Progress In Electromagnetics Research, Vol. 105, 171–191, 2010

DERIVATION OF KLEIN-GORDON EQUATION FROM MAXWELL'S EQUATIONS AND STUDY OF RELATIVISTIC TIME-DOMAIN WAVEGUIDE MODES

O. A. Tretyakov † and O. Akgun

Department of Electronics Engineering
Gebze Institute of Technology
Gebze, Kocaeli, Turkey

Abstract—An initial-boundary value problem for the system of Maxwell's equations with time derivative is formulated and solved rigorously for transient modes in a hollow waveguide. It is supposed that the latter has perfectly conducting surface. Cross section, S, is bounded by a closed singly-connected contour of arbitrary but smooth enough shape. Hence, the TE and TM modes are under study. Every modal field is a product of a vector function of transverse coordinates and a scalar amplitude dependent on time, t, and axial coordinate, z. It has been established that the study comes down to, eventually, solving two autonomous problems: *i) A modal basis problem.* Final result of this step is definition of complete (in Hilbert space, $L_2(S)$) set of functions dependent on transverse coordinates which originates a basis. *ii) A modal amplitude problem.* The amplitudes are generated by the solutions to Klein-Gordon equation (KGE), derived from Maxwell's equations directly, with t and z as independent variables. The solutions to KGE are invariant under relativistic Lorentz transforms and subjected to the causality principle. Special attention is paid to various ways that lead to analytical solutions to KGE. As an example, one case (among eleven others) is considered in detail. The modal amplitudes are found out explicitly and expressed via products of Airy functions with arguments dependent on t and z.

Corresponding author: O. A. Tretyakov (o.tretyakov@gmail.com).

† Also on leave from the Chair of Theoretical Radio Physics, Kharkov National University, Kharkov, Ukraine.

1. INTRODUCTION

In this work, a readily available technique is proposed for the study of electromagnetic field propagation in waveguides accompanied with transient processes in the time domain. This technique has been developed for solving an initial-boundary value problem for the system of Maxwell's equations with time derivative, ∂_t. It is known that Maxwell's equations with ∂_t are invariant under relativistic Lorentz transforms. The proposed technique leads to the complete set of TE and TM waveguide modes, each of which has the same relativistic properties as well. It is worth noting that relativistic Lorentz transforms are inapplicable to the time-harmonic waveguide waves in the frequency domain studied more than a century.

All time-domain methods, apart from the numerical ones, are based on obtaining Klein-Gordon equation (KGE) from Maxwell's equations with ∂_t, in one way or the other, since KGE is a relativistic equation as well. To our best knowledge, Gabriel was the first who succeeded in it starting directly from the relativistic covariant form of Maxwell's equations and resting on a new theorem on a class of four-dimensional skew-symmetric tensors [1]. His work was concerned with analysis of a network formalism within the concepts of electromagnetism.

Another approach to time-domain electromagnetics was developed in 80s [2, 3]. The key point of the studies is to extract some self-adjoint operators acting on the transverse waveguide coordinates only, and to preserve time as an independent variable in the process of solving Maxwell's equations with ∂_t. This procedure leads to derivation of KGE with time, t, and axial coordinate, z, as independent variables. The solutions to KGE serve as the potentials generating time-dependent modal amplitudes that are also relativistic.

One more time-domain study on the propagation of transient electromagnetic waves in waveguides was performed in [4]. In this work, an elegant method relying on wave splitting technique has been proposed. The field in the waveguide is represented exactly as a time convolution of a Green function and a source function. Exact numerical data can be obtained effectively at small distances between source point and the point of observation. However, when the distances are large, the convolution integral requires an approximation yielding loss of accuracy in the long run.

Comprehensive time-domain studies of nonsinusoidal signals describable by Walsh functions was performed by Harmuth[‡]. To this

[‡] Dr. Henning F. Harmuth is the author of more than 150 journal publications and of fifteen books. Complete information about his input in science and digital communication

purpose, he amended Maxwell's equations by introducing magnetic current density as a real physical quantity and also adding magnetic equivalent of Ohm's law to the set of constitutive relations. It is possible, however, to study propagation of digital signals in waveguides and oscillations in cavities excited by Walsh signals still remaining within the scope of generally accepted theory [6, 7].

It is pertinent to note that there are other interesting publications listed chronologically [8–18]. It is possible to find herein useful mathematical details on the analytical methods and description of various physical phenomena extracted from the analytical solutions.

Certainly, powerful computational methods delivered with the impressive numerical results in the last decades. However, it is appropriate to mention here that a physical explanation of the numerical data can be achieved only under the presence of analytical results.

Our method[§] is based on solving differential Maxwell's equations presentable in a transverse-longitudinal form where ∂_t participates. The sets of TE and TM time-domain modes, each of which is complete in Hilbert space, $L_2(S)$, have been derived. Every modal field is a product of a vector function of transverse coordinates and a scalar amplitude dependent on time, t, and axial coordinate, z. Study of the waveguide modes comes down to solving two autonomous problems, ultimately. The first is a modal basis problem. The vector functions of transverse coordinates are elements of the modal basis. Actually, they are the same as for the time-harmonic modes and hence, this facilitates the time-domain studies. The second one is a modal amplitude problem. The modal amplitudes both for TE and TM modes are generated by the solutions to KGE with appropriate initial conditions. This equation, derived directly from Maxwell's equations, involves all information about the type of modes (TE or TM) and the contour of waveguide cross section (i.e., its form and size). There are two ways in solving KGE rigorously. A standard way results in a convolution of time-domain Green function and a source function [19]. However, the convolution integral fails in calculations when t and z become relatively large due to fast oscillations of the kernel in the integrand. We solved this problem in a new way where the convolution integral does not participate. It is explained in Section 4. We succeeded in an analytical expansion of the initial data, given on time semi-axis, $t \geq 0$, over the whole domain of propagation, $0 \leq z \leq ct$, where c is the light velocity

technology can be found in the special issue of the journal *Electromagnetic Phenomena,* which is devoted to his 80 years anniversary, by link [5].

[§] Rather simple mathematical technique was used for development, as compared to our previous and other time-domain methods. Only vector analysis, backgrounds of the partial and ordinary differential equation theories and electromagnetic field theory are needed.

in vacuum. Eventually, the solutions are expressed explicitly in terms of well-studied special functions of mathematical physics. This way is equivalent to analytical calculation of the convolution integral without approximations of its kernel.

In this article, we consider a special case where the modal amplitudes are expressible explicitly via Airy functions. It is worth noting that Airy functions appear as well in the study of Airy light beams in optics. Comprehensive information on this topic can be found in recent article [20]. In the light beam theory, Airy equation appears as a result of paraxial *approximation* of a scalar wave equation. In our theory, however, Airy equation is obtained as an exact form of KGE which is derived *rigorously* from Maxwell's equations with ∂_t.

2. STATEMENT OF THE PROBLEM

In this work, we consider an ideal hollow (i.e., medium-free) waveguide having a cross-section domain, S, invariable along the waveguide axis, Oz. The domain S is bounded by a closed singly-connected contour, L. We also assume that the contour has an arbitrary, but smooth enough shape implying that none of its possible inner angles (i.e., measured within S) exceed π. In the rest of the work, we will make use of a right-handed triplet $(\mathbf{z}, \mathbf{l}, \mathbf{n})$ of mutually orthogonal unit vectors ($\mathbf{z} \times \mathbf{l} = \mathbf{n}$, and so on) where the vector $\mathbf{n}$ is the outer normal to the domain S, $\mathbf{z}$ is oriented along the axis Oz and $\mathbf{l}$ is tangential to the contour L. We now denote a point of observation within the waveguide by a three-component vector $\mathbf{R}$ and an observation time t.

2.1. Standard Formulation of the Problem

We have to solve the system of vectorial Maxwell's equations

$$\nabla \times \mathcal{E}(\mathbf{R}, t) = -\mu_0\, \partial_t \mathcal{H}(\mathbf{R}, t) \quad \text{and} \quad \nabla \times \mathcal{H}(\mathbf{R}, t) = \epsilon_0\, \partial_t \mathcal{E}(\mathbf{R}, t) \quad (1)$$

for the electric and magnetic field strength vectors $\mathcal{E}$ and $\mathcal{H}$, respectively. The scalar Maxwell's equations

$$\nabla \cdot \mathcal{E}(\mathbf{R}, t) = 0 \quad \text{and} \quad \nabla \cdot \mathcal{H}(\mathbf{R}, t) = 0 \tag{2}$$

will be used as well, although they are the corollaries of the Eq. (1).

Equations (1) and (2) hold within the waveguide volume, except for its surface. Assuming that the surface has the physical properties of the perfect electric conductor, we subject the field components to the following boundary conditions

$$(\mathbf{n} \cdot \mathcal{H})|_L = 0, \quad (\mathbf{l} \cdot \mathcal{E})|_L = 0 \quad \text{and} \quad (\mathbf{z} \cdot \mathcal{E})|_L = 0 \tag{3}$$

where the center dot stands for the scalar multiplication.

As being partial differential equations (PDEs) of the hyperbolic kind, a solution to Eq. (1) should be subjected to some given initial conditions as

$$\mathcal{E}(\mathbf{R},0)=\mathbf{0} \quad \text{and} \quad \mathcal{H}(\mathbf{R},0)=\mathbf{0}\,. \tag{4}$$

Thus, we have to solve the initial-boundary value problem (1)–(4) in a class of the *real-valued* functions, provided that the expected solution has appropriate physical properties of integrability. To this aim, we introduce the following quadric characteristics for the real-valued vectorial fields $\mathcal{E}$ and $\mathcal{H}$

$$\begin{aligned} W(z,t) &= \frac{1}{2S}\int_S \{\epsilon_0(\mathcal{E}\cdot\mathcal{E})+\mu_0(\mathcal{H}\cdot\mathcal{H})\}\,ds \\ \mathcal{P}_z(z,t) &= \frac{1}{S}\int_S \mathbf{z}\cdot[\mathcal{E}\times\mathcal{H}]\,ds \end{aligned} \tag{5}$$

where the center cross stands for the vectorial multiplication of the vectors. The integration is performed at a fixed instant of time t over the waveguide cross section S, located at a fixed coordinate z. Quantity $\mathcal{P}_z(z,t)$ is z-component of the Poynting's vector and $W(z,t)$ is the energy density of the field (both averaged over S), which are stored in the same point z and instant t. The Poynting Theorem, applied to Eq. (1) , yields the following differential equation

$$\partial_z\mathcal{P}_z(z,t)+\partial_t W(z,t)=0. \tag{6}$$

The differential form for the law of conservation of the averaged energy given in (6) is valid for the time-domain solutions. Square integrability in Eq. (5) specifies the global property of the expected solution, whereas the differentiability by the variables z and t in Eq. (6) specifies the local property.

Maxwell's equations (1) are invariant under the relativistic Lorentz transforms. Hence, solution to the problem should be found out in compliance with the *causality principle.* This aspect will be discussed in the later sections.

2.2. Transverse-longitudinal Decompositions

The three-component position vector, $\mathbf{R}$, and the vectorial operator nabla, ∇, can be presented as

$$\mathbf{R}=\mathbf{r}+\mathbf{z}\,z \quad \text{and} \quad \nabla=\nabla_\perp+\mathbf{z}\,\partial_z \tag{7}$$

where $\mathbf{r}$ is a projection of the vector $\mathbf{R}$ on the domain S, the differential procedure $\nabla_\perp$ acts on the transverse coordinates $(\mathbf{r})$, but it takes the variable z as a constant. In a similar fashion, we present the field vectors $\mathcal{E}$ and $\mathcal{H}$ as the sums of their transverse and longitudinal components

$$\begin{aligned}\mathcal{E}\left(\mathbf{R},t\right)&=\mathbf{E}\left(\mathbf{r},z,t\right)+\mathbf{z}\,E_z\left(\mathbf{r},z,t\right)\\ \mathcal{H}\left(\mathbf{R},t\right)&=\mathbf{H}\left(\mathbf{r},z,t\right)+\mathbf{z}\,H_z\left(\mathbf{r},z,t\right).\end{aligned}\tag{8}$$

Decompose vectorial Maxwell's equations (1) onto their transverse and longitudinal parts and add Eq. (2) to that result. These efforts can be grouped into two systems of equations. One of them includes only E_z component

$$\begin{cases}\nabla_\perp E_z=\mu_0\partial_t[\mathbf{H}\times\mathbf{z}]+\partial_z\mathbf{E} & (a)\\ \epsilon_0\partial_t E_z=\nabla_\perp\cdot[\mathbf{H}\times\mathbf{z}] & (b)\\ \partial_z E_z=-\nabla_\perp\cdot\mathbf{E}. & (c)\end{cases}\tag{9}$$

The other one includes only H_z component

$$\begin{cases}\nabla_\perp H_z=\epsilon_0\partial_t[\mathbf{z}\times\mathbf{E}]+\partial_z\mathbf{H} & (a)\\ \mu_0\partial_t H_z=\nabla_\perp\cdot[\mathbf{z}\times\mathbf{E}] & (b)\\ \partial_z H_z=-\nabla_\perp\cdot\mathbf{H}. & (c)\end{cases}\tag{10}$$

As is easy to see, Eq. (2) are placed in the lines (c) in (9) and (10).

The first pair of the boundary conditions (3) yields

$$\left(\mathbf{n}\cdot\mathbf{H}\right)|_L=0\quad\text{and}\quad\left(\mathbf{l}\cdot\mathbf{E}\right)|_L=0.\tag{11}$$

Condition $E_z|_L=0$ in Eq. (3) yields the following pair

$$\left(\nabla_\perp\cdot[\mathbf{H}\times\mathbf{z}]\right)|_L=0\quad\text{and}\quad\left(\nabla_\perp\cdot\mathbf{E}\right)|_L=0.\tag{12}$$

This follows from the equations which are placed in the lines (b) and (c) in (9). Later on, we shall establish under which stipulation the boundary conditions (12) are the corollaries of the conditions (11).

3. SOLVING THE PROBLEM

We approach the problem of solving (9)–(12) by the method of incomplete separation of the variables [19]. In our case, it implies that every field component participating in (9) and (10) can be presented as a product of two factors. One should be dependent on the transverse coordinates ($\mathbf{r}$) only and the other one should be dependent on the variables (z, t) solely.

3.1. Complete Set of the *TE* Time-domain Modes

The TE-modes are specified by the condition $E_z\left(\mathbf{r},z,t\right)\equiv 0$. Inserting this condition to the system (9), the equations placed in the lines (b) and (c) should give identity $0=0$ each. This suggests

$$\begin{aligned}\mathbf{E}\left(\mathbf{r},z,t\right)&=V\left(z,t\right)\left[\nabla_\perp\psi\left(\mathbf{r}\right)\times\mathbf{z}\right]\\ \mathbf{H}\left(\mathbf{r},z,t\right)&=I\left(z,t\right)\nabla_\perp\psi\left(\mathbf{r}\right)\end{aligned}\tag{13}$$

because the identity $[\nabla_\perp \times \nabla_\perp \psi(\mathbf{r})] \equiv \mathbf{0}$ holds. The functions $\psi(\mathbf{r})$, $V(z,t)$ and $I(z,t)$ should be found out thereafter.

Consider now the system (10). Choose H_z as follows $H_z = A(z,t)\psi(\mathbf{r})$, where $A(z,t)$ is one more unknown function which should be found out later. Substitution of the expression for H_z and the vectors **E** and **H**, given in (13), into the Eqs. (b) and (c) in (10) yields, respectively,

$$\begin{aligned} -\mu_0 \psi(\mathbf{r})\, \partial_t A(z,t) &= [-\nabla_\perp^2 \psi(\mathbf{r})] V(z,t) \\ \psi(\mathbf{r})\, \partial_z A(z,t) &= [-\nabla_\perp^2 \psi(\mathbf{r})] I(z,t)\ . \end{aligned} \tag{14}$$

Finally, Eq. (a) of the system (10) supplies with

$$A(z,t) = \epsilon_0 \partial_t V(z,t) + \partial_z I(z,t)\ . \tag{15}$$

Each condition in (11) for the vectors **E** and **H** yields the same boundary condition for the function $\psi(\mathbf{r})$ as

$$\partial_{\mathbf{n}} \psi(\mathbf{r})\,|_L = 0 \tag{16}$$

where $\partial_{\mathbf{n}} = \mathbf{n} \cdot \nabla_\perp$ is the normal derivative on the contour L.

Substitution of the definitions (13) for the vectors into the boundary conditions (12) both yields identity $0 = 0$ each.

It is clear from (14) that the potential $\psi(\mathbf{r})$ should be a twice differentiable function. Following Sturm-Liouville Theorems, the *boundary condition* (16) suggests using Neumann boundary eigenvalue problem for transverse Laplacian, $\nabla_\perp^2$, as

$$\nabla_\perp^2 \psi_n(\mathbf{r}) + \nu_n^2 \psi_n(\mathbf{r}) = 0 \quad \text{and} \quad \partial_{\mathbf{n}} \psi_n(\mathbf{r})\,|_L = 0 \tag{17}$$

where $\mathbf{r} \in S$, the eigenvalues $\nu_n^2 \geq 0$ $(n = 0, 1, 2, \ldots)$ are the discrete values of a spectral parameter ν. Discreteness of the spectrum is caused by the boundedness of the domain S. The elements of spectrum $\{\nu_n^2\}_{n=0}^{\infty}$ are real numbers. The spectrum has a single point of condensation located at infinity. The subscript n regulates the distribution of the eigenvalues on a real axis in increasing order of their numerical values.

The set $\{\psi_n(\mathbf{r})\}_{n=0}^{\infty}$ of the eigensolutions to the problem (17), corresponding to all eigenvalues from the set $\{\nu_n^2\}_{n=0}^{\infty}$, is complete in $L_2(S)$ (for a proof of completeness, the readers are referred to [3]). Hence, any potential $\psi(\mathbf{r})$ satisfying (16) can be presented as a decomposition with making use of the functions $\psi_n(\mathbf{r})$ as the basis elements. To this aim, the functions $\psi_n(\mathbf{r})$ should be normalized later.

The complete set of functions ψ_n generates a complete set of the TE-modes in the time domain, see [3]. In (13) and (14), an eigenfunction ψ_n can be taken as the potential ψ. Substituting

$[-\nabla_\perp^2 \psi_n] = \nu_n^2 \psi_n$ to the formula (14) and taking $A(z,t) = \nu_n^2 h_n(z,t)$, we obtain, respectively,

$$V_n(z,t) = -\mu_0 \partial_t h_n(z,t) \quad \text{and} \quad I_n(z,t) = \partial_z h_n(z,t) . \tag{18}$$

Substitution of (18) into Eq. (15) and inserting $\nu_n^2 h_n$ for A result in a differential equation for $h_n(z,t)$

$$\partial_{ct}^2 h_n(z,t) - \partial_z^2 h_n(z,t) + \nu_n^2 h_n(z,t) = 0 \tag{19}$$

where $\partial_{ct} = \partial / c\partial t$ and $c = 1/\sqrt{\epsilon_0 \mu_0}$. Eq. (19) is known as *Klein-Gordon equation* (*KGE*) in mathematical physics [21].

To sum it up, all the field components can be listed as

$$\begin{aligned} E'_{z\,n}(\mathbf{r},z,t) &= 0 \\ \mathbf{E}'_n(\mathbf{r},z,t) &= [-\partial_t h_n(z,t)]\, \mu_0 \nabla_\perp \psi_n(\mathbf{r}) \times \mathbf{z} \\ \mathbf{H}'_n(\mathbf{r},z,t) &= [\partial_z h_n(z,t)]\, \nabla_\perp \psi_n(\mathbf{r}) \\ H'_{z\,n}(\mathbf{r},z,t) &= [\nu_n h_n(z,t)]\, \nu_n \psi_n(\mathbf{r}) \end{aligned} \tag{20}$$

where $n = 1, 2, \ldots$ and prime ($'$) is used for notation of the TE-modes. Note that the terms in the square brackets of (20) are the modal amplitudes in physical sense. In order to provide the field components with appropriate physical dimensions, it is necessary to normalize the functions $\psi_n(\mathbf{r})$ properly. Making use of Eq. (5), the functions $\psi_n(\mathbf{r})$ for $n \neq 0$ can be normalized as

$$\frac{\mu_0 \nu_n^2}{S} \int_S \psi_n^2(\mathbf{r})\, ds = 1 . \tag{21}$$

This enables us to rewrite the quadric characteristics of the field (20) in the simplest form as

$$\begin{aligned} W'_n(z,t) &= \left[(\partial_{ct} h_n)^2 + (\partial_z h_n)^2 + \nu_n^2 h_n^2 \right] /2 \\ \mathcal{P}'_{z\,n}(z,t) &= (-\partial_t h_n)(\partial_z h_n) . \end{aligned} \tag{22}$$

This coincides with our previous result obtained by a more general method requiring a deep mathematical background, see in [6].

The infinite set (20) of TE-modes is generated by the eigensolutions to the problem (17), corresponding to all the eigenvalues $\nu_n^2 \neq 0$ *only.* However, the problem (17) has one more solution, $\psi_0(\mathbf{r})$, corresponding to the eigenvalue $\nu_0^2 = 0$, which is distinct from zero. Minimum-Maximum Theorem for the harmonic functions asserts that it is a constant: $\psi_0(\mathbf{r}) = C$ for $\mathbf{r} \in L$ and $\mathbf{r} \in S$. In turn, the solution ψ_0 generates a one-component modal field as

$$\mathcal{E}'_0(\mathbf{r},z,t) = \mathbf{0} \quad \text{and} \quad \mathcal{H}'_0(\mathbf{r},z,t) = \mathbf{z}\, h_0(z,t)\; C. \tag{23}$$

In the case of the hollow waveguide, the function $h_0(z,t)$ is a constant, as well. This fact follows from the equations which are placed in the lines (b) and (c) in the system (10). Without loss of generality, we may assume that $C = h_0 = 1$.

Potentials $\psi_n(\mathbf{r})$, generating the modal field patterns in the waveguide cross section, coincide with those known for the time-harmonic waves. It can be illustrated with the following example.

Example 1 The typical contour of waveguide cross section is rectangular, say, $0 \le x \le a$, $0 \le y \le b$. This contour is smooth enough; none of its inner angles (equal to $\pi/2$ each) exceed π. The eigensolutions to Neumann problem (17) are

$$\psi_n(\mathbf{r}) \equiv \psi_n(x, y) = A'_n \cos(\pi p x/a) \cos(\pi q y/b) \tag{24}$$

where A'_n is the normalization constant, p and q are the integers. The eigenfunctions (24) correspond to the eigenvalues

$$\nu_n^2 = (\pi p/a)^2 + (\pi q/b)^2 . \tag{25}$$

Normalization condition (21) yields the constants A'_n as

$$A'_n = \sqrt{(2-\delta_{p,0})(2-\delta_{q,0})} / (\nu_n \sqrt{\mu_0}) \tag{26}$$

where $\delta_{a,b}$ is Kronecker delta. Any combination of the integers $p = 0, 1, 2, \ldots$ and $q = 0, 1, 2, \ldots$ is available, provided that $p + q \neq 0$. The subscript n, which was introduced above as a regulator of positions of the eigenvalues on the real axis, can be interpreted as the doublet: $n \to (p, q)$.

Thus, we have established the complete set of the waveguide TE-modes in the time domain. Their field components are given in Eqs. (20) and (23). The set of TE-modes is complete in $L_2(S)$. This fact has been proven also resting on Weyl Theorem on orthogonal detachments of $L_2(S)$ [3]. The modal amplitudes depend on the axial coordinate z and time t. They can be found via solving a one-dimensional KGE (19). This equation is invariant under the relativistic Lorentz transforms. Hence, all the TE-time-domain modes obtained above have the same relativistic properties as well.

3.2. Complete Set of the *TM* Time-domain Modes

The TM-modes are specified by the condition $H_z(\mathbf{r}, z, t) \equiv 0$. Under this condition, the systems (9) and (10) can be solved in a similar fashion as was used for the TE-modes.

The set of TM-modes is generated by the complete set of eigensolutions to Dirichlet boundary eigenvalue problem

$$\begin{aligned} \nabla_\perp^2 \phi_n(\mathbf{r}) + \kappa_n^2 \phi_n(\mathbf{r}) &= 0 \\ \phi_n(\mathbf{r})|_L &= 0 \\ \frac{\epsilon_0 \kappa_n^2}{S} \int_S \phi_n^2(\mathbf{r})\, ds &= 1 \end{aligned} \tag{27}$$

where $\mathbf{r} \in S$, the eigenvalues $\kappa_n^2 \geq 0$ $(n = 0, 1, 2, \ldots)$ are discrete values of a spectral parameter κ, and $\phi_n(\mathbf{r})$ are the eigenfunctions corresponding to these eigenvalues. For $n = 0$, $\kappa_0^2 = 0$ is also an eigenvalue and $\phi_0(\mathbf{r})$ is an eigenfunction corresponding to it. When the contour L is singly connected, then $\phi_0(\mathbf{r}) = 0$. This follows from the Minimum-Maximum Theorem for the harmonic functions.

The complete set $\{\phi_n(\mathbf{r})\}_{n=1}^{\infty}$ of the eigenfunctions ϕ_n generates, respectively, the complete set of the TM-modes. The field components are

$$
\begin{aligned}
H''_{z\,n}(\mathbf{r},z,t) &= 0 \\
\mathbf{H}''_n(\mathbf{r},z,t) &= [-\partial_t e_n(z,t)]\, \mathbf{z} \times \epsilon_0 \nabla_\perp \phi_n(\mathbf{r}) \\
\mathbf{E}''_n(\mathbf{r},z,t) &= [\partial_z e_n(z,t)]\, \nabla_\perp \phi_n(\mathbf{r}) \\
E''_{z\,n}(\mathbf{r},z,t) &= [\kappa_n e_n(z,t)]\, \kappa_n \phi_n(\mathbf{r})
\end{aligned}
\tag{28}
$$

where $n = 0, 1, 2, \ldots$ and double prime $('')$ is used for notation of the TM-modes. The time dependent modal amplitudes in the square brackets are specified by the function $e_n(z,t)$ obeying KGE as well

$$
\partial_{ct}^2 e_n(z,t) - \partial_z^2 e_n(z,t) + \kappa_n^2\, e_n(z,t) = 0\,. \tag{29}
$$

The quadric field characteristics for the TM-modes

$$
\begin{aligned}
W''_n(z,t) &= \left[(\partial_{ct} e_n)^2 + (\partial_z e_n)^2 + \kappa_n^2 e_n^2\right]/2 \\
\mathcal{P}''_{z\,n}(z,t) &= (-\partial_t e_n)(\partial_z e_n)
\end{aligned}
\tag{30}
$$

results from the normalization condition involved in (27).

Boundary conditions (12) follow from conditions (11). This is because the potentials ϕ_n obey Helmholtz equation present in (27).

In addition to Example 1, the potentials $\phi_n(\mathbf{r})$ specifying the modal field patterns in the waveguide cross section can be found also.

Example 2 Solving Dirichlet problem (27) for the same rectangular cross section $0 \leq x \leq a$, $0 \leq y \leq b$ results in

$$
\phi_n(\mathbf{r}) \equiv \phi_n(x,y) = A''_n \sin(\pi p x/a) \sin(\pi q y/b) \tag{31}
$$

where $p = 1, 2, \ldots$ and $q = 1, 2, \ldots$. These eigensolutions correspond to the following eigenvalues

$$
\kappa_n^2 = (\pi p/a)^2 + (\pi q/b)^2\,. \tag{32}
$$

Again, $n \to (p,q)$. The normalization constants A''_n are equal to

$$
A''_n = 2/\left(\kappa_n \sqrt{\epsilon_0}\right). \tag{33}
$$

Thus, we have obtained the complete set of the TM-modes. Completeness in $L_2(S)$ has been proven also in [3]. The TM-modes possess all the appropriate relativistic properties in the time domain similarly to the TE-modes.

3.3. Several General Remarks

1. The presented theory can be extended on the waveguides with *multi-connected* cross sections. For example, a coaxial waveguide is in general usage, practically. Its cross section has a doubly-connected domain S. In the studies of the multi-connected waveguides, the eigensolutions to Dirichlet boundary eigenvalue problem (27) corresponding to the eigenvalue $\kappa_0^2 = 0$ should be carefully considered. In this case, the formulation of (27) should be set as follows

$$\nabla_\perp^2 \phi_k^{(i)}(\mathbf{r}) = 0 \quad \text{and} \quad \phi_k^{(i)}(\mathbf{r})\,|_{L_i} = C_i \tag{34}$$

where $i, k = 1, 2, \ldots, N$, N is a number of components in the multi-connected contour L, L_i are the components of $L : L_1, L_2, \ldots, L_N$. The constants C_i are *distinct* for different components L_i . A complete set of a finite number of linearly independent solutions to (34) generates the finite set of the TEM-modes. The constant C_i can be specified via applying the Gramm-Schmidt orthonormalization procedure for the functions $\phi_k^{(i)}$. Useful details can be found in [3].

2. In the case of the time-harmonic fields propagating along Oz axis, for example, KGE has the following particular solution

$$e_n(z,t) = A_n\left[\sin\left(\omega t - \gamma_n z + \varphi_n\right)\right] \tag{35}$$

where A_n and φ_n are real constants, $\gamma_n = \sqrt{\omega^2\epsilon_0\mu_0 - k_n^2}$, ω and k_n^2 are parameters, ω is a given frequency. For the TM-modes, k_n^2 should be replaced by the eigenvalue κ_n^2, but for the TE-modes k_n^2 is ν_n^2.

Condition $\gamma_n = 0$ specifies the cut-off frequencies of the time-harmonic modes. The eigenvalues ν_n^2 specify the cut-off frequencies ω_n' of the TE-modes, $\omega_n' = c\nu_n$. In turn, the eigenvalues κ_n^2 specify the cut-off frequencies ω_n'' of the TM-modes, $\omega_n'' = c\kappa_n$.

3. In various books, KGE is given in the following canonical form

$$\partial_\tau^2 f(\xi,\tau) - \partial_\xi^2 f(\xi,\tau) + f(\xi,\tau) = 0. \tag{36}$$

The KGEs, both (19) and (29), have this form after scaling real time t and coordinate z as follows

$$\begin{aligned} \tau = \nu_n c\, t = 2\pi\, t/T_n', \quad \xi = \nu_n\, z = 2\pi\, z/\lambda_n' \quad \Leftarrow TE \\ \tau = \kappa_n c\, t = 2\pi\, t/T_n'', \quad \xi = \kappa_n\, z = 2\pi\, z/\lambda_n'' \quad \Leftarrow TM \end{aligned} \tag{37}$$

where T_n' and T_n'' are the periods corresponding to the cut-off frequencies $\omega_n' = \nu_n c$ and $\omega_n'' = \kappa_n c$, respectively. Similarly, λ_n' and λ_n'' are the cut-off wavelengths. When the scaled variables are chosen as $\tau = \nu_n ct$ and $\xi = \nu_n z$, the solution $f(\xi,\tau)$ to KGE (36) is $h_n(\xi,\tau)$, otherwise, $f(\xi,\tau)$ is $e_n(\xi,\tau)$.

Equation (36) should be supplemented with a pair of "initial" conditions as

$$f\left(\xi,\tau\right)|_{\xi=0} = \varsigma\left(\tau\right) \quad \text{and} \quad \partial_\tau f\left(\xi,\tau\right)|_{\xi=0} = \upsilon\left(\tau\right) \tag{38}$$

where the functions $\varsigma(\tau)$ and $\upsilon(\tau)$ playing role of the source functions are given on semi-axis $\tau \geq 0$ but $\varsigma(\tau) = \upsilon(\tau) = 0$ if $\tau < 0$.

Thus, the variables τ and ξ have different numerical scaling t and z, respectively, for different modal fields. However, the TE- and TM-modal fields can be rewritten in a common form which enables studying the amplitudes of the field components *in parallel.* Simple manipulations with Eqs. (20), (28) yield

$$\begin{aligned} E'_{z\,n}\left(\mathbf{r},\xi,\tau\right) &= 0 \\ \nu_n^{-1}\sqrt{\epsilon_0/\mu_0}\,\mathbf{E}'_n\left(\mathbf{r},\xi,\tau\right) &= \mathcal{A}\left(\xi,\tau\right)\,\nabla_\perp\psi_n\left(\mathbf{r}\right)\times\mathbf{z} \\ \nu_n^{-1}\mathbf{H}'_n\left(\mathbf{r},\xi,\tau\right) &= \mathcal{B}\left(\xi,\tau\right)\,\nabla_\perp\psi_n\left(\mathbf{r}\right) \\ \nu_n^{-1}H'_{z\,n}\left(\mathbf{r},\xi,\tau\right) &= f\left(\xi,\tau\right)\,\nu_n\psi_n\left(\mathbf{r}\right)\,; \end{aligned} \tag{39}$$

$$\begin{aligned} H''_{z\,n}\left(\mathbf{r},\xi,\tau\right) &= 0 \\ \kappa_n^{-1}\sqrt{\mu_0/\epsilon_0}\,\mathbf{H}''_n\left(\mathbf{r},\xi,\tau\right) &= \mathcal{A}\left(\xi,\tau\right)\,\mathbf{z}\times\nabla_\perp\phi_n\left(\mathbf{r}\right) \\ \kappa_n^{-1}\mathbf{E}''_n\left(\mathbf{r},\xi,\tau\right) &= \mathcal{B}\left(\xi,\tau\right)\,\nabla_\perp\phi_n\left(\mathbf{r}\right) \\ \kappa_n^{-1}E''_{z\,n}\left(\mathbf{r},\xi,\tau\right) &= f\left(\xi,\tau\right)\,\kappa_n\phi_n\left(\mathbf{r}\right) \end{aligned} \tag{40}$$

where the amplitudes $\mathcal{A}(\xi,\tau)$ and $\mathcal{B}(\xi,\tau)$ of the transverse field components are generated by the amplitude $f(\xi,\tau)$ of the longitudinal fields *identically* as $\mathcal{A}(\xi,\tau) = -\partial_\tau f(\xi,\tau)$ and $\mathcal{B}(\xi,\tau) = \partial_\xi f(\xi,\tau)$.

4. The quadric characteristics (22) and (30) both have the following forms in terms of the new notation

$$\begin{aligned} k_n^{-2}\,W\left(\xi,\tau\right) &= \left(\mathcal{A}^2+\mathcal{B}^2+f^2\right)/2 \\ k_n^{-2}\,P_z\left(\xi,\tau\right) &= c\,\mathcal{A}\,\mathcal{B} \end{aligned} \tag{41}$$

where k_n^2 is ν_n^2 for the TE-modes; otherwise, it is κ_n^2. The conservation of energy law (6) should be read as

$$\partial_\xi P_z\left(\xi,\tau\right) + c\,\partial_\tau W\left(\xi,\tau\right) = 0. \tag{42}$$

In the particular case (35), the function $f(\xi,\tau)$ is

$$f\left(\xi,\tau\right) = \sin\left(\eta\tau - \vartheta\xi + \varphi\right) \tag{43}$$

where $\eta = \omega/\omega_n^c \geq 1$, ω_n^c is a cut-off frequency, $\vartheta = \sqrt{\eta^2 - 1} \geq 0$, the constant $A_n = 1$. Calculations by formulas (41) yield

$$\begin{aligned} k_n^{-2}\,W\left(\xi,\tau\right) &= \vartheta^2\cos^2\left(\eta\tau-\vartheta\xi+\varphi\right)+1/2 \\ k_n^{-2}\,P_z\left(\xi,\tau\right) &= c\,\vartheta\eta\cos^2\left(\eta\tau-\vartheta\xi+\varphi\right)\,. \end{aligned} \tag{44}$$

Calculations of $\bar{W}$ (energy density averaged over the period T corresponding to the frequency, ω) and $\bar{P}_z$ (Poynting's vector averaged in the same way) yield

$$k_n^{-2}\,\bar{W} = k_n^{-2}\frac{1}{T}\int_{\theta}^{\theta+T} W\left(\xi,\tau\right)\,d\tau = \eta^2/2$$
$$k_n^{-2}\,\bar{P}_z = k_n^{-2}\frac{1}{T}\int_{\theta}^{\theta+T} P_z\left(\xi,\tau\right)\,d\tau = c\,\vartheta\eta/2 \qquad (45)$$

where coordinate ξ is arbitrary, θ is a constant, period T in time τ is equal to $2\pi/\eta$. Relation $\bar{P}_z/\bar{W}$ specifies an averaged velocity $\bar{v}$ of the wave energy transferred along the waveguide. The results of integration (45) yield

$$\bar{v} = c\sqrt{\omega^2 - \omega_c^2}/\omega \quad \text{and} \quad v_{ph} = c\,\omega/\sqrt{\omega^2 - \omega_c^2} \qquad (46)$$

where the phase velocity v_{ph} is calculated by the standard manner. These results coincide with the classical ones.

5. The KGE obeys the fundamental principle of relativistic physics, i.e., form of KGE is invariant in any inertial reference frame. It implies that KGE must maintain its form under the action of the *Poincare group* within the framework of the Group Theory. This fact enables us to use remarkable properties of symmetry, which the solutions to this equation possess. These properties were disclosed by Miller and published in his monographs [22].

4. NON-SINUSOIDAL ANALYTICAL SOLUTIONS

We will now explain one of Miller's ideas which will be used in this article. Miller has proposed interpreting an expected solution $f\left(\xi,\tau\right)$ to Eq. (36) as a function of new independent variables u and v yet unknown. It is supposed, however, that u and v are twice differentiable functions of the "old" variables $\left(\xi,\tau\right)$. The solution $f\left(\xi,\tau\right)$ to KGE can be read as $f\left[u\left(\xi,\tau\right),v\left(\xi,\tau\right)\right]$. The latter, being substituted to Eq. (36), yields an equation somewhat "ugly" in form

$$\begin{aligned}&\Big\{\Big[(\partial_\tau u)^2 - (\partial_\xi u)^2\Big]\,\partial_u^2 + \Big[(\partial_\tau v)^2 - (\partial_\xi v)^2\Big]\,\partial_v^2\\ &+ 2[(\partial_\tau v)\,(\partial_\tau u) - (\partial_\xi v)\,(\partial_\xi u)]\,\partial_{u\,v}^2\\ &+ \left[\partial_\tau^2 u - \partial_\xi^2 u\right]\,\partial_u + \left[\partial_\tau^2 v - \partial_\xi^2 v\right]\partial_v + 1\Big\}\,f\left(u,v\right) = 0 \qquad (47)\end{aligned}$$

which is equivalent, nevertheless, to that elegant Eq. (36).

Miller's crucial idea follows. *i)* The functions $u\left(\xi,\tau\right)$ and $v\left(\xi,\tau\right)$ should be found out using the properties of symmetry of KGE. This enables us to express the coefficients in the square brackets as some given functions of the variables u and v. *ii)* Substituting these functions of (u, v) into (47) results in the factorization of the solution as $f(u, v) = U(u)\, V(v)$. Eventually, $U(u)$ and $V(v)$ appear as some known special functions of mathematical physics with the arguments dependent on (ξ, τ). This enables us to obtain new analytical solutions (unknown yet, to our best knowledge) for the time-domain modes.

Miller has established a complete set consisting of 11 pairs of the inverse functions $\tau(u, v)$ and $\xi(u, v)$ needed for derivation of $u(\xi, \tau)$ and $v(\xi, \tau)$, see Appendix A. The **1**st pair leads to the time-harmonic solutions. The **2**nd pair was used for the studies of digital signal propagation in waveguides [6]. The pairs **3)** and **4)** are still under study. We consider here which solution can be supplied with the pair placed in **5)**.

4.1. Time-domain Modes Expressible via Airy Functions

Inversion of the formulas given in **5)** yields

$$u + v = (\tau + \xi)/2 \quad \text{and} \quad u - v = \pm\sqrt{\tau - \xi}\,. \tag{48}$$

When we read the double sign $(\pm)$ as minus $(-)$, then

$$\begin{aligned} u &= (\tau + \xi)/4 - \sqrt{\tau - \xi}/2 \\ v &= (\tau + \xi)/4 + \sqrt{\tau - \xi}/2\,. \end{aligned} \tag{49}$$

Calculations of the coefficients standing in the square brackets in (47) result in

$$\left[(\partial_\tau u)^2 - (\partial_\xi u)^2\right] = -\left[(\partial_\tau v)^2 - (\partial_\xi v)^2\right] = \frac{1}{4(u - v)}. \tag{50}$$

All the other coefficients are zeros. Substitution of these coefficients to Eq. (47) makes its form more attractive as

$$\frac{\partial^2}{\partial u^2} f(u, v) + 4u\, f(u, v) = \frac{\partial^2}{\partial v^2} f(u, v) + 4v\, f(u, v)\,. \tag{51}$$

It is evident now that $f(u, v)$ is a product of two functions, $U(u)\, V(v)$, which should be found. Substitution of the product to Eq. (51) and simple manipulations result in

$$\frac{1}{U(u)} \frac{d^2}{du^2} U(u) + 4u = \frac{1}{V(v)} \frac{d^2}{dv^2} V(v) + 4v = 4\alpha \tag{52}$$

where α is a constant of separation of the variables.

At this point, it is convenient to slightly change notation for the variables u and v. When we introduce the new ones as

$$\bar{u} = \sqrt[3]{4}\,(\alpha - u) \quad \text{and} \quad \bar{v} = \sqrt[3]{4}\,(\alpha - v)\,, \tag{53}$$

then Eq. (52) yields standard Airy differential equation‖

$$\frac{d^2}{d\bar{u}^2}U(\bar{u}) - \bar{u}\,U(\bar{u}) = 0 \quad \text{and} \quad \frac{d^2}{d\bar{v}^2}V(\bar{v}) - \bar{v}\,V(\bar{v}) = 0 \tag{54}$$

the solutions $U(\bar{u})$ and $V(\bar{v})$ of which depend on different arguments, however. Airy functions, as the solutions to Eq. (54), are real-valued, provided that the constant parameter α in their arguments (53) is chosen as a real number.

Suppose that the solution should have a *beginning in time:* say, at $t = 0$. Physically, it implies that the field sources do not act before the instant $t = 0$.

In compliance with the causality principle, a final solution to KGE should be presented as the piecewise function

$$f(\xi,\tau) = \begin{cases} 0 & \text{if} \quad \tau < 0 \\ U(\bar{u})\,V(\bar{v}) & \text{if} \quad 0 \le \xi \le \tau \\ 0 & \text{if} \quad \xi > \tau\,. \end{cases} \tag{55}$$

The first line in Eq. (55) is written in accordance with a so-called *weak* causality condition. This states that all fields are zero for $t < 0$ when the sources of the field are zero at that time. The lower lines correspond to the *strong* causality condition. This follows from the axiom of the special relativity theory, which asserts that any electromagnetic field transfers a signal with the velocity of light c. This implies jointly that the modal fields are zero beyond the distance $z = ct$ along axis Oz from a source (located at $z = 0$) that turns on at instant $t = 0$. The solution $f(\xi,\tau) = U(\bar{u})\,V(\bar{v})$ assumes implicitly that the initial conditions (38) are chosen as $\varsigma(\tau) = U(\bar{u})\,V(\bar{v})\,|_{\xi=0}$ and $\upsilon(\tau) = -\partial_\tau\left[U(\bar{u})\,V(\bar{v})\right]|_{\xi=0}\,.$ Hence, our procedure has sense of *expansion* of these initial values, given for $\xi = 0$, over whole domain of propagation $0 \le \xi \le \tau < +\infty$, which is equivalent to *exact* calculation of the convolution integral.

4.2. Numerical Examples

Each equation in (54) has two linearly independent solutions

$$\begin{aligned} U(\bar{u}) &= a_1 Ai(\bar{u}) + b_1 Bi(\bar{u}) \\ V(\bar{v}) &= a_2 Ai(\bar{v}) + b_2 Bi(\bar{v}) \end{aligned} \tag{56}$$

where $a_{1,2}$ and $b_{1,2}$ are arbitrary constants, $Ai(*)$ and $Bi(*)$ are Airy functions. Their arguments, as the functions of time, τ, and coordinate,

‖ Airy equation $y''(x) - x\,y(x) = 0$ yields two linearly independent solutions as $Ai(x)$ and $Bi(x)$. In mathematics, they were named as Airy special functions after the British astronomer George Biddell Airy. The Airy functions describe an image of a star (a point source of light) as it appears in a telescope. Airy functions are also solutions to Schrodinger equation (quantum mechanics) for a particle confined within a triangular potential well.

ξ, are

$$\bar{u} = \sqrt[3]{4}\left[\alpha - (\tau + \xi)/4 + \sqrt{\tau - \xi}/2\right]$$
$$\bar{v} = \sqrt[3]{4}\left[\alpha - (\tau + \xi)/4 - \sqrt{\tau - \xi}/2\right]. \tag{57}$$

Denote for a while $\bar{u}$ and $\bar{v}$ both as x. When x is positive, $Ai\,(x)$ is positive, convex and *decreasing* exponentially to zero. Function $Bi\,(x)$ for positive x is also positive and convex, but is *increasing* exponentially. When x is negative, $Ai\,(x)$ and $Bi\,(x)$ oscillate around zero with ever-increasing frequency and ever-decreasing amplitude. Function $Bi\,(x)$ is defined as the solution with the same amplitude of oscillation as $Ai\,(x)$ (as x goes to $-\infty$), which differs in phase by $\pi/2$. In Fig. 1, the results of calculations of $Ai\,(\bar{u})$ and $Bi\,(\bar{u})$ are exhibited provided that $\alpha = 10$ and $\tau = 51$. It is worth noting that the positive values of the variable ξ provide negative values of the argument $\bar{u}$ if $\xi > -\tau + 4\alpha + 2\sqrt{\tau - \xi}$. The functions $Ai\,(\bar{u})$ and $Bi\,(\bar{u})$, which have the negative values of the argument, oscillate both with weakly decreasing amplitudes as $\bar{u}^{-1/4}$ in average.

Observation of (57) enables concluding that all possible combinations of the products of Airy functions

$$\begin{array}{ll} f_1(\xi,\tau) = Ai\,(\bar{u})\ Ai\,(\bar{v}) & f_2(\xi,\tau) = Ai\,(\bar{u})\ Bi\,(\bar{v}) \\ f_3(\xi,\tau) = Bi\,(\bar{u})\ Ai\,(\bar{v}) & f_4(\xi,\tau) = Bi\,(\bar{u})\ Bi\,(\bar{v}) \end{array} \tag{58}$$

can have physical sense as the solution $f(\xi,\tau)$, provided that a value of α is chosen properly.

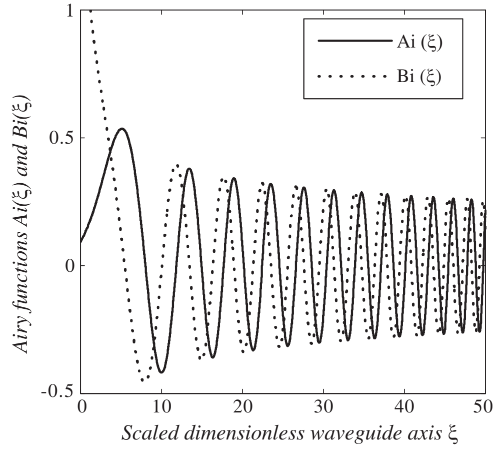

Figure 1. Dependence of Airy functions on the waveguide coordinate ξ.

Consider an interesting case when time, τ, and coordinate, ξ, both are in vicinity of the *characteristic line* of Eq. (36) , $\tau = \xi$. The variables τ and ξ are coupled by the condition $\tau = \xi + \delta$ where $\delta \ll 1$ is a small parameter. In Fig. 2, the dotted line corresponds to the function $f_1(\xi, \tau)$ specifying the longitudinal component of a three-component modal field. In Eqs. (39) and (40) , it is either the amplitude of $\nu_n^{-1} H'_{z\,n}$-component for the TE-modes or it is the amplitude of $\kappa_n^{-1} E''_{z\,n}$-component for the TM-modes. The dash-line corresponds to function $\mathcal{B} = \partial_\xi f_1(\xi, \tau)\,|_{\xi=\tau-\delta}$ specifying a transverse component of a three-component modal field. In Eqs. (39) and (40) , it is either the amplitude of field $\nu_n^{-1}\mathbf{H}'_n$ for the TE-modes or it is the amplitude of field $\kappa_n^{-1}\mathbf{E}''_n$ for the TM-modes. The solid line corresponds to the function $\mathcal{A} = -\partial_\tau f_1(\xi, \tau)\,|_{\xi=\tau-\delta}$ specifying the amplitude of a two-component modal field. In Eqs. (39) and (40) , it is either the amplitude of $\nu_n^{-1}\sqrt{\epsilon_0/\mu_0}\,\mathbf{E}'_n$-component for the TE-modes or it is the amplitude of $\kappa_n^{-1}\sqrt{\mu_0/\epsilon_0}\,\mathbf{H}''_n$-component for the TM-modes.

In Fig. 3, the modal amplitudes generated by the function $f_2(\xi, \tau)$ are exhibited. System of the line notation is the same as given for Fig. 2.

A reader can find some applications of Airy functions to various fields of classical and quantum physics in [23]. These examples and possible others, undoubtedly, have the waveguide analogies that will be discussed elsewhere.

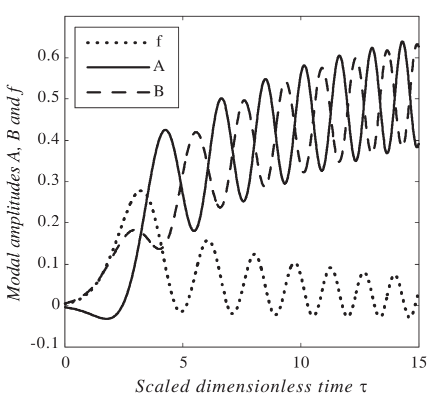

Figure 2. The modal amplitudes generated by the function f_1 $(\tau,\, \xi)$.

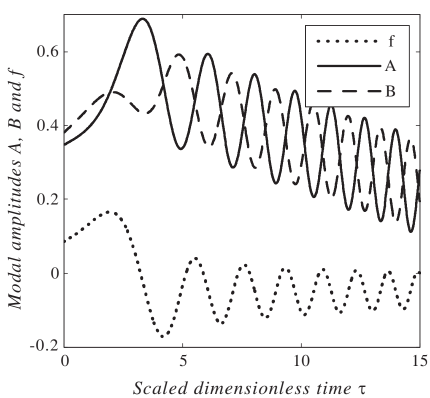

Figure 3. The modal amplitudes generated by the function f_2 $(\tau,\, \xi)$.

5. CONCLUSION

After all the efforts made in the previous sections, study of the time-domain modes comes down to solving two *autonomous* problems.

i) The modal basis problem. Determination of the complete set of the modal field patterns in the waveguide cross section is the same, in the final analysis, as for the classical time-harmonic modes. The two-dimensional Dirichlet and Neumann boundary eigenvalue problems for transverse Laplacian should be solved. This results in two complete sets of eigensolutions: $\{\phi_n(\mathbf{r})\}_{n=0}^{\infty}$ and $\{\psi_n(\mathbf{r})\}_{n=0}^{\infty}$, respectively. The eigenfunctions generate the basis elements and exhibit the field pattern configurations of the modal field components. Practically, one has a wide choice of the methods for solving these problems. The already known results for the typical waveguides can also be used after providing them with appropriate normalizations, which are given in Eqs. (21) and (27). Additionally, the physical dimension of *force* (i.e., newton N) should be assigned to the constant at the right hand sides of the normalization conditions. This provides electric and magnetic strength vectors with required dimensions Vm^{-1} and Am^{-1}, respectively. Meanwhile, the modal amplitudes $\mathcal{A}$, $\mathcal{B}$ and f remain dimensionless in Eqs. (39) and (40).

ii) The modal amplitude problem. We have derived a pair of potentials which generate explicit formulas for the modal amplitudes in the time domain. These potentials satisfy KGE with independent variables (z,t). The modal amplitudes $\mathcal{A}$, $\mathcal{B}$ and f of the TE and TM modes can be studied *in parallel* after replacing the independent variables (z,t) by dimensionless (ξ,τ).

Solving Klein-Gordon equation explicitly: The standard explicit solution to KGE as a convolution integral can be found in handbooks, e.g., [19]. However, exact calculations of that integral are often troublesome in realistic situations. Therefore, a new technique is proposed in present article for solving KGE analytically without recourse to the convolution integral. This technique is based on the properties of symmetry of KGE which were disclosed within the framework of Group Theory [22]. It turns out that the solution to KGE can be presented as a product of two functions, i.e., $f(\xi,\tau) = U(u(\xi,\tau))\,V(v(\xi,\tau))$ if and only if the functions $u(\xi,\tau)$ and $v(\xi,\tau)$ are specified in pairs in compliance with the properties of symmetry. A complete set of possible pairs is listed in Appendix A. The simplest pair is given in case 1) as $u = \tau$ and $v = \xi$ which leads to the time-harmonic fields. This emphasizes that classical time-harmonic fields are only one particular case. Hence, the other ten cases open new lines of analytical studies in the time-domain electromagnetics.

In this article, case 5) is studied in detail where the modal amplitude $f(\xi,\tau)$ is expressed as a product of two Airy functions with argument u and v. The other modal amplitudes are $\mathcal{A}=-\partial_\tau f$ and $\mathcal{B}=\partial_\xi f$. Actually, this solution is an exact expansion of the initial data given on time semi-axis $\tau\geq 0$ as $f(\xi,\tau)|_{\xi=0}$ and $\partial_\tau(\xi,\tau)|_{\xi=0}$ over the whole domain of propagation, $0\leq\xi\leq\tau$. This is equivalent to exact analytical calculation of the convolution integral without approximations of its kernel.

APPENDIX A.

A complete list of substitutions $u(\xi,\tau)$ and $v(\xi,\tau)$ which factorize solution to Eq. (47) as $f(u,v)=U(u)V(v)$.

1) $\tau=u$ and $\xi=v$, where $-\infty<u<\infty$, $-\infty<v<\infty$, yield $f(u,v)$ as a product of the exponential functions.

2) $\tau=u\cosh v$ and $\xi=u\sinh v$ with $0\leq u<\infty$, $-\infty<v<\infty$ yield a product of an exponential and Bessel functions.

3) $\tau=\left(u^2+v^2\right)/2$ and $\xi=uv$ with $0\leq u<\infty$, $-\infty<v<\infty$ yield $f(u,v)$ as a product of parabolic cylinder functions.

4) $\tau=uv$ and $\xi=\left(u^2+v^2\right)/2$ with $0\leq u<\infty$, $-\infty<v<\infty$ yield a product of parabolic cylinder functions.

5) $\tau+\xi=2(u+v)$ and $\tau-\xi=(u-v)^2$ with $-\infty<u,v<\infty$ yield a product of Airy functions.

6) $\tau+\xi=\cosh[(u-v)/2]$ and $\tau-\xi=\sinh[(u+v)/2]$ with $-\infty<u,v<\infty$ yield a product of Mathieu functions.

7) $\tau+\xi=2\sinh(u-v)$ and $\tau-\xi=\exp(u+v)$ with $-\infty<u,v<\infty$ yield a product of Bessel functions.

8) $\tau+\xi=2\cosh(u-v)$ and $\tau-\xi=\exp(u+v)$ with $-\infty<u,v<\infty$ yield a product of Bessel functions.

9) $\tau=\sinh u\cosh v$ and $\xi=\cosh u\sinh v$ with $-\infty<u,v<\infty$ yield a product of Mathieu functions.

10) $\tau=\cosh u\cosh v$ and $\xi=\sinh u\sinh v$ with $-\infty<u<\infty$, $0\leq v<\infty$ yield a product of Mathieu functions.

11) $\tau=\cos u\cos v$ and $\xi=\sin u\sin v$ with $0<u<2\pi$, $0\leq v<\pi$ yield a product of Mathieu functions.

The substitutions **1)–11)** specify some *orthogonal* systems of coordinates (u,v). Besides, there are some *non-orthogonal* systems which enable to separate the variables u and v as well in the KGE, see paper [24].

REFERENCES

1. Gabriel, G. J., "Theory of electromagnetic transmission structures, Part I: Relativistic foundation and network formalism," *Proc. IEEE*, Vol. 68, No. 3, 354–366, 1980.
2. Tretyakov, O. A., "Evolutionary waveguide equations," *Sov. J. Comm. Tech. Electron. (English Translation of Elektrosvyaz and Radiotekhnika)*, Vol. 35, No. 2, 7–17, 1990.
3. Tretyakov, O. A., "Essentials of nonstationary and nonlinear electromagnetic field theory," *Analytical and Numerical Methods in Electromagnetic Wave Theory*, M. Hashimoto, M. Idemen, O. A. Tretyakov (eds.), Chap. 3, Science House Co. Ltd., Tokyo, 1993.
4. Kristensson, G., "Transient electromagnetic wave propagation in waveguides," *Journal of Electromagnetic Waves and Applications*, Vol. 9, No. 5–6, 645–671, 1995.
5. http://www.emph.com.ua/18/index.htm.
6. Aksoy, S. and O. A. Tretyakov, "Evolution equations for analytical study of digital signals in waveguides," *Journal of Electromagnetic Waves and Applications*, Vol. 17, No. 12, 1665–1682, 2003.
7. Aksoy, S. and O. A. Tretyakov, "The evolution equations in study of the cavity oscillations excited by a digital signal," *IEEE Trans. Antenn. Propag.*, Vol. 52, No. 1, 263–270, Jan. 2004.
8. Slater, J. C., "Microwave electronics," *Rev. Mod. Phys.*, Vol. 18, No. 4, 441–512, 1946.
9. Kisun'ko, G. V., *Electrodynamics of Hollow Systems*, VKAS-Press, Leningrad, 1949 (in Russian).
10. Kurokawa, K., "The expansion of electromagnetic fields in cavities," *IRE Trans. Microwave Theory Tech.*, Vol. 6, 178–187, 1958.
11. Felsen, L. B. and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice Hall, Englewood Cliffs, NJ, 1973.
12. Borisov, V. V., *Transient Electromagnetic Waves*, Leningrad Univ. Press, Leningrad, 1987 (in Russian).
13. Tretyakov, O. A., "Evolutionary equations for the theory of waveguides," *IEEE AP-S Int. Symp. Dig.*, Vol. 3, 2465–2471, Seattle, Jun. 1994.
14. Shvartsburg, A. B., "Single-cycle waveforms and non-periodic waves in dispersive media (exactly solvable models)," *Phys. Usp.*, Vol. 41, No. 1, 77–94, Jan. 1998.
15. Slivinski, A. and E. Heyman, "Time-domain near-field analysis

of short-pulse antennas — Part I: Spherical wave (multipole) expansion," *IEEE Trans. Antenn. Propag.*, Vol. 47, 271–279, Feb. 1999.

16. Nerukh, A. G., I. V. Scherbatko, and M. Marciniak, *Electromagnetics of Modulated Media with Application to Photonics*, National Institute of Telecommunications Publishing House, Warsaw, Poland, 2001.
17. Geyi, W., "A time-domain theory of waveguides," *Progress In Electromagnetics Research*, Vol. 59, 267–297, 2006.
18. Erden, F. and O. A. Tretyakov, "Excitation by a transient signal of the real-valued electromagnetic fields in a cavity," *Phys. Rev. E*, Vol. 77, No. 5, 056605, May 2008.
19. Polyanin, A. D. and A. V. Manzhirov, *Handbook of Mathematics for Engineers and Scientists,* Chapman & Hall/CRC Press, Boca Raton, FL, 2006.
20. Torre, A., "A note on the Airy beams in the light of the symmetry algebra based approach," *J. Opt. A: Pure Appl. Opt.*, Vol. 11, 125701, Sep. 2009.
21. Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists,* Chapman & Hall/CRC Press, Boca Raton, FL, 2002.
22. Miller, W. Jr., *Symmetry and Separation of Variables,* Addison-Wesley Publication Co., Boston, MA, 1977.
23. Vallée, O. and M. Soares, *Airy Functions and Applications to Physics,* Imperial College Press, London, England, 2004.
24. Kalnins, E., "On the separation of variables for the Laplace equation in two- and three-dimensional Minkowski space," *SIAM J. Math. Anal.*, Vol. 6, 340–374, 1975.