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OPTIMAL IMPEDANCE MATCHING FOR CAPACITY MAXIMIZATION OF MIMO SYSTEMS WITH COUPLED ANTENNAS AND NOISY AMPLILFIERS

W.-F. Tsen and H.-J. Li

Graduate Institute of Communication Engineering
Department of Electrical Engineering
National Taiwan University
Taipei, Taiwan, R.O.C.

Abstract—The impedance matching problem in the presence of signal and noise coupling in compact MIMO arrays is addressed. By maximizing an upper bound of the ergodic capacity for an $N \times N$ MIMO system with signal and noise coupling at the receiver in high signal-to-noise ratio (SNR) scheme, a set of equations is formulated to find the optimal matching circuit. A closed-form result for the optimum matching circuit is given. For two-element arrays, we show numerically that significant performance improvement can be achieved by introducing the optimal matching.

1. INTRODUCTION

The use of multiple-input multiple-output (MIMO) system is known to improve the capacity and reliability of wireless communication links. However, correlation of signals at different antenna elements can considerably decrease the capacity of a MIMO system. In particular, the integration of MIMO technique into compact devices can reduce the signal correlation by distorting the radiation patterns of each element [1, 2]. However, it also induces a mismatch between the antennas and their corresponding source and load impedances [3], which is detrimental to the capacity performance.

Several works have examined the impact on channel capacity due to signal mutual coupling [4, 5]. Other works have employed matching circuit to control the reception characteristics. The optimal

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Corresponding author: W.-F. Tsen (d90942003@ntu.edu.tw).

performance can be achieved if a coupled matching circuit is utilized [2]. Nevertheless, it is not easy to integrate this solution into MIMO systems as multiple circuit components have to be interconnected across the antenna ports. As a result, a suboptimal and simple uncoupled (or individual port) matching circuit is recently studied [6]. However, all these works assume statistically independent port noise.

Recent works address the subject of noise coupling. In [7], a realistic noise model for the receiver amplifier is introduced. The model is used to compute the noise covariance matrix and the channel capacity of a MIMO system. They demonstrate that matching for minimum noise figure is superior to matching for maximum power transfer. [8] also derives the matching circuit optimized for output SNR maximization of a beamformer. The impact of thermal noise and amplifier noise is included in determining the channel capacity of compact MIMO systems in [9, 10], respectively. When thermal noise dominates, matching circuit offers no improvement in capacity.

In this paper, we present a circuit model for a noisy MIMO receiver which includes arbitrary signal and noise coupling. After obtaining the MIMO capacity with spatially correlated noise, the optimum matching circuit for capacity maximization in high SNR regime is derived by extending the ideas in [8]. Numerical simulations are presented to validate the accuracy of our analysis.

A note on notation: We use boldface to denote matrices and vectors, the superscripts T, $*$ and H represent matrix transpose, complex conjugate, conjugate transpose, respectively. The notations $\mathrm{tr}(\mathbf{X})$, $\mathrm{E}[\mathbf{X}]$, $\det(\mathbf{X})$ and $[\mathbf{X}]_{ij}$, denote the trace, expectation, determinant and the (i, j)-th element of the matrix $\mathbf{X}$, respectively. respectively. where $E[\cdot]$ is the expectation. The notations $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ are used to denote the real and imaginary part of a complex number/matrix, respectively. $\mathbf{I}_N$ is an $N \times N$ identity matrix. For Hermitian matrices the notation $\mathbf{X} \geq 0$ implies that $\mathbf{X}$ is positive semidefinite.

2. SYSTEM MODEL

2.1. MIMO Circuit Model

Consider a frequency-flat MIMO system with N transmitting and N receiving antennas. In this paper, we focus mutual coupling effect on the receive end. The receiver architecture is given in Fig. 1. When the antennas are open circuited, the signals impinging on the antennas produce an open circuit voltage $v_{oc,i}$ across the input port of the ith antenna. The voltages can then be represented by Thevenin equivalent circuit voltage sources, $\mathbf{v}_{oc}= [v_{oc,1},\dots,v_{oc,N}]^T$, whereas the Thevenin

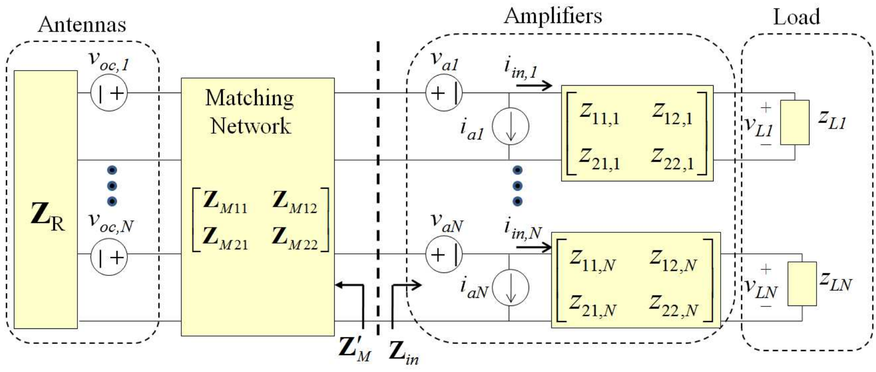


Figure 1. Receiver model.

equivalent impedance matrix $\mathbf{Z}_R = \mathbf{R}_R + j\mathbf{X}_R \in \mathbb{C}^{N\times N}$ is the mutual impedance matrix of the receive antennas. $\mathbf{v}_{oc}$ can be expressed by

$$\mathbf{v}_{oc} = \sqrt{\frac{P}{N}}\mathbf{H}\mathbf{x} + \mathbf{v}_E \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^N$ is the transmitted signal; P is the total energy at the transmitter over one symbol period; $\mathbf{H} \in \mathbb{C}^{N\times N}$ is the channel matrix, assumed to be spatially correlated Rayleigh fading. Using the well established Kronecker model [11], the propagation channel is represented by

$$\mathbf{H} = \mathbf{\Psi}^{1/2}\mathbf{H}_w \tag{2}$$

where $\mathbf{\Psi}$ is the receive correlation matrix, $\mathbf{H}_w \in \mathbb{C}^{N\times N}$ containing i.i.d $CN(0,1)$ elements. $\mathbf{v}_E = [v_{e,1}, \ldots, v_{e,N}]^T$ denotes the external noise received by the antennas. Applying the generalized Nyquist's thermal noise theorem [12], the external noise covariance matrix is given by

$$\mathbf{N}_E = E\left\{\mathbf{v}_E\mathbf{v}_E^H\right\} = 4kBT\mathbf{R}_R \tag{3}$$

where k is the Boltzman's constant, T is the standard noise temperature, and B is the instantaneous bandwidth of observation in Hz.

The antennas are connected to a $2N$-port matching network, with N ports on the antenna side and N ports on the amplifier side. Any two terminals of a network can be reduced to one voltage source in series with one impedance. The voltage that appears across these two terminals is the Thevenin voltage. The Thevenin equivalent impedance is the impedance between two terminals with all sources disabled. Extending the Thevenin's theorem to the $2N$-port system,

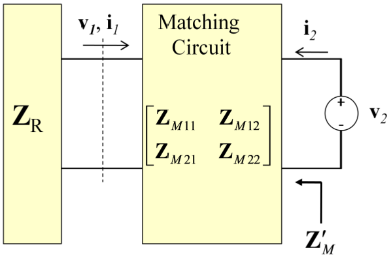


Figure 2. Circuit model to compute Z'_M.

the antennas plus matching circuit shown in Fig. 2 can be represented by a Thevenin equivalent circuit with equivalent Thevenin impedance $\mathbf{Z'}_M$. The port equations for the circuit shown in Fig. 2 are as follows

$$\mathbf{v}_1 = -\mathbf{Z}_R\mathbf{i}_1 \tag{4}$$
$$\mathbf{v}_1 = \mathbf{Z}_{M11}\mathbf{i}_1 + \mathbf{Z}_{M12}\mathbf{i}_2 \tag{5}$$
$$\mathbf{v}_2 = \mathbf{Z}_{M21}\mathbf{i}_1 + \mathbf{Z}_{M22}\mathbf{i}_2 \tag{6}$$

Using (4)–(6), we can express $\mathbf{v}_2$ as a function of $\mathbf{I}_2$, i.e., $\mathbf{v}_2 = \mathbf{Z'}_M\mathbf{i}_2$, and

$$\mathbf{Z'}_M = -\mathbf{M}\mathbf{Z}_{M12} + \mathbf{Z}_{M22} \tag{7}$$

where $\mathbf{M} = \mathbf{Z}_{M21}(\mathbf{Z}_R+\mathbf{Z}_{M11})^{-1}$.

The matching circuit is in turn connected to the amplifiers. An amplifier is usually represented by a simple noise circuit cascaded with a noise-free circuit [13, 14]. For the MIMO network in Fig. 1, the input impedance looking into the amplifiers with terminations can be written as $\mathbf{Z}_{in} = \text{diag}\{z_{in,1}, \dots, z_{in,N}\}$, where

$$z_{in,i} = z_{11,i} - \frac{z_{12,i}z_{21,i}}{z_{L,i} + z_{22,i}} \tag{8}$$

And the relation between the current $\mathbf{i}_{in}= [i_{in,1}, \dots, i_{in,N}]^T$ and the voltage across the load can be derived using the port equations as follows:

$$\mathbf{v}_L = \mathbf{Z}_{21}\mathbf{i}_{in} + \mathbf{Z}_{22}\mathbf{i}_L \tag{9}$$
$$\mathbf{v}_L = -\mathbf{Z}_L\mathbf{i}_L \tag{10}$$

Combinig (9) and (10), $\mathbf{v}_L = [v_{L,1}, \dots, v_{L,N}]^T$ is given by

$$\mathbf{v}_L = \mathbf{C}_L\mathbf{i}_{in}, \quad \mathbf{C}_L = \mathbf{Z}_L\left(\mathbf{Z}_L + \mathbf{Z}_{22}\right)^{-1}\mathbf{Z}_{21} \tag{11}$$

where $\mathbf{Z}_L= \text{diag}\{z_{L1}, \dots, z_{LN}\}$ and $\mathbf{Z}_{ij}= \text{diag}\{z_{ij,1}, \dots, z_{ij,N}\}$, i, $j=$ 1 or 2. We assume that different amplifiers are not correlated with

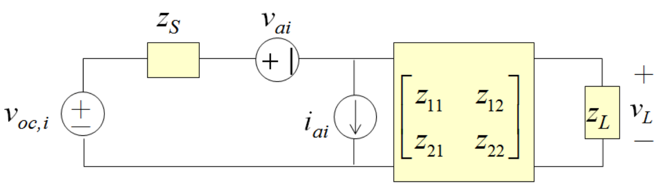


Figure 3. A SISO receiver model.

each other. The noise generated by the amplifiers is characterized by two correlated voltage and noise sources, $\mathbf{v}_a = [v_{a1}, \ldots, v_{aN}]^T$ and $\mathbf{i}_a = [i_{a1}, \ldots, i_{aN}]^T$, respectively. The variances of these sources and their correlation are given by

$$\frac{E\left\{\mathbf{v}_a \mathbf{v}_a^H\right\}}{4KTB} = \mathbf{R}_n \tag{12}$$

$$\frac{E\left\{\mathbf{i}_a \mathbf{i}_a^H\right\}}{4KTB} = \mathbf{G}_n \tag{13}$$

$$\frac{E\left\{\mathbf{v}_a \mathbf{i}_a^H\right\}}{4KTB} = \mathbf{S}_n \tag{14}$$

where $\mathbf{R}_n = \mathrm{diag}\{r_{n1}, \ldots, r_{nN}\}$, $\mathbf{G}_n = \mathrm{diag}\{g_{n1}, \ldots, g_{nN}\}$, $\mathbf{S}_n = \mathrm{diag}\{y_{n1} r_{n1}, \ldots, y_{nN} r_{nN}\}$, and r_{ni}, g_{ni} and y_{ni} are the noise resistance, conductance and admittance of ith amplifier, respectively.

An important metric associated with an amplifier is noise figure, F, defined by the ratio of total output noise power to the noise power due to the input noise only, when a source impedance is connected to the input port. From this definition, the noise figure of a SISO receiver when a source impedance $z_s = r_s + jx_s$ is connected to the input port, as shown in Fig. 3 can be shown as $F_i = 1 + \left(r_{ni} + g_{ni}\left|z_s\right|^2 + y_{ni}^* r_{ni} z_s^* + y_{ni} r_{ni} z_s\right)/r_S$. To derive the $z_s = z_{opt,i}$ such that F_i achieves its minimum value $F_{i,\min}$, we set the partial derivatives of F_i with respect to r_s and x_s to zero:

$$\frac{\partial F_i}{\partial r_s} = \frac{\partial F_i}{\partial x_s} = 0 \tag{15}$$

Thus the equations to be solved for r_{opt} and x_{opt} are as follows

$$g_{ni} x_s - y_{Ii} r_{ni} = 0 \tag{16}$$

$$g_{ni} r_s^2 - g_{ni} x_s^2 + 2 y_{Ii} r_{ni} x_s - r_{ni} = 0 \tag{17}$$

Solving (16) and (17), we can have

$$F_{\min,i} = 1 + 2\left(\sqrt{g_{ni}r_{ni} - y_{Ii}^2 r_{ni}^2} + y_{Ri}r_{ni}\right) \tag{18}$$

$$z_{opt,i} = \frac{\sqrt{g_{ni}r_{ni} - y_{Ii}^2 r_{ni}^2} + jy_{Ii}r_{ni}}{g_{ni}} \tag{19}$$

where $y_{Ri} = \mathrm{Re}\{y_{ni}\}$ and $y_{Ii} = \mathrm{Im}\{y_{ni}\}$.

2.2. MIMO Capacity

Solving the circuit in Fig. 1 for the load voltage, we obtain

$$\mathbf{v}_L = \mathbf{Q}\left(\mathbf{Hx} + \mathbf{n}\right) \tag{20}$$

where $\mathbf{Q} = \mathbf{C}_L\left[\mathbf{Z}_{M21}\left(\mathbf{Z}_R + \mathbf{Z}_{M11}\right)\mathbf{Z}_{M12} - \left(\mathbf{Z}_{M22} + \mathbf{Z}_{in}\right)\right]^{-1}$, $\mathbf{n}$ is the total noise component

$$\mathbf{n} = \mathbf{v}_E - \mathbf{M}^{-1}\left(\mathbf{v}_a + \mathbf{Z'}_M\mathbf{i}_a\right) \tag{21}$$

In radio communication system, the total noise can be considered as a superposition of two uncorrelated noise sources: 1) external noise received by the antennas and 2) internal noise due to the amplifiers and passive components. Combining (3) and (12) $\sim$ (14), the total noise covariance matrix $\mathbf{N}_{tot} = E\{\mathbf{nn}^H\}$ is given by

$$\mathbf{N}_{tot} = 4kTB\left[\mathbf{R}_R + \mathbf{M}^{-1}\left(\mathbf{R}_n + \mathbf{Z'}_M\mathbf{G}_n\mathbf{Z'}_M^H + \mathbf{S}_n^H\mathbf{Z'}_M^H + \mathbf{Z'}_M\mathbf{S}_n\right)\mathbf{M}^{-H}\right] \tag{22}$$

In this paper, we assume that only the receiver knows the channel state information. Since $\mathbf{Q}$ multiplies both the signal and noise component of $\mathbf{v}_L$, the ergodic capacity of the MIMO system is then

$$C = E\left[\log_2 \det\left(\mathbf{I} + \frac{P}{N}\mathbf{HH}^H\mathbf{N}_{tot}^{-1}\right)\right] \tag{23}$$

In particular, we consider a high-SNR regime, since the benefits of the use of MIMO systems are more pronounced at high SNR. Let $\mathbf{N} \equiv \mathbf{N}_{tot}/4kTB$, The ergodic capacity in (23) in high-SNR regime can be approximated by

$$C_{high\_SNR} \approx E\left[\log_2 \det\left(\frac{4kTBP}{N}\mathbf{HH}^H\right)\right] - \log_2 \det\left(\mathbf{N}\right) \tag{24}$$

3. OPTIMAL MATCHING CIRCUIT

3.1. Derivation of Optimal Matching Circuit

From the analysis in Section 2, the noise figure of an amplifier is minimized when a source is matched to z_{opt}. To extend the result to multiport circuits, we now prove that the optimal matching which maximizes the ergodic capacity of an $N \times N$ MIMO system in high SNR scenarios is $\mathbf{Z}'_M = \text{diag}\{z_{opt,1}, \ldots, z_{opt,N}\}$, and one particular circuit which achieves this goal is

$$\begin{bmatrix} \mathbf{Z}_{M11,opt} & \mathbf{Z}_{M12,opt} \\ \mathbf{Z}_{M21,opt} & \mathbf{Z}_{M22,opt} \end{bmatrix} = j \begin{bmatrix} -\mathbf{X}_R & \mathbf{R}_R^{1/2}\mathbf{R}_{opt}^{1/2} \\ \mathbf{R}_{opt}^{1/2}\mathbf{R}_R^{1/2} & \mathbf{X}_{opt} \end{bmatrix} \tag{25}$$

where $\mathbf{Z}_{opt} = \text{diag}\{z_{opt,1}, \ldots, z_{opt,N}\}$, $\mathbf{R}_{opt} = \text{Re}\{\mathbf{Z}_{opt}\}$ and $\mathbf{X}_{opt} = \text{Im}\{\mathbf{Z}_{opt}\}$.

An ideal matching circuit is lossless so it adds no noise. All elements of a lossless matching circuit must be purely imaginary, i.e., $\mathbf{Z}_{Mpq} = j\,\mathbf{X}_{Mpq}$ (p, $q = 1$ or 2, $\mathbf{X}_{Mpq}$ are real matrices), and satisfies $\mathbf{Z}_{M11} = -\mathbf{Z}_{M11}^H$, $\mathbf{Z}_{M22} = -\mathbf{Z}_{M22}^H$, $\mathbf{Z}_{M21} = -\mathbf{Z}_{M12}^T$ [9]. Let $\mathbf{Z}_{M11} = j(\mathbf{U} - \mathbf{X}_R)$, $\mathbf{Z}_{M22} = j\mathbf{V}$, $\mathbf{Z}_{M21} = j\mathbf{W}$ and $\mathbf{Z}_{M12} = j\mathbf{W}^T$, and using the property that logarithmic function is strictly increasing, the optimal problem can be stated as

$$\begin{bmatrix} \mathbf{Z}_{M11,opt} & \mathbf{Z}_{M12,opt} \\ \mathbf{Z}_{M21,opt} & \mathbf{Z}_{M22,opt} \end{bmatrix} = j \begin{bmatrix} \mathbf{U} - \mathbf{X}_R & \mathbf{W}^T \\ \mathbf{W} & \mathbf{V} \end{bmatrix} = \min_{\mathbf{Z}_M} \det(\mathbf{N}) \tag{26}$$

where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W} \in \mathbb{R}^{N\times N}$, where $\mathbf{U} = \mathbf{U}^T$ and $\mathbf{V} = \mathbf{V}^T$. To solve (26), we first substitute $\mathbf{Z}_{M11} = j(\mathbf{U} - \mathbf{X}_R)$, $\mathbf{Z}_{M22} = j\mathbf{V}$, $\mathbf{Z}_{M21} = j\mathbf{W}$ and $\mathbf{Z}_{M12} = j\mathbf{W}^T$ into (22), and set the partial derivatives of $h = \det(\mathbf{N})$ with respect to $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ to zero

$$\frac{\partial h}{\partial \mathbf{U}} = \frac{\partial h}{\partial \mathbf{V}} = \frac{\partial h}{\partial \mathbf{W}} = 0 \tag{27}$$

A general framework is introduced in [15] showing how to find the derivative of complex-valued scalar-, vector-, or matrix functions with respect to the complex-valued input parameter matrix and its complex conjugate. For functions of the type of $f(\mathbf{Z}, \mathbf{Z}^*)$, the way of arranging the partial derivatives is: If $df = \text{Tr}\left\{\mathbf{E}_0^T d\mathbf{Z} + \mathbf{E}_1^T d\mathbf{Z}^*\right\}$, then $(\partial/\partial\mathbf{Z}) = \mathbf{E}_0$ and $(\partial/\partial\mathbf{Z}^*) = \mathbf{E}_1$. Some of the most important rules on complex differentials used in this paper are listed in Table 1, assuming A, B, and a to be constants, and $\mathbf{Z}$, $\mathbf{Z}_0$, $\mathbf{Z}_1$ to be complex-valued matrix variables [15]. Since $\mathbf{N} = E\{\mathbf{n}\mathbf{n}^H\}/4kTB$ is positive definite, i.e., $\det(\mathbf{N}) \neq 0$, the equations to be solved for ($\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$)

Table 1. Important results for complex differentials.

Functions	$\mathbf{A}$	$a\mathbf{Z}$	$\mathbf{Z}_0 + \mathbf{Z}_1$	$\mathbf{Z}_0\mathbf{Z}_1$
Differential	$\mathbf{0}$	$a(d\mathbf{Z})$	$d\mathbf{Z}_0 + d\mathbf{Z}_1$	$(d\mathbf{Z}_0)\ \mathbf{Z}_1$ $+\ \mathbf{Z}_0(d\mathbf{Z}_1)$
Function	$\mathbf{Z}^H$	$\det(\mathbf{Z})$	$\mathrm{Tr}\{\mathbf{Z}\}$	$\mathbf{Z}^{-1}$
Differential	$(d\mathbf{Z})^H$	$\det(\mathbf{Z})$ $\mathrm{Tr}\{\ \mathbf{Z}^{-1}(d\mathbf{Z})\ \}$	$\mathrm{Tr}\{d\mathbf{Z}\}$	$\mathbf{Z}^{-1}(d\mathbf{Z})\mathbf{Z}^{-1}$

can be stated as (see APPENDIX A)

$$\left(\mathbf{W}^{-1}\mathbf{A}_1\mathbf{B}_1^H+\mathbf{W}^{-1}\mathbf{A}_2^H\mathbf{W}\right)\mathbf{N}^{-1}-\mathbf{N}^{-1}\left(\mathbf{B}_1\mathbf{A}_1\mathbf{W}^{-T}+\mathbf{W}^T\mathbf{A}_2\mathbf{W}^{-T}\right) = 0 \tag{28}$$

$$\left[\mathbf{N}^{-1}\left(\mathbf{W}^T\mathbf{G}_n+\mathbf{B}_1\mathbf{A}_2^H\right)-\mathbf{W}^{-1}\left(\mathbf{A}_1\mathbf{B}_1^H+\mathbf{A}_2^H\mathbf{W}\right)\mathbf{N}^{-1}\mathbf{B}_1\right]^T + \left[\left(\mathbf{G}_n\mathbf{W}+\mathbf{A}_2\mathbf{B}_1^H\right)\mathbf{N}^{-1}-\mathbf{B}_1^H\mathbf{N}^{-1}\left(\mathbf{B}_1\mathbf{A}_1+\mathbf{W}^T\mathbf{A}_2\right)\mathbf{W}^{-T}\right]=0 \tag{29}$$

$$\left(\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H+\mathbf{B}_2\right)\mathbf{N}^{-1}\mathbf{B}_1+\mathbf{B}_1^H\mathbf{N}^{-1}\left(\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H+\mathbf{B}_2\right)^H = 0 \tag{30}$$

where $\mathbf{A}_1 = \mathbf{R}_n + \mathbf{V}\mathbf{G}_n\mathbf{V} + j\left(\mathbf{V}\mathbf{S}_n - \mathbf{S}_n^H\mathbf{V}\right)$, $\mathbf{A}_2 = \mathbf{S}_n - j\mathbf{G}_n\mathbf{V}, \mathbf{B}_1 = (\mathbf{R}_R + j\mathbf{U})\,\mathbf{W}^{-1}$ and $\mathbf{B}_2 = j\left(\mathbf{G}_n\mathbf{W} + \mathbf{S}_n\mathbf{B}_1^H\right)$. (28)–(30) are complicated equations and not easily solvable, however, after some tedious but essentially straightforward analysis, and using the fact that $\mathbf{Z}_1\mathbf{Z}_2 = \mathbf{Z}_2\mathbf{Z}_1$ if $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are diagonal matrices, we can see that $\mathbf{U} = \mathbf{0}$, $\mathbf{V} = \mathbf{X}_{opt}$, $\mathbf{W} = (\mathbf{R}_{opt}\mathbf{R}_A)^{1/2}$ satisfies (28)–(30).

The same matching circuit has appeared in [7], with no proof of optimality. [8] uses the matching circuits of the above form, and proves it to be optimal for the output SNR maximization of a beamformer. Since the array is mutually coupled, noise exiting the input of one amplifier scatters between array elements and is presented at the inputs of all other ports. Because of this noise coupling, it is not obvious that a matching circuit which minimizes the noise figure of each amplifier individually is optimal. In this paper, we prove that the optimal impedance matching for capacity maximization of MIMO systems with coupled antennas and noisy amplifiers is the one which decouple the array and present isolated, individually noise-matched ports to the amplifier inputs.

3.2. Capacity Upperbound

To study the effect of matching circuit, we use the fact that $\mathbf{N}$, $\mathbf{R}_R$ are positive definite matrices, and factor the term inside the parentheses of (22)

$$\begin{aligned}
&\mathbf{R}_n + \mathbf{Z'}_M \mathbf{G}_n \mathbf{Z'}_M^H + \mathbf{S}_n^H \mathbf{Z'}_M^H + \mathbf{Z'}_M \mathbf{S}_n \\
&= \left(\mathbf{Z'}_M - \mathbf{Z}_{opt}\right) \mathbf{G}_N \left(\mathbf{Z'}_M - \mathbf{Z}_{opt}\right)^H + (\mathbf{F}_{\min} - \mathbf{I}_N) \frac{\mathbf{Z'}_M + \mathbf{Z'}_M^H}{2} \\
&\geq (\mathbf{F}_{\min} - \mathbf{I}_N) \frac{\mathbf{Z'}_M + \mathbf{Z'}_M^H}{2}
\end{aligned} \tag{31}$$

with equality if and only if $\mathbf{Z'}_M = \mathbf{Z}_{opt}$. Using the property: If $\mathbf{Z}_1$, $\mathbf{Z}_2 \in \mathbb{C}^{N\times N}$ are Hermitian and $\mathbf{Z}_1 \geq \mathbf{Z}_2$, then 1) $\mathbf{T}^H\ \mathbf{Z}_1\mathbf{T} \geq \mathbf{T}^H\ \mathbf{Z}_2\mathbf{T}$ for all $\mathbf{T} \in M_n$; 2) $\mathbf{Z}_2^{-1} \geq \mathbf{Z}_1^{-1}$; 3) $\det(\mathbf{Z}_1) \geq \det(\mathbf{Z}_2)$ [16], and applying (31) to (22) and (23) yields

$$\mathbf{N} \geq \mathbf{N}_{\min} = \mathbf{R}_R^{1/2} \mathbf{F}_{\min} \mathbf{R}_R^{1/2} \tag{32}$$

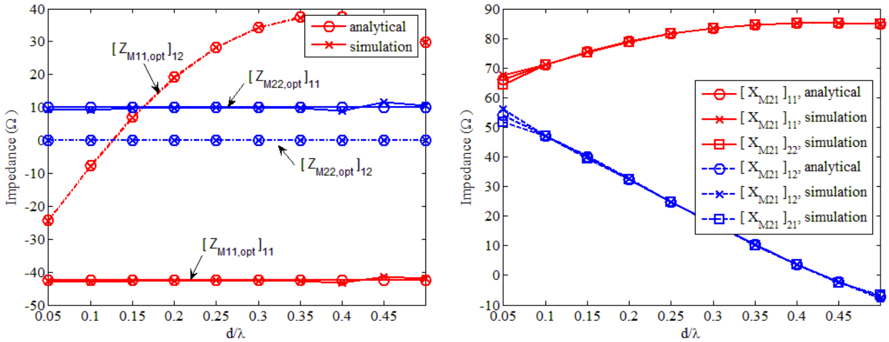
$$C \leq E\left[\log_2 \det\left(\mathbf{I}_N + \frac{4kTBP}{N} \mathbf{H}\mathbf{H}^H \mathbf{N}_{\min}^{-1}\right)\right] \tag{33}$$

with equality if and only if $\mathbf{Z'}_M = \mathbf{Z}_{opt}$. $\mathbf{F}_{\min} = \mathrm{diag}\{F_{\min,1}, \ldots, F_{\min,N}\}$. From the analysis in this paper, we see that the capacity for any matching circuit with $\mathbf{Z'}_M \neq \mathbf{Z}_{opt}$ is less than that achieved by optimal matching. Though the optimal matching circuit $\mathbf{Z}_{opt}$ is derived for capacity maximization in high SNR regime in Sections 2 and 3.1, the capacity upper bound expression in (33) is independent of SNR.

4. NUMERICAL RESULTS

In this section, we provide a numerical study to evaluate the matching performance of identical $\lambda/2$ dipoles with fixed self impedance $z_{11} = r_{11} + jx_{11}$. Results are given for $N = 2$. Induced EMF method is used to fill $\mathbf{Z}_R$ [17]. Based on minimum scattering antenna theory, the spatial correlation coefficient between two antennas in three-dimensional (3D) isotropic scattering can be calculated using the normalized resistance as $\rho_{12} = r_{12}/\sqrt{r_{11}r_{22}}$ [1], where r_{12} is the mutual resistance and r_{ii} is the self resistance of port i. Thus, given d, the normalized resistance will be used to fill $\mathbf{\Psi} = \mathbf{R}_R/r_{11}$. For the simulation in this section, we consider 2×2 MIMO systems with identical LNAs.

First, we compare the optimal impedance circuit derived analytically with simulated results. The noise statistics of the amplifier is characterized by $r_{ni} = 10\,\Omega$, $g_{ni} = 1\,\mathrm{mS}$ and $y_{ni} = (2 + j1)\,\mathrm{mS}$, $i = 1, 2$, this is equivalent to a noise figure of 1 dB. The simulated $\{\mathbf{Z}_{M11,opt},\ \mathbf{Z}_{M12,opt},\ \mathbf{Z}_{M21,opt},\ \mathbf{Z}_{M22,opt}\}$ in (26) which minimizes



(a) (b)

Figure 4. (a) Analytical and simulation $\mathbf{Z}_{M11,opt}$, $\mathbf{Z}_{M22,opt}$. (b) Analytical and simulation $\mathbf{Z}_{M21,opt}$.

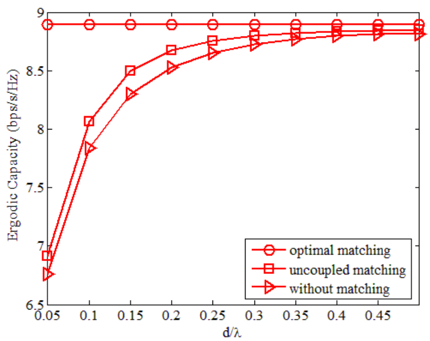


Figure 5. The ergodic capacity of MIMO system ($r_n = 10\,\Omega$, $g_n = 1\,\text{mS}$ and $y_n = (2 + j1)\,\text{mS}$).

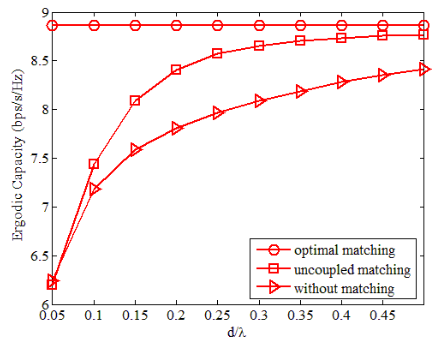


Figure 6. The ergodic capacity of MIMO system ($r_n = 15\,\Omega$, $g_n = 9\,\text{mS}$ and $y_n = (9 + j1)\,\text{mS}$).

$\det(\mathbf{N})$ is obtained via numerical search. Figs. 4(a) and (b) shows the analytical and numerical results. The close match of numerical and analytical results ensures the validity of our derivation in Section 3.

Secondly, we study the effect of matching on MIMO capacity. An optimal matching circuit given in Section 3 is difficult to implement in practice. A practical alternative is to apply each receive antenna the two-port matching circuit that achieves the minimum noise figure for that antenna in isolation. This is called self matching and is accomplished by the circuit $\mathbf{Z}_{M11} = -jx_{11}\mathbf{I}_2$, $\mathbf{Z}_{M12} = \mathbf{Z}_{M21} = -j\sqrt{r_{opt}r_{11}}\mathbf{I}_2$ and $\mathbf{Z}_{M22} = jx_{opt}\mathbf{I}_2$. To demonstrate the effect of matching circuit on spectral efficiency and to illustrate the impact of amplifiers on the capacity of mutual coupled antennas, the mean capacity for optimal matching, self matching and without matching is depicted in Fig. 5. To further study the effect of amplifiers, the mean capacity of MIMO systems with identical amplifiers characterized by $\{r_n, g_n, y_n, f_{\min}\} = \{15\,\Omega, 9\,\text{mS}, (9+j1)\,\text{mS}, 3\,\text{dB}\}$ for optimal-, self-, and without-matching is depicted in Fig. 6. We generated 10,000 instances of MIMO channel and collect the mean capacity. And the reference SISO SNR is 10 dB. It is observed that the performance with optimal matching surpasses self-matching and without-matching schemes. The performance disparity between optimal-matching and other matching schemes is more distinctive for amplifiers with larger noise figure. As antenna separation d increases, the performance difference decreases. Substituting (2), (32) and $\mathbf{\Psi} = \mathbf{R}_R/r_{11}$ into (23), the ergodic capacity with optimal matching in 3D isotropic scattering environment $C_{opt,3D}$ is

$$C_{opt,3D} = E\left[\log_2 \det\left(\mathbf{I}_N + \frac{4kTBP}{Nr_{11}}\mathbf{H}_w\mathbf{H}_w^H\mathbf{F}_{\min}^{-1}\right)\right] \tag{34}$$

Thus, $C_{opt,3D}$ is a function on transmit power, P, antenna self resistance, r_{11}, and noise parameters of amplifiers $\mathbf{F}_{\min}$. $C_{opt,3D}$ is independent of antenna spacing and this can be seen in Figs. 5 and 6.

5. CONCLUSION

This paper derives the optimal matching circuit for capacity maximization of a $N \times N$ compact MIMO system with signal and noise coupling in high SNR scenario. We show that significant improvement can be realized when employing the optimal matching circuit for capacity maximization in compact MIMO systems. The matching circuit is proved to be the one which decouple the array and present isolated, individually noise-matched ports to the amplifier inputs. We have shown that the analytical and simulation results agree well with each other through the example of ideal dipole. We conclude that the performance of MIMO system can be significantly improved by integrating $\mathbf{Z}_{M,opt}$ into compact antenna arrays.

APPENDIX A. DERIVATION OF $\partial H/\partial \mathbf{U}$, $\partial H/\partial \mathbf{V}$, $\partial F/\partial \mathbf{W}$

A.1. Derivation of $\partial h/\partial \mathbf{U}$

Substituting $\mathbf{Z}_{M11} = j(\mathbf{U} - \mathbf{X}_R)$, $\mathbf{Z}_{M22} = j\mathbf{V}$, $\mathbf{Z}_{M21} = j\mathbf{W}$ and $\mathbf{Z}_{M12} = j\mathbf{W}^T$ into (22), thus $\mathbf{N}$ can be rewritten as follows:

$$\begin{aligned}\mathbf{N} \;=\; & \mathbf{R}_R + \mathbf{W}^T\mathbf{G}_n\mathbf{W} + (\mathbf{R}_R + j\mathbf{U})\,\mathbf{W}^{-1}\mathbf{A}_1\mathbf{W}^{-T}(\mathbf{R}_R - j\mathbf{U}) \\ & +\mathbf{W}^T\mathbf{A}_2\mathbf{W}^{-T}(\mathbf{R}_R - j\mathbf{U}) + (\mathbf{R}_R + j\mathbf{U})\,\mathbf{W}^{-1}\mathbf{A}_2^H\mathbf{W}\end{aligned} \tag{A1}$$

where $\mathbf{A}_1 = \mathbf{R}_n + \mathbf{V}\mathbf{G}_n\mathbf{V} + j\mathbf{V}\mathbf{S}_n - j\mathbf{S}_n^H\mathbf{V}$ and $\mathbf{A}_2 = \mathbf{S}_n - j\mathbf{G}_n$. Treating $\mathbf{U}$ as a variable, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ as constants, and using the results in Table 1, we can deduce the following formula

$$d\mathbf{N} = j\,(d\mathbf{U})\,\mathbf{W}^{-1}\mathbf{A}_1\mathbf{W}^{-T}\,(\mathbf{R}_R - j\mathbf{U}) - j\,(\mathbf{R}_R + j\mathbf{U})\mathbf{W}^{-1}\mathbf{A}_1\mathbf{W}^{-T}(d\mathbf{U}) - j\mathbf{W}^T\mathbf{A}_2\mathbf{W}^{-T}\,(d\mathbf{U}) + j\,(d\mathbf{U})\,\mathbf{W}^{-1}\mathbf{A}_2^H\mathbf{W} \tag{A2}$$

Thus $dh = \det(\mathbf{N})\mathrm{Tr}\{\mathbf{N}^{-1}d\mathbf{N}\}$ is

$$dh = \det(\mathbf{N})\,\mathrm{Tr}\left\{\left[j\left(\mathbf{W}^{-1}\mathbf{A}_1\mathbf{B}_1^H + \mathbf{W}^{-1}\mathbf{A}_2^H\mathbf{W}\right)\mathbf{N}^{-1} - j\mathbf{N}^{-1}\left(\mathbf{B}_1\mathbf{A}_1\mathbf{W}^{-T} + \mathbf{W}^T\mathbf{A}_2\mathbf{W}^{-T}\right)\right]d\mathbf{U}\right\} \tag{A3}$$

where $\mathbf{B}_1 = (\mathbf{R}_R + j\mathbf{U})\,\mathbf{W}^{-1}$. Thus, the derivatives with respect to $\mathbf{U}$ of h:

$$\frac{\partial h}{\partial \mathbf{U}} = \det(\mathbf{N})\left[j\left(\mathbf{W}^{-1}\mathbf{A}_1\mathbf{B}_1^H + \mathbf{W}^{-1}\mathbf{A}_2^H\mathbf{W}\right)\mathbf{N}^{-1} - j\mathbf{N}^{-1}\left(\mathbf{B}_1\mathbf{A}_1\mathbf{W}^{-T} + \mathbf{W}^T\mathbf{A}_2\mathbf{W}^{-T}\right)\right]^T \tag{A4}$$

A.2. Derivation of $\partial h/\partial \mathbf{W}$

In (A1), treating $\mathbf{W}$ as a variable, $\mathbf{V}$, $\mathbf{U}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ as constants, we can deduce the following formula

$$d\mathbf{N} = (d\mathbf{W})^T\,\mathbf{G}_n\mathbf{W} + \mathbf{W}^T\mathbf{G}_n\,(d\mathbf{W}) - \mathbf{B}_1\,(d\mathbf{W})\,\mathbf{W}^{-1}\mathbf{A}_1\mathbf{B}_1^H - \mathbf{B}_1\mathbf{A}_1\mathbf{W}^{-T}\,(d\mathbf{W})^T\,\mathbf{B}_1^H + (d\mathbf{W})^T\,\mathbf{A}_2\mathbf{B}_1^H - \mathbf{W}^T\mathbf{A}_2\mathbf{W}^{-T}\,(d\mathbf{W})^T\,\mathbf{B}_1^H - \mathbf{B}_1\,(d\mathbf{W})\,\mathbf{W}^{-1}\mathbf{A}_2^H\mathbf{W} + \mathbf{B}_1\mathbf{A}_2^H\,(d\mathbf{W}) \tag{A5}$$

Thus, we can deduce

$$dh = \det(\mathbf{N})\,\mathrm{Tr}\Big\{\left[\mathbf{N}^{-1}\left(\mathbf{W}^T\mathbf{G}_n + \mathbf{B}_1\mathbf{A}_2^H\right) - \mathbf{W}^{-1}\left(\mathbf{A}_1\mathbf{B}_1^H + \mathbf{A}_2^H\mathbf{W}\right)\mathbf{N}^{-1}\mathbf{B}_1\right](d\mathbf{W}) \left[\left(\mathbf{G}_n\mathbf{W} + \mathbf{A}_2\mathbf{B}_1^H\right)\mathbf{N}^{-1} - \mathbf{B}_1^H\mathbf{N}^{-1}\left(\mathbf{B}_1\mathbf{A}_1 + \mathbf{W}^T\mathbf{A}_2\right)\mathbf{W}^{-T}\right] (d\mathbf{W})^T\Big\} \tag{A6}$$

From this, the derivatives with respect to $\mathbf{W}$ of h can be derived

$$\frac{\partial h}{\partial \mathbf{W}} = \det(\mathbf{N})\left\{\left[\mathbf{N}^{-1}\left(\mathbf{W}^T\mathbf{G}_n + \mathbf{B}_1\mathbf{A}_2^H\right) - \mathbf{W}^{-1}\left(\mathbf{A}_1\mathbf{B}_1^H + \mathbf{A}_2^H\mathbf{W}\right)\mathbf{N}^{-1}\mathbf{B}_1\right]^T + \left[\left(\mathbf{G}_n\mathbf{W} + \mathbf{A}_2\mathbf{B}_1^H\right)\mathbf{N}^{-1} - \mathbf{B}_1^H\mathbf{N}^{-1}\left(\mathbf{B}_1\mathbf{A}_1 + \mathbf{W}^T\mathbf{A}_2\right)\mathbf{W}^{-T}\right]\right\} \tag{A7}$$

A.3. Derivation of $\partial h/\partial \mathbf{V}$

(A1) can be rewritten as

$$\begin{aligned}\mathbf{N} \;=\; & \mathbf{R}_R + \mathbf{B}_1\mathbf{R}_n\mathbf{B}_1^H + \mathbf{W}^T\mathbf{G}_n\mathbf{W} + \mathbf{W}^T\mathbf{S}_n\mathbf{B}_1^H + \mathbf{B}_1\mathbf{S}_n^H\mathbf{W} \\ & + \mathbf{B}_1\mathbf{V}\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H + \mathbf{B}_1\mathbf{V}\mathbf{B}_2 + \mathbf{B}_2^H\mathbf{V}\mathbf{B}_1^H \end{aligned} \tag{A8}$$

where $\mathbf{B}_2 = j\left(\mathbf{G}_n\mathbf{W} + \mathbf{S}_n\mathbf{B}_1^H\right)$. Treating $\mathbf{V}$ as a variable, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{W}$ and $\mathbf{U}$ as constants, we can deduce

$$d\mathbf{N} = \mathbf{B}_1(d\mathbf{V})\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H + \mathbf{B}_1\mathbf{V}\mathbf{G}_n(d\mathbf{V})\mathbf{B}_1^H + \mathbf{B}_1(d\mathbf{V})\mathbf{B}_2 + \mathbf{B}_2^H(d\mathbf{V})\mathbf{B}_1^H \tag{A9}$$

Thus, we can deduce

$$\begin{aligned} dh = \det(\mathbf{N})\,\mathrm{Tr}\Big\{ & \Big[\left(\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H + \mathbf{B}_2\right)\mathbf{N}^{-1}\mathbf{B}_1 \\ & + \mathbf{B}_1^H\mathbf{N}^{-1}\left(\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H + \mathbf{B}_2\right)^H\Big](d\mathbf{V})\Big\} \end{aligned} \tag{A10}$$

From (A10), the derivatives with respect to $\mathbf{V}$ of h can be derived

$$\frac{\partial h}{\partial \mathbf{N}} = \det(\mathbf{N})\left[\left(\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H + \mathbf{B}_2\right)\mathbf{N}^{-1}\mathbf{B}_1 + \mathbf{B}_1^H\mathbf{N}^{-1}\left(\mathbf{G}_n\mathbf{V}\mathbf{B}_1^H + \mathbf{B}_2\right)^H\right]^T \tag{A11}$$

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