \sigma E_x \quad (4c)$$

$$\mu \frac{\partial H_y}{\partial t} = \sigma E_y \quad (4d)$$

where  $\sigma = 2\omega \kappa \sqrt{\mu_0 \varepsilon_0}$ .

From (4a)–(4d), one obtains two sets of equations

$$\begin{cases} \varepsilon \frac{\partial E_x}{\partial t} = -\sigma H_x \\ \mu \frac{\partial H_x}{\partial t} = \sigma E_x \end{cases} \quad (5a)$$

$$\begin{cases} \varepsilon \frac{\partial E_y}{\partial t} = -\sigma H_y \\ \mu \frac{\partial H_y}{\partial t} = \sigma E_y \end{cases} \quad (5b)$$

From Equation (5a), one obtains

$$\frac{\partial^2 E_x}{\partial t^2} + \frac{\sigma}{\mu\varepsilon} E_x = 0 \quad (6)$$

The equation can be written as

$$\frac{\partial^2 f}{\partial t^2} + \alpha^2 f = 0 \quad (7)$$

where  $\alpha^2 = \frac{\sigma}{\mu\varepsilon}$  and  $f$  denotes  $E_x$ .

It is easy to know the general solution of (7) is

$$f = c_1 \cos(\alpha t) + c_2 \sin(\alpha t) \quad (8)$$

and

$$\frac{\partial f}{\partial t} = -\alpha c_1 \sin(\alpha t) + \alpha c_2 \cos(\alpha t) \quad (9)$$

Use the initial conditions, that is, the electromagnetic field values of  $E_x^{n+1/2}$  and  $H_x^{n+1/2}$  at the moment of  $n + 1/2$ , to solve the coefficients  $c_1$  and  $c_2$ .

$$E_x^{n+1/2} = c_1 \cos\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) + c_2 \sin\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) \quad (10)$$

and  $\frac{\partial f}{\partial t} = \frac{\partial E_x}{\partial t} = -\frac{2\sigma}{\varepsilon} H_x$ , so

$$-\frac{2\sigma}{\varepsilon} H_x^{n+1/2} = -\alpha c_1 \sin\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) + \alpha c_2 \cos\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) \quad (11)$$

From (10) and (11),  $c_1$  and  $c_2$  can be solved.

$$\begin{aligned} c_1 &= \cos\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) E_x^{n+1/2} + \frac{\sigma}{\alpha\varepsilon} \sin\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) H_x^{n+1/2} \\ c_2 &= \sin\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) E_x^{n+1/2} - \frac{\sigma}{\alpha\varepsilon} \cos\left(\alpha\left(n + \frac{1}{2}\right)\Delta t\right) H_x^{n+1/2} \end{aligned} \quad (12)$$

Use (8) to obtain the electric field value of moment  $n + 1$ .

$$E_x^{n+1} = c_1 \cos(\alpha(n + 1)\Delta t) + c_2 \sin(\alpha(n + 1)\Delta t) \quad (13)$$

Substituting the values of  $c_1$  and  $c_2$  in (12) into (10), one obtains

$$E_x^{n+1} = \cos \gamma E_x^{n+1/2} - \frac{\sigma}{\alpha \varepsilon} \sin \gamma H_x^{n+1/2}, \quad (14a)$$

where  $\gamma = \frac{\alpha \Delta t}{2}$ .

Using (11), one obtains

$$H_x^{n+1} = \cos \gamma H_x^{n+1/2} + \frac{\alpha \varepsilon}{\sigma} \sin \gamma E_x^{n+1/2} \quad (14b)$$

Using the similar procedure, one obtains

$$E_y^{n+1} = \cos \gamma E_y^{n+1/2} - \frac{\sigma}{\alpha \varepsilon} \sin \gamma H_y^{n+1/2} \quad (14c)$$

$$H_y^{n+1} = \cos \gamma H_y^{n+1/2} + \frac{\alpha \varepsilon}{\sigma} \sin \gamma E_y^{n+1/2} \quad (14d)$$

(14a)–(14d) are the iteration process of the second substep  $n + 1/2 \rightarrow n + 1$ .

### 3. NUMERICAL STABILITY ANALYSIS

To use Von Neumann's method to verify the unconditional stability in the combining process, substitute (15a)–(15d) into the first substep [1]:

$$E_x^n(k) = E_{0x} p^n \exp\{j(Kk\Delta z)\} \quad (15a)$$

$$E_y^n(k) = E_{0y} p^n \exp\{j(Kk\Delta z)\} \quad (15b)$$

$$H_x^n(k) = H_{0x} p^n \exp\{j(Kk\Delta z)\} \quad (15c)$$

$$H_y^n(k) = H_{0y} p^n \exp\{j(Kk\Delta z)\} \quad (15d)$$

where  $j = \sqrt{-1}$ ,  $p$  denotes the amplification factor and  $K$  denotes the wavenumber. One obtains:

$$\begin{bmatrix} q-1 & 0 & 0 & jc_1(q+1) \\ 0 & q-1 & -jc_1(q+1) & 0 \\ 0 & -jc_2(q+1) & q-1 & 0 \\ jc_2(q+1) & 0 & 0 & q-1 \end{bmatrix} \begin{bmatrix} E_{0x} \\ E_{0y} \\ H_{0x} \\ H_{0y} \end{bmatrix} = 0 \quad (16)$$

where  $q = p^{\frac{1}{2}}$

$$c_1 = \frac{\Delta t \sin \frac{k\Delta z}{2}}{\varepsilon \Delta z}, \quad c_2 = \frac{\Delta t \sin \frac{k\Delta z}{2}}{\mu \Delta z}$$

For Formula (16), the condition of non-zero solutions for the homogeneous equations is that the coefficient determinant must be zero. Therefore, the roots of  $q$  for the first substep is obtained. Then

utilizing  $p = q^2$ , one gets the roots of amplification factor  $p_1 - p_4$ . The forms are:

$$p_1 = p_2 = \left( \frac{c_1 c_2 - 1 + j2\sqrt{c_1 c_2}}{c_1 c_2 + 1} \right)^2$$

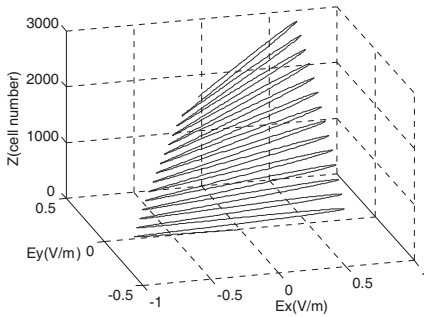
$$p_3 = p_4 = \left( \frac{c_1 c_2 - 1 - j2\sqrt{c_1 c_2}}{c_1 c_2 + 1} \right)^2$$

It is easy to confirm that:  $|p_1| = |p_2| = |p_3| = |p_4| = 1$ . The second substep is solved by using the analytical solution, which is consistently stable. Therefore, the combining process of the two substeps within a time step is unconditionally stable. The results of the numerical simulation also verified the correctness of this conclusion.

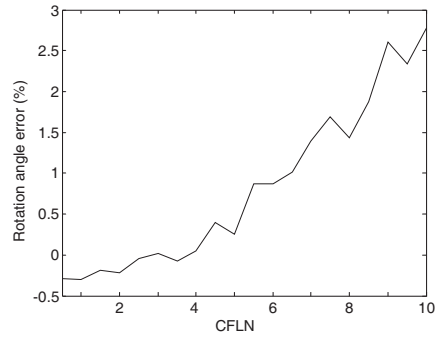
## 4. NUMERICAL RESULTS

### 4.1. Homogeneous Chiral Media

The polarization rotates when the mono-frequency linear polarized wave propagates in the homogeneous chiral medium. The parameters of the chiral medium are the same as those in [10], that is,  $\varepsilon_r = 2$ ,  $\mu_r = 1$ ,  $k = -9.42 \times 10^{-3}$ . The sinusoidal source is at the centre of the computational domain and its frequency is 3 GHz. The cell size is set to  $\Delta z = 1/3$  mm and the time-step size is set to  $\Delta t = 1$  ps. There are 8000 cells and the observation points are 1500 and 3000 cells, respectively, away from the source. The mesh boundaries were sufficiently remote so that no reflection appeared in the simulation.



**Figure 1.** Polarization rotation of a propagating wave in a chiral medium.



**Figure 2.** Relative errors of rotation angles in the homogeneous chiral medium with different CFLN values.

**Table 1.** Angle of rotation.

Distance		$1500\Delta z$	$3000\Delta z$
Angle of rotation (degrees)	Theoretical	16.965	33.929
	Literature [10]	16.891	34.107
	This work	16.957	33.916
Relative error (%)	Literature [10]	-0.44	+0.52
	This work	-0.05	-0.04
Computing time (second)	Literature [10]	16.968	
	This work	9.750	

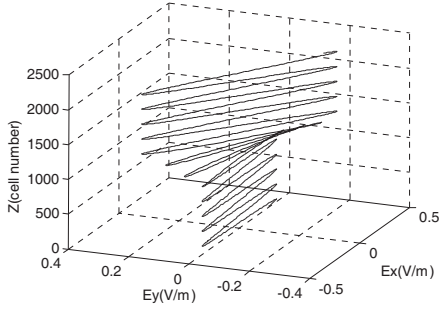
The obtained results are compared with the results from the method developed in [10], as is shown in Table 1. It can be seen that, with the above-mentioned parameters, the method proposed in this work showed very accurate results, and the time spent in computing is much less than that in [10], saving 42.5%. Figure 1 shows the rotation of the electric field suffered by a linear polarized wave that propagates in a chiral medium with the above-mentioned parameters. Figure 2 shows the relative errors of the rotation angles corresponding to 3000 cells with different CFLN values ( $\Delta t = CFLN * \Delta t_{CFL}$ ,  $\Delta t_{CFL} = 1.5724 \text{ ps}$ ). It can be seen from Figure 2 that with different values of CFLN ranging from 1 to 10, the absolute value of the relative error is always less than 3.0%.

#### 4.2. Chiral Slab

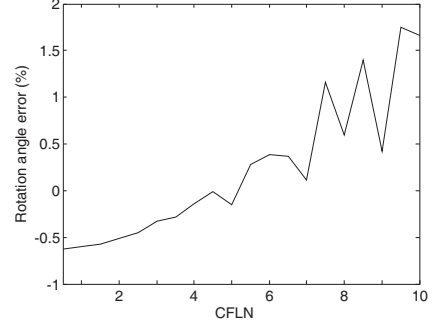
On both sides of the chiral slab is free space. The polarization will rotate when a linear polarized wave goes through the chiral slab. The parameters of the chiral media are as follows:  $\varepsilon_r = 4$ ,  $\mu_r = 1$ ,  $k = 0.06$ . The frequency of the sinusoidal source is 10 GHz. The cell size is set to  $\Delta z = 140 \mu\text{m}$  and there are 3000 cells altogether. The chiral slab, with 300 cells, is located at the centre of the computational domain, the slab length being 42 mm. The time step size is set to  $\Delta t = CFLN * \Delta t_{CFL}$  and  $\Delta t_{CFL} = 4.6667 \times 10^{-13} \text{ s}$ . The first-order Mur boundary is employed [17]. The observation point is on the right side of the chiral slab and one cell away from the slab. When a linear polarized wave goes through the chiral slab, the theoretical value of the angle of polarization rotation is 30.24 degrees [4].

Figure 3 shows the rotation of the electric field suffered by a linear polarized wave that propagates through the above-mentioned chiral

slab when  $CFLN = 1$  (There are 1000 cells both in front of and behind the chiral slab, as is shown in Figure 3.). Figure 4 shows the relative error of the rotation angles with different CFLN values, from which we can see that the relative errors of the rotation angles are always less than 2.0% with the CFLN values ranging from 1 to 10.



**Figure 3.** Polarization rotation of a propagating wave through a chiral slab.



**Figure 4.** Relative errors of rotation angles for the chiral slab with different CFLN values.

## 5. CONCLUSION

This work has proposed a LOD-like scheme which characterizes analytical solutions and has applied it to the computing of the 1-D chiral media. It has been verified numerically through the computation of the rotation angles of the polarization when a linear polarized wave propagates in a homogeneous chiral media and through a chiral slab respectively. Both theoretical analysis and numerical simulation have proved that the developed method is unconditionally stable. Compared with the ADI-like method, lower calculation work is the distinctive feature of the proposed method and it is based on the time domain. Although the LOD-like method is only applied to the 1-D chiral media in this work, in fact, it can be extended to the 2-D and 3-D situations.

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