ON THE COMPENSATION OF PROBE POSITIONING ERRORS WHEN USING A NONREDUNDANT CYLINDRICAL NF-FF TRANSFORMATION

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Abstract—Two different approaches for compensating the probe positioning errors in a near-field–far-field transformation with cylindrical scanning using a nonredundant number of measurements are presented and experimentally validated in this paper. In order to evaluate the uniformly distributed samples from the irregularly spaced ones, the former makes use of the singular value decomposition method, whereas the latter employs an iterative technique. In both the cases, the near-field data needed by a standard near-field-far-field transformation are efficiently evaluated via an optimal sampling interpolation algorithm.

1. INTRODUCTION

In the last fifteen years, the theoretical results on the nonredundant representations of electromagnetic (EM) fields [1] have allowed the development of efficient near-field-far-field (NF-FF) transformation techniques, which generally require a number of NF data remarkably lower than the standard ones. As a matter of fact, the NF data needed by the standard NF-FF transformations are accurately recovered by interpolating the minimum set of measurements via optimal sampling interpolation (OSI) expansions. A substantial time saving can be so achieved making these transformations more and more appealing, since nowadays the measurement time is very much greater than the computational one.

The mathematical foundation to achieve this result relies on the quasi-bandlimitation properties of EM fields [2] and their

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nonredundant sampling representations [1]. In fact, according to [1], the EM fields radiated by finite size sources enclosed in a convex domain bounded by a rotational surface Σ and observed on a surface $\mathcal{M}$ having the same rotational symmetry can be very well approximated by spatially bandlimited functions, provided that a proper phase factor is singled out by the field expression and proper parameterizations are employed to describe $\mathcal{M}$. Since, as shown in [3], the voltage acquired by a nondirective probe has the same effective bandwidth of the field radiated by the antenna under test (AUT), these sampling representations can be applied also to the voltage measured by such a kind of probe.

From a practical viewpoint, it may be impossible to get regularly spaced NF measurements due to an inaccurate control of the positioning systems. On the other hand, the actual measurement positions can be accurately read by optical devices. Moreover, the finite resolution of the positioning devices prevents the possibility to locate exactly the receiving probe at the points fixed by the sampling representation. In light of these considerations, the development of an accurate and stable reconstruction algorithm from nonuniformly distributed data becomes relevant.

The problem of the interpolation from nonuniform samples has been well investigated in the (one-dimensional) case of bandlimited functions defined over the real axis. Several sufficient conditions ensuring the possibility of recovering a function from its nonuniform samples have been stated in [4]. The stability in a nonuniform sampling process, i.e., the requirement that small errors affecting the sample values produce small errors in the recovered functions, has been exhaustively investigated in [5, 6], wherein it has been shown that a stable sampling cannot be performed at an "average" rate lower than the Nyquist one. Some closed-form expressions, which allow the interpolation of bandlimited functions from nonuniform samples, have been reported in [7]. However, they are valid only for particular sampling points arrangements, are cumbersome and not user friendly. Moreover, they become more and more unstable as the sample points distribution deviates from the uniform one. The two-dimensional nonuniform sampling has not attracted an equal consideration in literature. However, it has been shown [8] that also in such a case a stable sampling cannot be performed at a rate lower than the Nyquist one. As it is clearly pointed out in [9], wherein a more exhaustive discussion can be found, a nonuniform sampling algorithm, useful for practical applications, must be computationally manageable, stable, and accurate. Therefore, it is more convenient to recover the uniform samples from the irregularly spaced ones than to resort to a direct

interpolation formula. As a matter of fact, once the uniform samples have been determined, the value at any point of the scanning surface can be recovered by an accurate and stable OSI formula.

In this framework, two different approaches have been proposed. The former [10, 11] relies on the singular value decomposition (SVD) method and has been applied when the two-dimensional problem can be reduced to find the solution of two independent one-dimensional ones. Moreover, it may take advantage of data redundancy for increasing the algorithm stability [10, 11]. The latter [9, 12] is based on an iterative technique which has been found convergent only if it is possible to build a biunique correspondence, associating at each uniform sampling point the nearest nonuniform one.

The aim of this paper is to present and experimentally validate these two techniques in the case of the cylindrical scanning. The choice of such a scanning is mainly due to the availability of a NF cylindrical facility at the Antenna Characterization Lab of University of Salerno, but such a validation implicitly confirms the applicability of these approaches to other NF scanning geometries. It is worthy to note that, unlike done in [12], the iterative technique is here applied to a nonredundant NF-FF transformation with cylindrical scanning.

2. SAMPLING REPRESENTATION OF THE VOLTAGE ON A CYLINDER

Let us consider an AUT as enclosed in a convex domain bounded by a rotational surface Σ and a non directive probe scanning a cylinder of radius d in the NF region. According to [1], the "reduced voltage"

$$\tilde{V}(\xi) = V(\xi) \, e^{j\gamma(\xi)}, \tag{1}$$

where $\gamma(\xi)$ is a proper phase function and ξ is an optimal parameter used to describe each of the curves $\mathcal{C}$ (generatrices and azimuthal rings) representing the cylindrical surface $\mathcal{M}$, can be well approximated by a spatially bandlimited function. The related bandlimitation error becomes negligible as the bandwidth exceeds a critical value W_ξ [1] and, accordingly, it can be effectively controlled by choosing a bandwidth equal to $\chi' W_\xi$, where $\chi' > 1$ is the bandwidth enlargement factor.

When $\mathcal{C}$ is a generatrix, by adopting $W_\xi = \beta \ell'/2\pi$ (β is the wavenumber and ℓ' is the length of the curve $\mathcal{C}''$ obtained as intersection between the meridian plane passing through the observation point P and Σ), we get:

$$\gamma = \frac{\beta}{2} \left[R_1 + R_2 + s'_1 - s'_2 \right]; \qquad \xi = \frac{\pi}{\ell'} \left[R_1 - R_2 + s'_1 + s'_2 \right] \tag{2}$$

where $R_{1,2}$ are the distances from P to the tangency points $P_{1,2}$ on $\mathcal{C}''$ and $s'_{1,2}$ are their arclength coordinates.

When $\mathcal{C}$ is a ring, γ is constant and then it is possible to choose for it the value relevant to the generatrix passing through P. Moreover, it is convenient to choose the azimuthal angle φ as parameter. The corresponding bandwidth is given [1] by

$$W_\varphi = \frac{\beta}{2} \max_{z'} \left(\sqrt{(z-z')^2 + (d+\rho'(z'))^2} - \sqrt{(z-z')^2 + (d-\rho'(z'))^2} \right) \tag{3}$$

$\rho'(z')$ being the equation of Σ in cylindrical coordinates.

In the following, for sake of simplicity, Σ is assumed coincident with the smallest spherical surface of radius a able to contain the AUT (see Fig. 1). In such a case, for a generatrix, it results [1, 13]

$$W_\xi = \beta a; \quad \gamma = \beta\sqrt{r^2 - a^2} - \beta a \cos^{-1}(a/r); \quad \xi = \vartheta \tag{4}$$

whereas, the azimuthal bandwidth (3) becomes $W_\varphi = \beta a \sin\vartheta$.

By taking into account these results, the reduced voltage at P on the generatrix fixed by φ can be evaluated via the following OSI expansion:

$$\tilde{V}(\vartheta, \varphi) = \sum_{n=n_0-q+1}^{n_0+q} \tilde{V}(\vartheta_n, \varphi)\, \Omega_N(\vartheta - \vartheta_n)\, D_{N''}(\vartheta - \vartheta_n) \tag{5}$$

where $n_0 = \text{Int}\left[(\vartheta - \Delta\vartheta/4)/\Delta\vartheta\right]$ is the index of the sample nearest to the output point, $2q$ is the number of retained samples, and

$$\vartheta_n = n\Delta\vartheta + \Delta\vartheta/4; \quad \Delta\vartheta = 2\pi/(2N'' + 1) \tag{6}$$

$$N'' = \text{Int}\left(\chi\, N'\right) + 1; \quad N' = \text{Int}\left(\chi' W_\xi\right) + 1 \tag{7}$$

$\chi > 1$ being the oversampling factor needed to control the truncation error. Moreover,

$$D_{N''}(\vartheta) = \frac{\sin\left[(2N''+1)\vartheta/2\right]}{(2N''+1)\,\sin(\vartheta/2)};$$
$$\Omega_N(\vartheta) = \frac{T_N\left[2\cos^2(\vartheta/2)/\cos^2(\vartheta_0/2) - 1\right]}{T_N\left[2/\cos^2(\vartheta_0/2) - 1\right]} \tag{8}$$

are the Dirichlet and Tschebyscheff Sampling functions, respectively, $T_N(\cdot)$ being the Tschebyscheff polynomial of degree $N = N'' - N'$ and $\vartheta_0 = q\Delta\vartheta$.

It is worth noting that the OSI expansion (5) is different from that used in [13], in order to have a symmetrical distribution of rings with respect to $z = 0$.

The intermediate samples $\tilde{V}(\vartheta_n, \varphi)$ can be evaluated by interpolating the samples measured on the rings via a quite analogous OSI formula.

The following two-dimensional OSI expansion thus results:

$$\tilde{V}(\vartheta, \varphi) = \sum_{n=n_0-q+1}^{n_0+q} \{\Omega_N(\vartheta - \vartheta_n) D_{N''}(\vartheta - \vartheta_n)$$

$$\cdot \sum_{m=m_0-p+1}^{m_0+p} \tilde{V}(\vartheta_n, \varphi_{m,n}) \Omega_{M_n}(\varphi - \varphi_{m,n}) D_{M_n''}(\varphi - \varphi_{m,n})\} \quad (9)$$

where $m_0 = \text{Int}(\varphi/\Delta\varphi_n)$, $2p$ is the number of retained samples along φ

$$\varphi_{m,n} = m\Delta\varphi_n = 2m\pi/(2M_n'' + 1); \quad M_n'' = \text{Int}\left(\chi M_n'\right) + 1 \quad (10)$$

$$M_n' = \text{Int}\left(\chi^* W_{\varphi_n}\right) + 1; \quad \chi^* = 1 + \left(\chi' - 1\right)(\sin\vartheta_n)^{-2/3} \quad (11)$$

$M_n = M_n'' - M_n'$, and the other symbols have the same meaning as in (5).

3. FROM NONUNIFORM TO UNIFORM SAMPLES

Two different approaches, which allow the reconstruction of the uniform samples from the nonuniform acquired ones, will be described in this section by highlighting all their features.

3.1. SVD Approach

Let us now suppose that the nonuniformly distributed samples lie on rings not regularly spaced (see Fig. 1). This represents a realistic hypothesis in a cylindrical NF facility, where the acquisition is made by rings as required to exploit the possibility of reducing the number of NF data on non-central rings, offered by the described non-redundant representation. In such a hypothesis, the starting two-dimensional problem is reduced to find the solution of two independent one-dimensional problems.

In particular, the uniformly spaced samples on each nonuniform ring are recovered as follows. Given a sequence of $J_k \geq 2M_k'' + 1$ nonuniform sampling points (θ_k, ϕ_j) on the nonuniform ring at height $z(\theta_k)$, the reduced voltage $\tilde{V}(\theta_k, \phi_j)$ at each nonuniform sampling point can be expressed in terms of the unknown uniform ones by means of the OSI expansion along φ, thus obtaining a linear system that can be rewritten in matrix form as $\underline{\underline{A}}\,\underline{x} = \underline{b}$, where $\underline{b}$ is the sequence
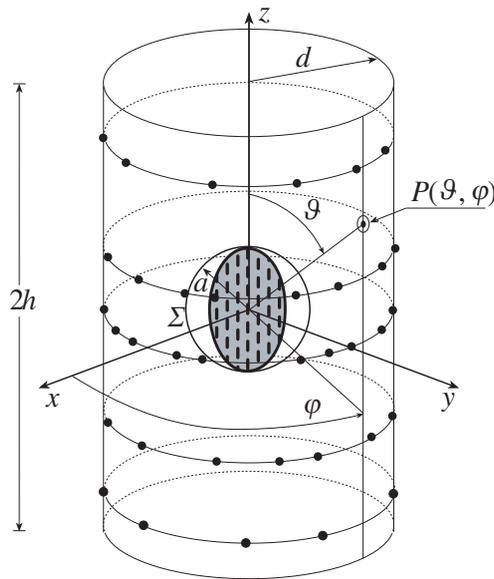
Figure 1. Geometry of the problem.

$\tilde{V}(\theta_k, \phi_j)$ of the known nonuniform samples, $\underline{x}$ is the sequence of the unknown uniformly distributed samples $\tilde{V}(\theta_k, \varphi_m)$, and $\underline{A}$ is the $J_k \times (2M_k'' + 1)$ matrix, whose elements are given by the weight functions in the considered OSI expansion:

$$a_{jm} = \Omega_{M_k} (\phi_j - \varphi_m) D_{M_k''} (\phi_j - \varphi_m) \tag{12}$$

Note that, for a fixed row j, these elements are equal to zero if m is out of the range $[m_0(\phi_j) - p + 1, m_0(\phi_j) + p]$.

Once the uniform samples on the irregularly spaced rings have been recovered via the SVD algorithm, the OSI expansion along φ is employed for determining the intermediate samples $\tilde{V}(\theta_k, \varphi)$ at the intersection points between the nonuniform rings and the generatrix through P. Since these samples are again nonuniformly distributed, the voltage at P can be found by recovering the regularly spaced intermediate samples again via SVD and then interpolating them by means of the OSI formula (5).

In both the one-dimensional problems, the displacements between the uniform and nonuniform samples are assumed such that to each uniform sampling position must correspond at least a nonuniform one whose distance is less than one half the uniform sampling spacing ($\Delta\vartheta$ or $\Delta\varphi_n$) in order to avoid a strong ill-conditioning of the related linear system [10, 11].

To minimize the computational effort for recovering the uniformly spaced samples on the uniform rings, it is convenient to reconstruct the same number N_φ of samples on each of them. Thus, the samples are aligned along the generatrices and, as a consequence, the number of systems to be solved is minimized. Obviously, N_φ is fixed by the ring corresponding to the maximum azimuthal bandwidth. Once the uniform samples (nonredundant in ϑ, but slightly redundant in φ) have been determined, the data needed to perform a standard NF-FF transformation (e.g., [14]) can be evaluated by using the OSI expansion (9) properly modified to take into account the redundancy in φ.

3.2. Iterative Approach

Let us now remove the hypothesis that the nonuniformly distributed samples lie on rings not regularly spaced. Thus, the starting two-dimensional problem can no longer be reduced to find the solution of two independent one-dimensional ones. Obviously, the SVD-based approach could be generalized to such a case, but the dimension of the involved matrix would become very large, thus requiring a huge computational effort.

In order to overcome this drawback, it is convenient to resort to the iterative technique [12] and, accordingly, let us assume in the following that the nonuniformly distributed samples are such that it is possible to build a biunique correspondence, associating at each uniform sampling point the "nearest" nonuniform one. In such a case, by expressing the reduced voltage at each nonuniform sampling point $(\theta_k, \phi_{j,k})$ as a function of the unknown values at the nearest uniform ones $(\vartheta_n, \varphi_{m,n})$ via the two-dimensional OSI expansion (9), we get:

$$\tilde{V}(\theta_k, \phi_{j,k}) =$$
$$\sum_{n=n_0-q+1}^{n_0+q} \{\Omega_N (\theta_k - \vartheta_n) \ D_{N''} (\theta_k - \vartheta_n)$$
$$\cdot \sum_{m=m_0-p+1}^{m_0+p} \tilde{V}(\vartheta_n, \varphi_{m,n}) \, \Omega_{M_n} (\phi_{j,k} - \varphi_{m,n}) \, D_{M_n''} (\phi_{j,k} - \varphi_{m,n}) \} \,(13)$$

The resulting linear system can be rewritten as $\underline{\underline{A}}\underline{x} = \underline{b}$, where $\underline{b}$ is the sequence $\tilde{V}(\theta_k, \phi_{j,k})$ of the known nonuniform samples, $\underline{x}$ is the sequence of the unknown uniformly distributed samples $\tilde{V}(\vartheta_n, \varphi_{m,n})$, and $\underline{\underline{A}}$ is the matrix whose elements are given by the weight functions in the considered OSI expansion. Its dimension is $Q \times Q$, Q being the overall number of nonuniform/uniform samples.

By splitting the matrix $\underline{\underline{A}}$ in its diagonal and nondiagonal parts, $\underline{\underline{A}}_D$ and $\underline{\underline{\Delta}}$, respectively, multiplying both members of the matrix relation $\underline{\underline{A}}\,\underline{x} = \underline{b}$ by $\underline{\underline{A}}_D^{-1}$ and rearranging the terms, it results:

$$\underline{x} = \underline{\underline{A}}_D^{-1}\underline{b} - \underline{\underline{A}}_D^{-1}\underline{\underline{\Delta}}\,\underline{x} \tag{14}$$

The following iterative scheme is so obtained:

$$\underline{x}^{(\nu)} = \underline{x}^{(0)} - \underline{\underline{A}}_D^{-1}\underline{\underline{\Delta}}\,\underline{x}^{(\nu-1)} \tag{15}$$

where $\underline{x}^{(0)} = \underline{\underline{A}}_D^{-1}\underline{b}$ and $\underline{x}^{(\nu)}$ is the vector of the uniform samples estimated at the νth step. Necessary conditions for the convergence of the above scheme are that the modulus of each element on the principal diagonal of the matrix $\underline{\underline{A}}$ be not zero and greater than those of the other elements on the same row and column [12]. These conditions are surely verified in the assumed hypothesis of biunique correspondence between each uniform sampling point and the "nearest" nonuniform one.

By straightforward evaluations from (15), it results

$$\tilde{V}^{(\nu)}\left(\vartheta_n, \varphi_{m,n}\right) =$$

$$\frac{1}{\Omega_N\left(\theta_n - \vartheta_n\right) D_{N''}\left(\theta_n - \vartheta_n\right) \Omega_{M_n}\left(\phi_{m,n} - \varphi_{m,n}\right) D_{M_n''}\left(\phi_{m,n} - \varphi_{m,n}\right)}$$

$$\cdot \left\{ \tilde{V}\left(\theta_n, \phi_{m,n}\right) - \sum_{\substack{k=k_0-q+1 \\ (k \neq n)}}^{k_0+q} \sum_{\substack{j=j_0-p+1 \\ (j \neq m)}}^{j_0+p} \Omega_N\left(\theta_n - \vartheta_k\right) D_{N''}\left(\theta_n - \vartheta_k\right) \right.$$

$$\left. \cdot \Omega_{M_k}\left(\phi_{m,n} - \varphi_{j,k}\right) D_{M_k''}\left(\phi_{m,n} - \varphi_{j,k}\right) \tilde{V}^{(\nu-1)}\left(\vartheta_k, \varphi_{j,k}\right) \right\} \tag{16}$$

The OSI expansion (9) is then used to reconstruct the NF data needed by a standard probe compensated NF-FF transformation from the so recovered uniform NF samples.

4. EXPERIMENTAL VALIDATION

The above described NF-FF transformations from irregularly spaced data have been experimentally validated in the anechoic chamber available at the UNISA Antenna Characterization Lab. The chamber, whose dimensions are $8\,\text{m} \times 5\,\text{m} \times 4\,\text{m}$, is equipped with a rotating table MI-6111B and a vertical scanner, having height 240 cm, supplied by MI Technologies, LLC. The amplitude and phase measurements are performed via a vectorial network analyzer Anritsu 37247C. An open-ended MI-6970-WR90 rectangular waveguide is used as probe. The considered AUT is a MI-12-8.2 standard gain horn with aperture

$19.4\,\mathrm{cm} \times 14.4\,\mathrm{cm}$, located on the plane $x = 0$ of the adopted reference system (Fig. 1) and operating at the frequency of $10\,\mathrm{GHz}$. Such an AUT has been modelled as enclosed in a sphere having radius equal to 12 cm. The probe output voltages have been collected on a cylinder having $d = 43.8$ cm and $2h = 240$ cm.

In the first set of measurements, the nonuniformly distributed sampling points lie on rings not regularly spaced. In particular, the NF data have been acquired in such a way that the distance between the position of each nonuniform ring and the associated uniform one is random variable uniformly distributed in $(-\Delta\vartheta/2,\ \Delta\vartheta/2)$. In a similar way, the distances between the nonuniform sampling points and corresponding uniform ones on each of these rings are random variables uniformly distributed in $(-\Delta\varphi_n/2,\ \Delta\varphi_n/2)$.

The amplitudes of the probe voltage relevant to the generatrices at $\varphi = 0°$ and $\varphi = 30°$ reconstructed via the SVD-based approach are compared in Figs. 2 and 3 with those directly measured on the same generatrices. As can be seen, although the considered values of probe positioning errors are very pessimistic in an actual scanning, there is an excellent agreement between the reconstructed voltage (crosses) and the measured one (solid line), save for the peripheral zone wherein the error is caused both by the truncation of the scanning zone and the environmental reflections. The same nonuniform NF data have been employed to reconstruct the voltage on the generatrix at $\varphi = 0°$ by using the iterative-based approach (see Fig. 4). As can be seen, the
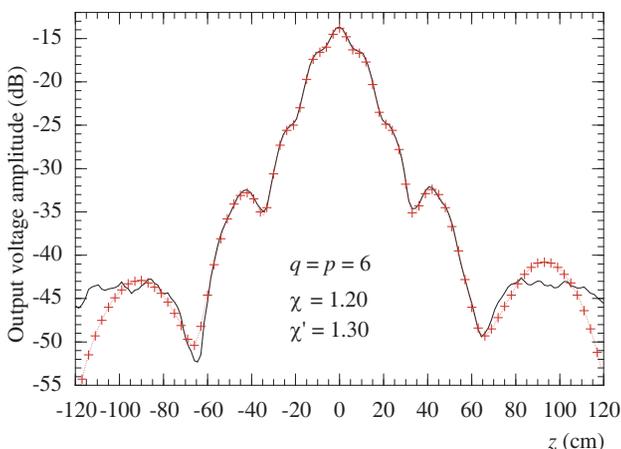
Figure 2. Amplitude of the probe voltage on the generatrix at $\varphi = 0°$. Solid line: measured. Crosses: recovered from irregurarly spaced NF data via the SVD-based approach.
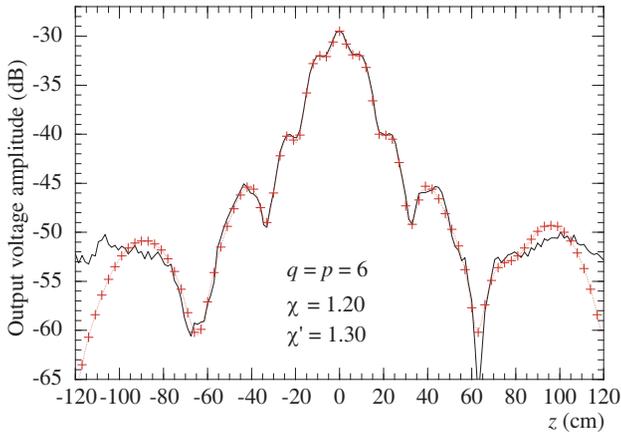
Figure 3. Amplitude of the probe voltage on the generatrix at $\varphi = 30°$. Solid line: measured. Crosses: recovered from irregurarly spaced NF data via the SVD-based approach.
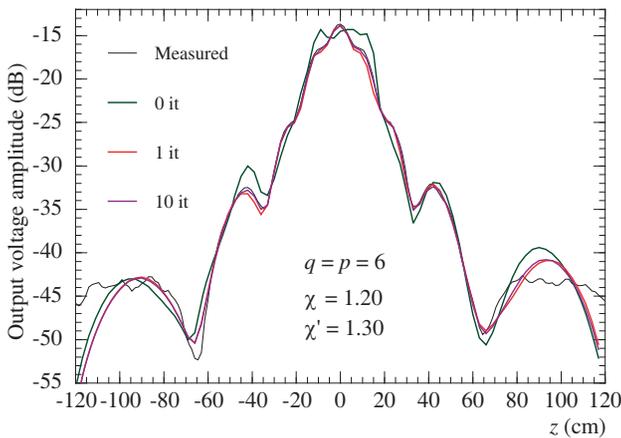

Figure 4. Amplitude of the probe voltage on the generatrix at $\varphi = 0°$. Black line: measured. Coloured lines: recovered from irregurarly spaced NF data via the iterative approach.

reconstruction obtained by employing 10 iterations coincides with that relevant to the use of the SVD approach. Moreover, in the same figure it is shown the intermediate results when 0 and 1 iterations are used, in order to give an insight on the convergence of the technique. Note that all the reconstructions have been obtained by using $\chi' = 1.30$, $\chi = 1.20$, and $p = q = 6$.
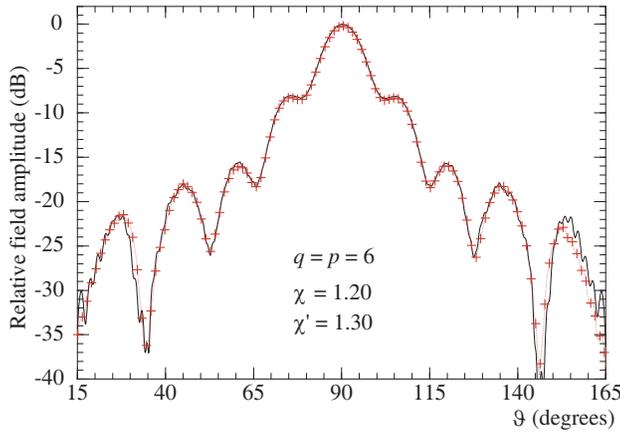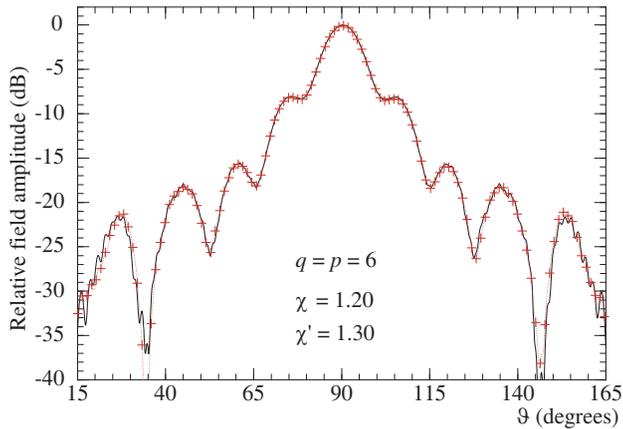
Figure 5. E-plane pattern. Solid line: reference. Crosses: recovered from irregurarly spaced NF data via the SVD-based approach.
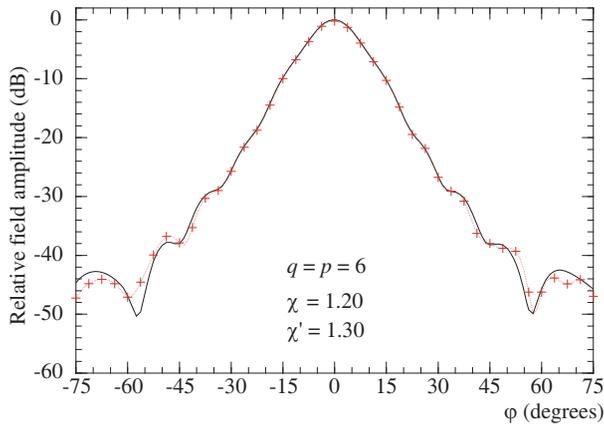

Figure 6. H-plane pattern. Solid line: reference. Crosses: recovered from irregurarly spaced NF data via the SVD-based approach.

The overall effectiveness of the described techniques is assessed by comparing the FF pattern in the principal planes reconstructed from the collected nonuniform NF data (#2 067) with that obtained from the data directly measured on the classical cylindrical grid (#11 520). In both the cases, the software package MI-3000 has been used to get the FF reconstructions. Obviously, once the uniformly distributed data have been recovered, the two-dimensional OSI algorithm has been employed for recovering the cylindrical data needed to perform the

NF-FF transformation. Figures 5 and 6 are relevant to the SVD-based approach and refer to the FF pattern reconstructions in the E and H planes, respectively. As can be seen, these reconstructions result to be very accurate, thus confirming the effectiveness of the approach. Practically identical results are obtained when using the iterative technique.
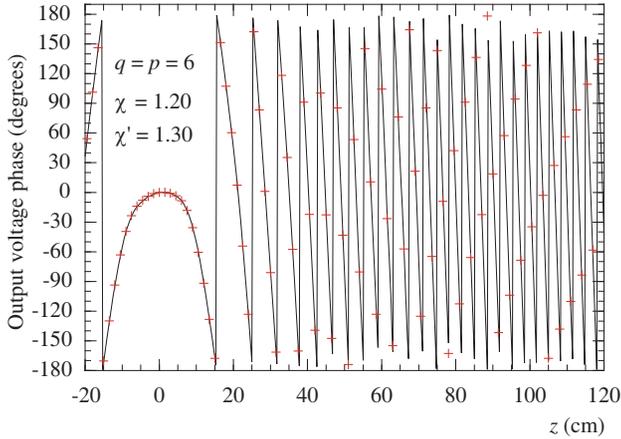

Figure 7. E-plane pattern. Solid line: reference. Crosses: recovered from an increased number of nonuniform NF data via the SVD-based approach.
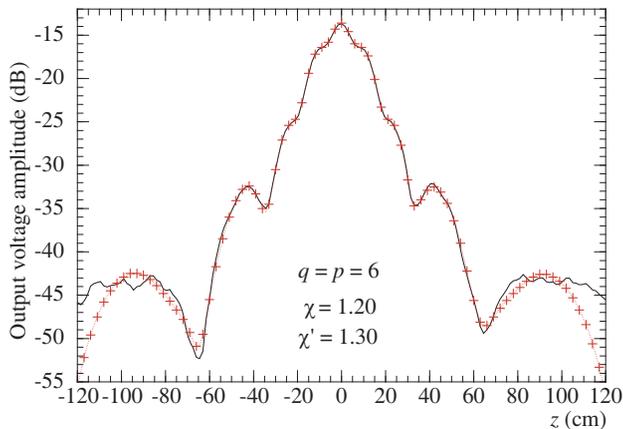

Figure 8. Amplitude of the probe voltage on the generatrix at $\varphi = 0°$. Solid line: measured. Crosses: recovered from irregurarly spaced NF data via the iterative approach.

When using the SVD-based technique, it is possible to exploit the data redundancy to increase the algorithm stability. This is confirmed by the better E-plane pattern reconstruction shown in Fig. 7, which has been obtained from an increased number (3 471) of irregularly spaced data.

Figure 9. Phase of the probe voltage on the generatrix at $\varphi = 0°$. Solid line: measured. Crosses: recovered from irregurarly spaced NF data via the iterative approach.
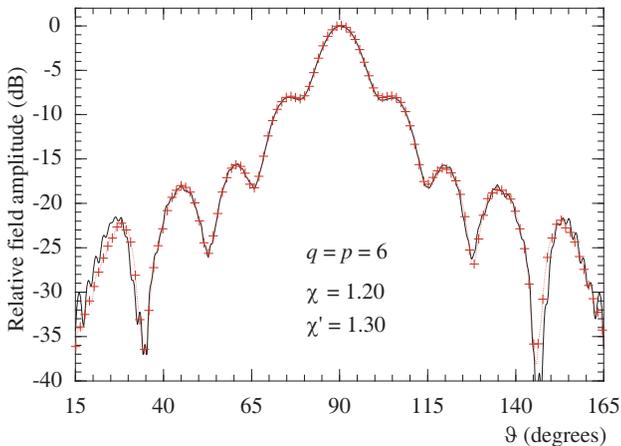
Figure 10. E-plane pattern. Solid line: reference. Crosses: recovered from irregurarly spaced NF data via the iterative approach.

Let us now turn to the validation of the iterative technique when the hypothesis of nonuniform sampling points that lie on rings is removed. To this end, the nonuniform samples have been acquired in such a way that the distances in ϑ and φ between the position of each nonuniform sample and the associated uniform one are random variables uniformly distributed in $(-\Delta\vartheta/3, \Delta\vartheta/3)$ and $(-\Delta\varphi_n/3, \Delta\varphi_n/3)$. The amplitude and phase of the reconstructed probe voltage relevant to the generatrix at $\varphi = 0°$ are compared in Figs. 8 and 9 with those directly measured. Note that the phase is shown only in the range $[-20\,\text{cm}, 120\,\text{cm}]$ in order to improve the readability. At last, the overall effectiveness of the technique is confirmed by the E-plane pattern reconstruction shown in Fig. 10.

5. CONCLUSIONS

In this paper, we have presented and experimentally validated two different techniques to compensate the probe positioning errors in a nonredundant NF-FF transformation with cylindrical scanning. The former makes use of the SVD method for recovering the uniformly distributed samples from the irregularly spaced ones and has been applied when the two-dimensional problem can be reduced to find the solution of two independent one-dimensional ones. The latter employs an iterative technique which requires a one-to-one correspondence, associating at each uniform sampling point the nearest nonuniform one. The validation has been performed at the Antenna Characterization Lab of University of Salerno, where a NF cylindrical facility is available and implicitly confirms the applicability of these approaches to other NF scanning geometries. Although the considered probe positioning errors are very pessimistic in an actual scanning, excellent results have been achieved both in the NF and FF reconstructions.

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