PERTURBATION METHOD FOR THE CALCULATION OF LOSSES INSIDE CONDUCTORS IN MICROWAVE STRUCTURES

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Abstract—A perturbation method based on the decoupling of propagation and diffusion phenomenons is proposed in order to calculate losses in microwave structures. Starting from the first problem in which the conducting regions are not described, a perturbation is calculated by solving a second problem restricted to the vicinity of the conductors; iterations between these problems can be performed when the perturbed solution is not sufficiently accurate. The perturbation approach is however more accurate than a method based on a surface impedance model, without introducing the huge calculations that appear when both conducting region and external medium are described in a single problem. 2D examples are presented using the finite element method and the integral equation method.

1. INTRODUCTION

The perturbation method presented here was originally introduced to solve problems in the low frequency range [1–3] but its scope is larger. Basically, a complex problem is solved in two steps. First, some details are removed, and a simplified problem is considered on the global

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computational domain. Then, these details are restored; this operation makes electromagnetic sources appear, and a second problem is solved on a local computational domain. If a significant coupling exists between the problems on the global and local domains, the process can be iterated, i.e., solutions on the global domain are alternated with solutions on the local domain until a "sufficient accuracy" was reached.

The application proposed here concerns a recurrent problem in the high frequency range, that is the accurate evaluation of losses in passive structures as for example microwave filters. The solution of this problem involves resonance phenomenons specific to the wave equation but also diffusion phenomenons inside the conductors. Two approaches are possible as will be detailed in the following.

In one approach, no hypothesis is made: the numerical result will be accurate but the computing cost will be huge. Indeed, a finite element method implies to mesh the entire structure, including the interior of conductors. Using an integral equation method, the interior of conductors is discretized into volume filaments to describe the volume current; this technique is for example used in the partial equivalent electric circuit method [4].

In the other approach, the conductors are supposed to be modeled by a surface impedance that takes into account the skin effect [5, 6]: the computing cost problem is expected to be reduced but the result will not be correct in particular in the corners of the metallic regions. Refinements of the impedance boundary conditions are however possible with some limitations. Asymptotic expansions of the surface impedance can be derived near corners and edges but these expressions are limited to 2D geometries and depend on the polarizations of the electric field [7]. A better accuracy can be achieved by changing the local relation on the boundary to a global one but those approaches are based on 2D approximations of the original problem and valuable for definite directions of the current [8]. Recently, a method that combines integral equations for the conducting region and the external medium has been proposed [9]; the skin effect is thus well captured but a specific treatment for the Green function integrals in the conductive medium is required to ensure the efficiency of the method.

The perturbation method makes possible to exploit the advantages of both approaches. In a first problem, the conducting regions are not described; the conductors are supposed to be perfectly metallic or modeled by a surface impedance. Then, in a second problem restricted to the vicinity of the conductors, the interior of the conductors is considered; this change makes surface or volumic current sources

appear, whose effects correct the solution found in the first problem. When the first problem is "too far" from the problem with the whole details, the correction brought by the second problem may be not sufficient. Then, going back to the problem on the global computational domain, the solution to the second problem induces new electromagnetic sources and the method can be iterated until accurate results are achieved.

In this paper, different cases of the perturbation method are treated. First, a 1D problem is proposed in order to show how the perturbation method makes possible to properly decouple wave and diffusion problems. Then, the perturbation method is applied in a 2D waveguide in which a conducting plate is introduced as an obstacle; the finite element method is used for the solution of the different sub-problems. Finally, the 2D problem is reassessed when the plate is very thin; an integral equation method can then be used to solve the problem at the large scale.

2. 1D ANALYTICAL PROBLEM

Consider, as illustrated Fig. 1, the reflection of an incident plane wave $E^{\mathrm{inc}} = E_0 e^{-jk_0 z}$ on a wall of high conductivity $\sigma \gg \omega \epsilon_0$; k_0 is the wavenumber in vacuum. For our purpose, the reflection will be characterized by the impedance Z seen at $z = -l$.

The exact solution of this problem is well known. On one hand, the electromagnetic field inside the conductor obeys a diffusion equation whose solution is

$$E_x = E_{x|z=0}\, e^{-\frac{1+j}{\delta} z}, \tag{1}$$

where $\delta = \sqrt{2/(\sigma\omega\mu_0)}$ is the skin depth.

The impedance Z_s at the surface of the conductor is given by

$$Z_s = \left(\frac{E_x}{H_y}\right)_{|z=0} = \frac{1+j}{\sigma\delta}. \tag{2}$$
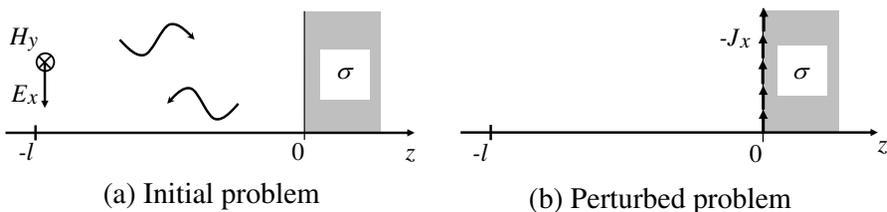
Figure 1. Application of the perturbation method in a 1D problem.

On the other hand, the electromagnetic field in the vacuum region is governed by a wave equation. Then, using the transmission line theory, one can express the impedance Z seen at $z = -l$:

$$Z_{\text{exact}} = Z_0 \frac{Z_s + jZ_0 \tan (k_0 l)}{Z_0 + jZ_s \tan (k_0 l)}. \tag{3}$$

where Z_0 is the impedance of a plane wave in vacuum.

In the application of the perturbation method, the wall is supposed in a first step to be a perfect conductor. One immediately finds the electromagnetic field $\{E_1, H_1\}$ of the first problem:

$$\begin{cases} E_1 = E_0 e^{-jk_0 z} - E_0 e^{+jk_0 z} & z < 0 \\ H_1 = E_0/Z_0 e^{-jk_0 z} + E_0/Z_0 e^{+jk_0 z} & z < 0. \end{cases} \tag{4}$$

This field induces at the interface vacuum/metal a surface current $J_x = H_{y\ |z=0} \times n_z = 2E_0/Z_0$ that does not appear in the exact solution. The perturbed problem consists then in canceling this surface current as illustrated Fig. 1. From a circuit point of view, this problem is equivalent to the situation where a current source $-J_x$ feeds two loads Z_0 and Z_s connected in parallel. When the metal is highly conducting, $Z_0 \gg Z_s$; then, the load Z_0 is insignificant, that is the radiating part of the field can be neglected in the perturbed problem. This assumption is an extension of the image theory which states that an electric current radiates no field when it is adjacent and tangential to a perfectly conducting plate [10]; an electric current radiates nearly no field when it is adjacent and tangential to a highly conducting plate. It is then possible to decouple the wave and the diffusion problems. Thus, the electromagnetic field $\{E_2, H_2\}$ of the perturbed problem is given by

$$\begin{cases} E_2 = Z_s H_2 & z > 0 \\ H_2 = 2E_0/Z_0 e^{-\frac{1+j}{\delta} z} & z > 0. \end{cases} \tag{5}$$

The solution of the whole problem is equal to the sum of the electromagnetic fields $\{E_1, H_1\}$ and $\{E_2, H_2\}$. The impedance Z can then be calculated from Poynting's theorem, that is to say by a volumic calculation of the magnetic and electric energies in the whole problem and of the power dissipated inside the conductor:

$$Z_{\text{perturbe}} = jZ_0 \tan (k_0 l) + \frac{Z_s}{\cos^2 (k_0 l)}, \tag{6}$$

where the first term is the impedance seen in the simplified problem and the second term is the contribution of the perturbed problem.

One can demonstrate that Z_{perturbe} is exactly equal to the Taylor series of Z_{exact} at the first order following $Z_s/Z_0 \times \tan (k_0 l)$; this

equality is consequently valid when $Z_s/Z_0 \ll 1$ and $\tan(k_0 l) \ll 1$. The term $\tan(k_0 l)$ is due to a spurious resonance that appears at $l = \lambda_0/4$ in the first problem. The physical interpretation of this result is that the current J_x induced in the first problem increases dramatically near the resonance; then the radiating part in the perturbed problem increases in the same way and cannot finally be neglected. Thus, one can define Z_s/Z_0 as the small parameter of the perturbation method: the perturbation remains small as long as Z_s/Z_0 is small except near the spurious resonances that appear in the solution of the problem.

Figure 2 gives a comparison of Z_{perturbe} to Z_{exact} in the microwave frequency range when the wall is made of copper ($\sigma = 5.7\,10^7\,\mathrm{S/m}$) and $l = 7\lambda_0/8$. It appears that the error increases slightly with the frequency; this behavior is expected since Z_s varies following $\sqrt{\omega}$. However, the relative error remains lower than 0.01% until 10 GHz.
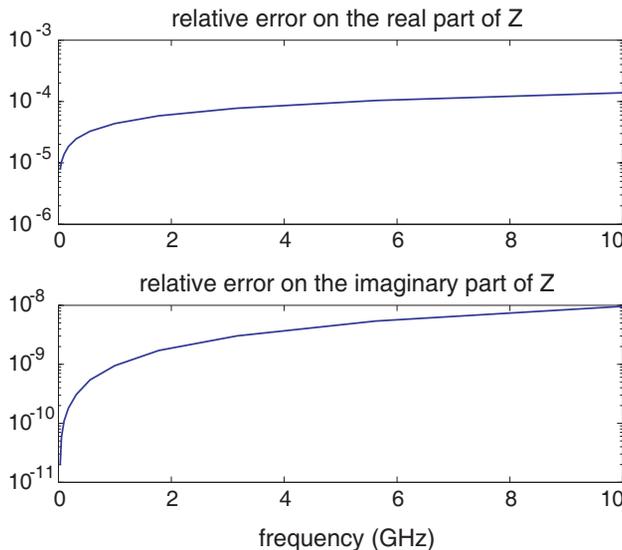
Figure 2. Relative error of Z_{perturbe} when the wall is made of copper and for $l = 7\lambda_0/8$.
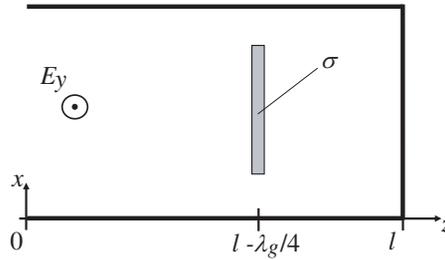
Figure 3. Geometry of the 2D numerical problem.

3. 2D NUMERICAL PROBLEM

3.1. Perturbation Approach Using the Finite Element Method

Consider, as illustrated Fig. 3, a 2D waveguide with perfect conducting walls ended by a short circuit and excited by the TE_1 mode around 2.45 GHz. A high conducting plate of thickness $1/10$ mm ($\approx 100\delta$ at 2.45 GHz) is positioned at 91.0 mm ($= \lambda_g/4$ at 2.45 GHz) from the short circuit.

It is expected that the structure exhibits a resonance around 2.45 GHz, characterized by a peak of losses since the plate is not perfectly conducting. Losses can be quantified by the formula $1 - |\Gamma|^2$ where Γ is the reflection coefficient of the TE_1 mode. This problem is numerically solved using a finite element method with Lagrange polynomial of degree 2; because of the symmetry, only a half of the structure is described. Different approaches to solve this problem are compared.

Firstly, the entire domain — both vacuum and conducting regions — is meshed in a single problem. The invariance in y direction leads to the following weak formulation without a proper treatment for the boundary conditions [11]:

$$\int_\Omega \left[\vec{\nabla} W \cdot \vec{\nabla} E_y - k_0^2 W E_y \right] ds + j\omega\mu_0\sigma \int_{\Omega_c} W E_y ds - \int_{\partial\Omega} W \frac{\partial E_y}{\partial n} dl = 0, \quad (7)$$

where Ω is the entire domain, n the unit outward normal to $\partial\Omega$, Ω_c the conducting region and W a test function.

The numerical difficulty relies on the difference of scales between the diffusion phenomenon ($\delta \approx 1\,\mu$m) and the propagation phenomenon ($\lambda_0 \approx 10$ cm). The mesh is built [12] heeding these characteristic lengths (see Fig. 4). Moreover, due to the skin effect, the electromagnetic field vanishes in the core of the plate; consequently,

(a) Mesh around the plate   (b) mesh inside the plate (in blue).
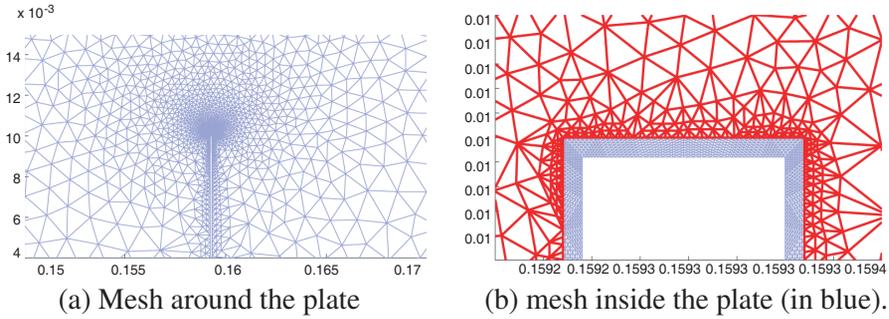
Figure 4. Mesh of the structure.

the interior of the plate is meshed on about 10δ from the border. The solution of this problem, referred as the *initial problem*, implies about 34000 degrees of freedom (that is 34000 unknowns).

Secondly, only the vacuum region is meshed and a surface impedance condition — see (2) — is applied on the boundary of the plate. The solution of this problem, referred as the Z_s *modeling problem*, implies about 18000 degrees of freedom.

Thirdly, the perturbation method is applied. In the first problem, the plate is supposed to be a perfect conductor. Consequently, only the vacuum region is meshed. The weak formulation of this problem is given by

$$\int_{\Omega_1} \left[\vec{\nabla}W \cdot \vec{\nabla}E_1 - k_0^2 W E_1 \right] ds - \int_{\partial\Omega_1} W \frac{\partial E_1}{\partial n} dl = 0, \qquad (8)$$

where $\Omega_1 = \Omega \setminus \Omega_c$ is the domain of the first problem.

In a second problem, a current source $-J_1$ is introduced at the interface vacuum/plate in order to cancel the electric current J_1 induced by the first problem. This source term is not calculated from the normal derivative of E_1 on $\partial\Omega_c$ but from the weak formulation:

$$\int_{\Omega_2} \left[\vec{\nabla}W \cdot \vec{\nabla}E_2 - k_0^2 W E_2 \right] ds + j\omega\mu_0\sigma \int_{\Omega_c} W E_2 ds - \int_{\partial\Omega_2} W \frac{\partial E_2}{\partial n} dl$$

$$= -\int_{\Omega_2} \left[\vec{\nabla}W \cdot \vec{\nabla}E_1 - k_0^2 W E_1 \right] ds + \int_{\partial\Omega_2} W \frac{\partial E_1}{\partial n} dl$$

$$= +j\omega\mu_0 \int_{\partial\Omega_c} W J_1 ds, \qquad (9)$$

where Ω_2 is the domain of the second problem whose definition is detailed in the following.

Here, the second problem is not only described inside the conducting region, but it is also extended in the vacuum region that surrounds the plate in order to take into account the reactive field induced in the vicinity of the plate; this is an important difference compared to the 1D problem solved in Section 2 in which only a plane wave propagates in the vacuum region. In the present problem, the extension of the domain in the vacuum region is chosen as a disk whose size is in relation with the dimensions of the plate; radiation conditions are enforced on the boundary of the disk. The solution of this problem, referred as the *perturbed problem n° 1*, implies about 18000 degrees of freedom (first problem) + 21000 degrees of freedom (second problem).

The results of the different methods are reported Fig. 5; moreover, the electric field in the *initial problem* and in the *perturbed problem n° 1* are given Fig. 6. In Fig. 5, it appears that there is a difference of the results given by the Z_s *modeling problem* and the *initial problem* that reaches 0.16 dB at the resonance, that is about 4%. The *perturbed problem n° 1* gives better results except near a frequency slightly lower than the resonance frequency: there is a spurious resonance. This frequency coincides with the resonance of the first problem where no losses are introduced; at this point, the induced current increases dramatically — like in the analytical problem presented in Section 2 — and the radiated field in the second problem becomes non-negligible.
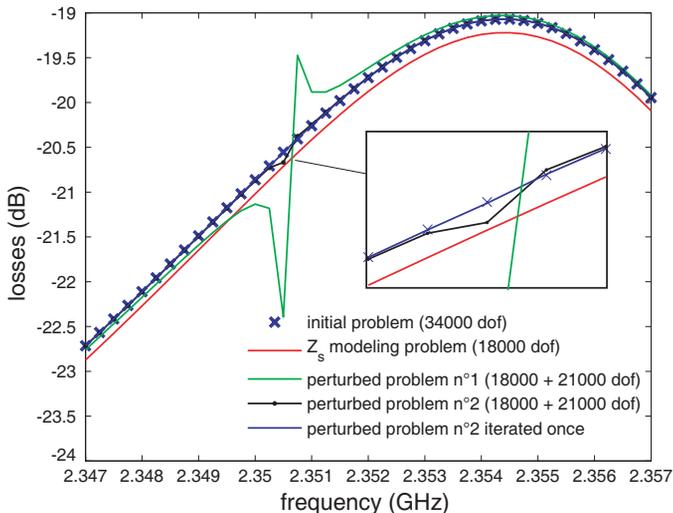
Figure 5. Results of the 2D numerical problem using the finite element method. The acronym "dof" means degrees of freedom.

(a) Initial problem

(b) Perturbed problem nº1
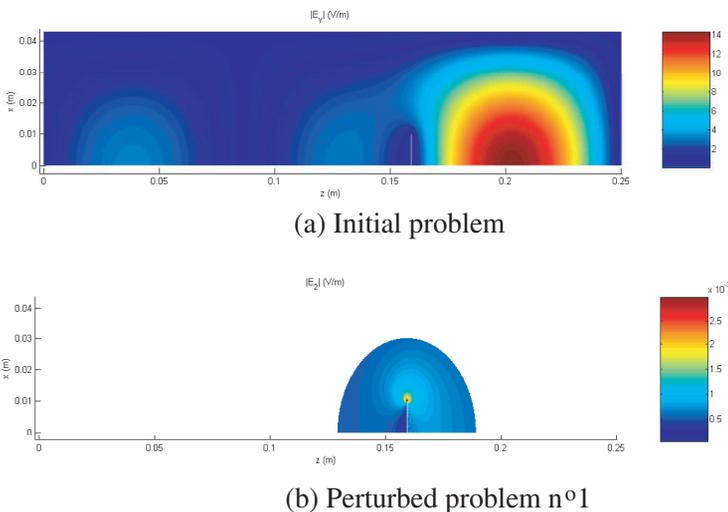
Figure 6. Electric field in different problems at $2.37\,\text{GHz}$ when the TE_1 mode is set at $z = 0$ with an amplitude of $1\,\text{V/m}$.

To avoid this difficulty, the first problem can be modified such that the plate is characterized by a surface impedance Z_s. In this situation, the second problem makes two kind of sources appear at the interface vacuum/metal: an electric current like in the *perturbed problem n° 1* but also a magnetic current since the electric field is not zero at the surface of the plate. Numerically, this problem is solved by introducing a function g continuous in $\Omega_2 \setminus \Omega_c$ such that the solution $E_1 + g$ is continuous everywhere in Ω_2:

$$\begin{cases} g(x, z) = -E_1(x, z) & \forall \{x, z\} \in \partial\Omega_c, \\ g(x, z) = 0 & \forall \{x, z\} \in \Omega_c. \end{cases} \tag{10}$$

Thus the magnetic source is canceled. Then, the electric current due to $E_1 + g$ is treated in a similar way as (9). The results of this problem, referred as the *perturbed problem n° 2*, are improved compared to the ones of *perturbed problem n° 1* (see Fig. 5). This was expected since the perturbation method improves the accuracy of the first problem that is the Z_s *modeling problem* here. However, there remains a residual error near the spurious resonance.

To cancel it, one has to iterate the perturbation method. Without iteration, the solution E_2 is not zero on the boundary of Ω_2 and consequently there are magnetic and electric currents on the border of the disk. Those currents can be canceled in a third problem described

in Ω_1. Then, the solution of this third problem can be corrected in the neighborhood of the plate like in the second problem and so on. A detailed algorithm is reported in Appendix A. The results of the *perturbed problem n°2* iterated once are given Fig. 5: they show that one iteration is enough to make the effects of the spurious resonance completely disappear.

3.2. Perturbation Approach Using an Integral Equation Method

When the thickness of the plate is not larger than a few skin depths δ, an integral equation formulation can be applied. In this approach, the plate is supposed infinitely thin; the problem consists then in calculating the surface current J_1 induced on the plate. Here, the solution can be more properly expanded using trigonometric polynomials instead of Lagrange polynomials. Applying the Method of Moments [11], the integral formulation is given by

$$\int\limits_{\Gamma_c} V E_0 dl + \int\limits_{\Gamma_c} V G * J_1 dl = Z'_s \int\limits_{\Gamma_c} V J_1 dl, \tag{11}$$

where Γ_c is the surface of the equivalent plate without thickness, E_0 the electric field due to the source when there is no plate, G the Green function for the model considered, Z'_s the surface impedance and V a test function.

The surface impedance Z'_s in (11) gives the relation between the total electric field and the surface current. Its expression is different from (2); considering a 1D plate with a thickness e, the surface impedance is derived assuming the diffusion phenomenon on both sides of the plate [8]:

$$Z'_s = \frac{1+j}{2\sigma\delta} \Big/ \tanh\left(\frac{1+j}{2\delta}e\right). \tag{12}$$

In addition, the Green function [13] can be reduced in the present case to a single series:

$$
\begin{aligned}
&G(x, z = l_1, x', z' \leq l_1) \\
&= -\frac{2}{a} \times \sum_{m=1,3,\ldots} Z_m \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{m\pi}{a}x'\right) \frac{\sinh\left(\gamma_m z'\right)}{\sinh\left(\gamma_m l_1\right)}, \\
&G(x, z = l_1, x', z' \geq l_1) \\
&= -\frac{2}{a} \times \sum_{m=1,3,\ldots} Z_m \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{m\pi}{a}x'\right) \frac{\sinh\left(\gamma_m (l - z')\right)}{\sinh\left(\gamma_m l_2\right)},
\end{aligned} \tag{13}
$$
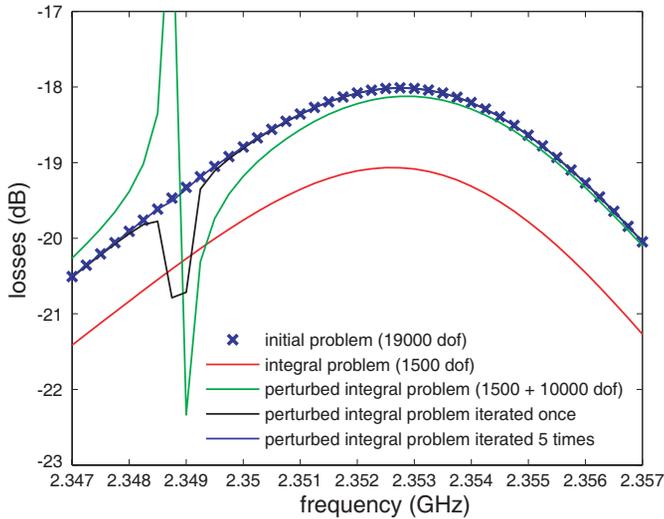
Figure 7. Results of the 2D numerical problem using an integral equation method. The acronym "dof" means degrees of freedom.

with

$$Z_m = \frac{j\omega\mu_0}{\gamma_m}\left(\frac{1}{\tanh\left(\gamma_m l_1\right)} + \frac{1}{\tanh\left(\gamma_m l_2\right)}\right)^{-1},$$

$$\gamma_m = \sqrt{\left(\frac{m\pi}{a}\right)^2 - k_0^2},$$

where a is the width of the waveguide, l_1 the distance of the plate to the origin, l_2 the distance of the plate to the short circuit and $l = l_1 + l_2$.

The solution of this problem, referred as the *integral problem*, implies about 1500 degrees of freedom. The results of this method are given Fig. 7. In this numerical example, the thickness of the plate is $e = 3\delta$. It appears that there is an error of about 1 dB (that is about 25%) compared to the *initial problem* solved here with $e = 3\delta$.

In order to improve the accuracy of the results, the perturbation method is applied. The solution E_1 of the *integral problem* is corrected introducing the thickness of the plate as described Fig. 8. The source terms of this second problem can be identified from Maxwell-Ampère equation. Indeed, the solution $\{E_1, H_1\}$ of the first problem is such that in Ω_c region:

$$\vec{\nabla} \times \vec{H}_1 = j\omega\epsilon_0\vec{E}_1 + \vec{J}_1. \tag{14}$$

Then, the solution $\{E_2, H_2\}$ of the second problem is built such that the complete solution $\{E_1 + E_2, H_1 + H_2\}$ is consistent with the
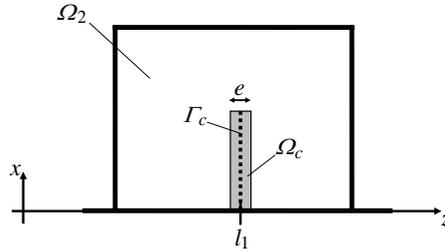
Figure 8. Second problem in the perturbation method using an integral method where Ω_2 is the computational domain, Ω_c the real conducting domain of the plate and Γ_c the surface of the equivalent infinitely thin plate.

electromagnetic properties of the conducting region Ω_c:

$$\vec{\nabla} \times \left(\vec{H}_1 + \vec{H}_2 \right) = j\omega\epsilon_0 \left(\vec{E}_1 + \vec{E}_2 \right) + \sigma \left(\vec{E}_1 + \vec{E}_2 \right). \qquad (15)$$

From (14) and (15), one finds the following equation:

$$\vec{\nabla} \times \vec{H}_2 = (j\omega\epsilon_0 + \sigma) \vec{E}_2 - \vec{J}_1 + \sigma\vec{E}_1. \qquad (16)$$

In this expression, it appears two kinds of sources for the second problem: a surface current $-J_1$, exactly like in the problem treated in Subsection 3.1, but also a volumic current σE_1. Consequently, the weak formulation of the second problem is given by

$$\int\limits_{\Omega_2} \left[\vec{\nabla} W \cdot \vec{\nabla} E_2 - k_0^2 W E_2 \right] ds + j\omega\mu_0\sigma \int\limits_{\Omega_c} W E_2 ds - \int\limits_{\partial\Omega_2} W \frac{\partial E_2}{\partial n} dl$$

$$= -\int\limits_{\Omega_2} \left[\vec{\nabla} W \cdot \vec{\nabla} E_1 - k_0^2 W E_1 \right] ds - j\omega\mu_0\sigma \int\limits_{\Omega_c} W E_1 ds + \int\limits_{\partial\Omega_2} W \frac{\partial E_1}{\partial n} dl. \ (17)$$

The domain Ω_2 is a rectangle here (see Fig. 8). The reason of this choice is that it simplifies the solution of the iterated problem as it will be discussed in the following paragraph. Perfect metal conditions are set on the boundary of this domain for the same reason. The solution of this problem, referred as the *perturbed integral problem*, implies about 1500 degrees of freedom (first problem) + 10000 degrees of freedom (second problem). The results of this method are given Fig. 7. The perturbation method introduces a spurious resonance larger than the one observed in the numerical example treated in Subsection 3.1.

To cancel this spurious resonance, one has to iterate the perturbation method. At this stage, the characteristics of $\partial\Omega_2$ plays

upon the complexity of the third problem that is solved in the entire domain using an integral formulation. Here, perfect metal conditions have been required for E_2 solution; then, there is only an electric current source in the third problem and consequently only a single operator is used in the integral formulation. Moreover, the current source is defined on the sides of a rectangular; then, the Green function G' is reduced to a single series in a similar way as in (13). Thus, the integral formulation of the third problem is given by

$$\frac{1}{j\omega\mu_0} \int\limits_{\Gamma_c} V G' * \frac{\partial E_2}{\partial n} |_{\partial\Omega_2} dl + \int\limits_{\Gamma_c} V G * J_3 dl = Z'_s \int\limits_{\Gamma_c} V J_3 dl. \quad (18)$$

Note that the current source on $\partial\Omega_2$ is calculated from the derivative of E_2.

The results for different iterations are given Fig. 7. One iteration is enough to converge except near the spurious resonance; the convergence around this critical frequency point is reached after five iterations.

4. CONCLUSION

A perturbation method has been successfully applied to calculate losses in microwaves resonating problems. On one hand, this approach properly decouples propagation and diffusion phenomenons and on the other hand its application in numerical problems makes possible to exploit the advantages of the different numerical methods, like the use of the weak formulation in the finite element method or the choice of the Green function in the integral equation method. However, contrarily to its application in low frequency problems, the method can generate spurious resonances, which enforce to iterate several times the scheme in the neighborhood of those frequency points.

APPENDIX A. ALGORITHM FOR THE ITERATIVE SCHEME OF THE PERTURBATION METHOD USING THE FINITE ELEMENT METHOD

In order to build the algorithm, the weak formulation of the different problems has to be explicitly detailed with the proper boundary conditions to avoid ambiguities. For a sake of simplicity, a formalism with bilinear forms can be used. In the first problem of the *perturbed*

problem n° 2, two bilinear forms a_1 and a_1' are introduced:

$$a_1(u,v) = \int_{\Omega_1} \left[\vec{\nabla}u \cdot \vec{\nabla}v - k_0^2 uv \right] ds + \frac{j\omega\mu_0}{Z_s} \int_{\partial\Omega_c} uv\, dl,$$

$$a_1'(u,v) = \int_{\Omega_1} \left[\vec{\nabla}u \cdot \vec{\nabla}v - k_0^2 uv \right] ds - \int_{\partial\Omega_1} u\frac{\partial v}{\partial n} dl,$$

(A1)

where $u, v \in H^1(\Omega_1)$. Z_s is given by (2).

The source term in the first problem of the *perturbed problem n° 2* can be introduced via a continuous function E_0 that set the TE_1 mode at $z = 0$. Then, the formulation given by (8) can be summarized as follow:

$$a_1(W, E_1) = -a_1'(W, E_0).$$

(A2)

In a similar way, the second problem of the *perturbed problem n° 2* can be formalized introducing two bilinear forms a_2 and a_2':

$$a_2(u,v) = \int_{\Omega_2} \left[\vec{\nabla}u \cdot \vec{\nabla}v - k_0^2 uv \right] ds + j\omega\mu_0\sigma \int_{\Omega_c} uv\, ds$$

$$+ \alpha \int_{\mathscr{C}} uv\, dl + \beta \int_{\mathscr{C}} \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} dl,$$

(A3)

$$a_2'(u,v) = \int_{\Omega_2} \left[\vec{\nabla}u \cdot \vec{\nabla}v - k_0^2 uv \right] ds - \int_{\partial\Omega_2} u\frac{\partial v}{\partial n} dl,$$

where $u, v \in H^1(\Omega_2)$. The constants α and β are related to the second order radiation conditions set on the boundary $\mathscr{C}$ of the disk that partially defines $\partial\Omega_2$ (remember that the core of the plate is not described because of the skin effect); these constants depend on the radius of the disk and on the wavenumber k_0 [14]. τ is the unit tangent on $\mathscr{C}$.

Then, the second problem of the *perturbed problem n° 2* can be expressed from the weak formulation given in (9) and the function g given in (10):

$$a_2(W, E_2) = -a_2'(W, E_1 + g(E_1)).$$

(A4)

To formalize the third problem, a function f is introduced in order to cancel the magnetic current due to E_2 that appears on $\mathscr{C}$. This function is continuous in Ω_2 and such that:

$$\begin{cases} f(x,z) = -E_2(x,z) & \forall\{x,z\} \in \mathscr{C}, \\ f(x,z) = 0 & \forall\{x,z\} \in \Omega_1 \setminus \Omega_2. \end{cases}$$

(A5)

Thus, the third problem can be summarized as follow:

$$a_1 (W, E_3) = -a_1' (W, E_2 + f (E_2)). \tag{A6}$$

The iterative scheme can then be built repeating (A4) and (A6) as presented in Algorithm 1. Note that a quantity ϵ is introduced in Algorithm 1 to quantify the convergence; in the present case, ϵ gives the relative error in the calculation of losses between two solutions E and E_{old}. Moreover, the initial solution E_0 is chosen such that $f (E_0) = 0$ so that the algorithm is consistent with (A2).

Algorithm 1 Algorithm of the *perturbed problem n°2*

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set E_0; $E_{old} = 0$; $\epsilon = \epsilon_{\max}$
$n = 1$
while $\epsilon \geq \epsilon_{\max}$ **do**
    $a_1 (W, E_n) = -a_1' (W, E_{n-1} + f (E_{n-1}))$
    $a_2 (W, E_{n+1}) = -a_2' (W, E_n + g (E_n))$
    $E = E_{old} + E_n + E_{n+1} + f (E_{n-1}) + g (E_n)$
    compute $\epsilon (E, E_{old})$
    $E_{old} = E$
    $n = n + 2$
end while

Algorithm 1 shows that the perturbation method presents some similarities with the multi-scale algorithm using patches of finite elements [15]. This method consists in calculating successive corrections to the solution in domains meshed at different scales. However, the multi-scale algorithm using patches of finite elements is not dedicated specially to the wave equation and consequently does not exploit the specific boundary conditions required in this kind of problems; moreover, this approach is limited to the finite element method, which is not the case of the perturbation method.

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