Progress In Electromagnetics Research B, Vol. 21, 299–328, 2010

ANALYTICAL METHODS IN THE THEORY OF THIN IMPEDANCE VIBRATORS

M. V. Nesterenko

Department of Radiophysics
V. N. Karazin Kharkov National University
4, Svobody Sq., Kharkov 61077, Ukraine

Abstract—The advantages and disadvantages of more extended approximated analytical methods of the integral equations solution for the current in thin perfectly conducting and impedance vibrators have been investigated in details in this paper. The solutions of the problem about the electromagnetic waves scattering by the thin vibrators with the distributed surface impedance, obtained with the help of the method of expansion of the searched function for the current in a series on small parameter. The method of consistent iterations and asymptotic averaging method are given. The comparison of the calculated results with the experimental data in the case of excitation of the vibrator in the centre by the point source of voltage is represented.

1. INTRODUCTION

A great number of publications (see, for example, [1–20]) are devoted to the investigation of the electrodynamic characteristics of material bodies of different configurations, on the surface of which the impedance boundary conditions are set. Thin vibrators, applied to antenna-waveguide engineering widely, take a special place among them. Both direct numerical and approximate analytical or numerical-analytical methods of the solution of the corresponding boundary problems of electrodynamics are applied to the mathematical analysis of the functional characteristics of different devices, the components of which are thin impedance vibrators. The undoubted advantage of analytical methods is that they are physically more visual in comparison with the numerical methods, and they permit to use available calculated resources more effectively. There is no comparative analysis of advantages and disadvantages of the known

Corresponding author: M. V. Nesterenko (Mikhail.V.Nesterenko@gmail.com).

approximated analytical methods of the integral equations solution for the current in thin impedance vibrators in the literary sources. At present, suggestions of new numerical-analytical approaches, allowing to eliminate existing disadvantages of the electromagnetic theory of impedance vibrator antennas, are also absent. The author's aim is to fill up these principle gaps in this paper.

2. GENERAL QUESTIONS OF THE THEORY OF THIN IMPEDANCE VIBRATORS IN SPATIAL-FREQUENCY REPRESENTATION

2.1. Problem Formulation and Initial Integral Equations

Let us formulate the general problem about scattering (radiation) of electromagnetic waves by the material body, having finite sizes. The problem geometry and accepted symbols are represented in Figure 1. Let the electromagnetic field of the set impressed sources $\{\vec{E}_0(\vec{r}), \vec{H}_0(\vec{r})\}$, depending on the time t as $e^{i\omega t}$ ($\vec{r}$ is the radius-vector of the observation point; $\omega = 2\pi f$ is the circular frequency; f is the frequency, measured in Hertz), influence the body of the volume V, which is bounded by the smooth closed surface S and is characterized by the homogeneous material parameters (the permittivity ε, permeability μ and conductivity σ). The body is located inside the electrodynamic volume V_1, bounded by the perfectly conducting (or impedance) surface S_1 (including the infinitely remote one) and filled by the medium with the parameters ε_1, μ_1 (being complex piece-constant functions of the coordinates in a more general case). The field of the impressed sources can be set as the field of the electromagnetic wave, incident on the body (the problem about scattering) or as the field of the electromotive forces (EMF), applied

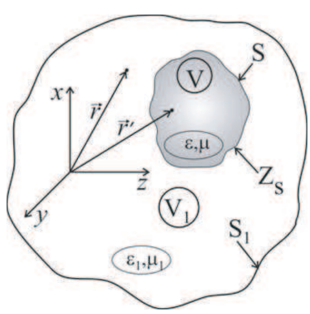

Figure 1. The problem general formulation.

to the body, different from null only in some part of the volume V (the problem about radiation), or as a combination of these fields in a general case. We need to define the full electromagnetic field $\{\vec{E}(\vec{r}),\, \vec{H}(\vec{r})\}$ in the volume V_1, satisfying the Maxwell's equations and boundary conditions on the surfaces S and S_1.

As known, the set problem can be investigated on the basis of the equations for the electromagnetic fields in a differential or an integral form. The advantage of the integral equations use consists in that their solutions satisfy the required boundary conditions on the body's surface automatically [21]. Besides they are differentially effective for the cases, when the boundary surfaces S and S_1 represent themselves as the coordinate surfaces in different coordinate systems: for example, S_1 — the surface of the waveguide with cylindrical symmetry, and the body's surface S can be of quite another symmetry in a general case.

That is why the mathematical model of the electromagnetic process in question is not created on the basis of the Maxwell's equations in their differential form, but on the basis of the integral equations of macroscopic electrodynamics, equivalent to the boundary problem in whole — to the Maxwell's equations together with the boundary conditions on the surfaces of the body S and of the electrodynamic volume S_1, in which the body is located. They have the form in the Gauss' unit system CGS [21, 22]:

$$\begin{aligned}\vec{E}\left(\vec{r}\right) &= \vec{E}_0\left(\vec{r}\right) + \left(\text{grad div} + k^2\varepsilon_1\mu_1\right)\vec{\Pi}^e\left(\vec{r}\right) - ik\mu_1\text{rot}\vec{\Pi}^m\left(\vec{r}\right),\\ \vec{H}\left(\vec{r}\right) &= \vec{H}_0\left(\vec{r}\right) + \left(\text{grad div} + k^2\varepsilon_1\mu_1\right)\vec{\Pi}^m\left(\vec{r}\right) + ik\varepsilon_1\text{rot}\vec{\Pi}^e\left(\vec{r}\right).\end{aligned} \tag{1}$$

Here $k = 2\pi/\lambda$ is the wave number, and λ is the wavelength in free space. $\vec{\Pi}^e(\vec{r})$ and $\vec{\Pi}^m(\vec{r})$ are the Hertz's electrical and magnetic vector potentials, correspondingly equal to

$$\begin{aligned}\vec{\Pi}^e\left(\vec{r}\right) &= \frac{(\varepsilon/\varepsilon_1 - 1)}{4\pi}\int\limits_V \hat{G}^e\left(\vec{r},\, \vec{r}'\right)\vec{E}\left(\vec{r}'\right)d\vec{r}',\\ \vec{\Pi}^m\left(\vec{r}\right) &= \frac{(\mu/\mu_1 - 1)}{4\pi}\int\limits_V \hat{G}^m\left(\vec{r},\, \vec{r}'\right)\vec{H}\left(\vec{r}'\right)d\vec{r}',\end{aligned} \tag{2}$$

$\hat{G}^e(\vec{r},\, \vec{r}')$ and $\hat{G}^m(\vec{r},\, \vec{r}')$ are the Green's electrical and magnetic tensor functions for the vector potential, satisfying the Helmholtz's vector equation and the corresponding boundary conditions on the surface S_1. Let us note that in the case when the surface S_1 is removed on infinity, the corresponding boundary conditions for $\hat{G}^{e,m}(\vec{r},\, \vec{r}')$ transit into the Zommerfeld's radiation condition.

Interpretation of the fields, located in the left part of Equation (1), is changed in the dependence on the observation point $\vec{r}$ location with the coordinates x, y, z, in which it is necessary to define the searched field. If the point $\vec{r}$ belongs to the volume V, then the fields $\vec{E}(\vec{r})$ and $\vec{H}(\vec{r})$ represent the internal fields for the body, that is, the same fields, located under the integrals signs in the right part of (1). Equation (1) are the Fredholm's inhomogeneous linear integral equations of the second kind in this case, and they have the unique mathematically correct solution. If the point $\vec{r}$, in which the field is searched, is outside the region V, then Equation (1) become equalities, defining the full electromagnetic field in the medium outside the material body via the set field of the impressed sources. These equalities solve the problem about scattering (radiation) of the electromagnetic waves by the bodies of finite sizes in a more general kind, if the fields inside these bodies are known.

It turns out to be expedient to express the full electromagnetic field in the volume V_1 via the fields tangential components on the surface S, bounding the volume V, at the solution of a great number of problems. As a result of the corresponding mathematical transformations [21, 22] in this case, transition to the Kirchhoff-Kotler's integral equations, which are completely equivalent to the initial Equation (1) ($k_1 = k\sqrt{\varepsilon_1\mu_1}$, $\vec{n}$ is the outer normal to the surface S) and turns out to be possible:

$$\begin{aligned}\vec{E}(\vec{r}) \;=\; & \vec{E}_0(\vec{r}) + \frac{1}{4\pi ik\varepsilon_1}\left(\text{grad div} + k_1^2\right)\int\limits_S \hat{G}^e(\vec{r},\vec{r}\,')\left[\vec{n}, \vec{H}(\vec{r}\,')\right] d\vec{r}\,' \\ & - \frac{1}{4\pi}\text{rot}\int\limits_S \hat{G}^m(\vec{r},\vec{r}\,')\left[\vec{n}, \vec{E}(\vec{r}\,')\right] d\vec{r}\,', \\ \vec{H}(\vec{r}) \;=\; & \vec{H}_0(\vec{r}) + \frac{1}{4\pi ik\mu_1}\left(\text{grad div} + k_1^2\right)\int\limits_S \hat{G}^m(\vec{r},\vec{r}\,')\left[\vec{n}, \vec{E}(\vec{r}\,')\right] d\vec{r}\,' \\ & + \frac{1}{4\pi}\text{rot}\int\limits_S \hat{G}^e(\vec{r},\vec{r}\,')\left[\vec{n}, \vec{H}(\vec{r}\,')\right] d\vec{r}\,'. \end{aligned} \tag{3}$$

The representation (3) is used at the electrodynamic problems solution in the cases, when the field on the surface of the material body is defined due to some additional physical considerations. For example, for good conducting bodies ($\sigma \to \infty$) the current, induced in them, is concentrated near the body's surface. Then, neglecting the thickness of the skin-layer, it is possible to use the Leontovich-Shukin's approximate impedance boundary condition [23]:

$$\left[\vec{n}, \vec{E}(\vec{r})\right] = \bar{Z}_S(\vec{r})\left[\vec{n}, \left[\vec{n}, \vec{H}(\vec{r})\right]\right], \tag{4}$$

where $\bar{Z}_S(\vec{r}) = \bar{R}_S(\vec{r}) + i\bar{X}_S(\vec{r}) = Z_S(\vec{r})/Z_0$ is the distributed surface impedance (normalized on the wave impedance of free space $Z_0 = 120\pi\,\text{Ohm}$), the value of which can be changed from one point to another on the body's surface in a general case. Locating the observation point $\vec{r}$ on the body's surface S, we obtain the following integral equation due to (3), (4)

$$
\begin{aligned}
Z_S(\vec{r})\vec{J}^e(\vec{r}) &= \vec{E}_0(\vec{r}) + \frac{1}{i\omega\varepsilon_1}\left(\text{grad div} + k_1^2\right)\int_S \hat{G}^e\left(\vec{r},\vec{r}'\right)\vec{J}^e\left(\vec{r}'\right)d\vec{r}' \\
&\quad + \frac{1}{4\pi}\,\text{rot}\int_S \hat{G}^m\left(\vec{r},\vec{r}'\right)Z_S\left(\vec{r}'\right)\left[\vec{n},\vec{J}^e\left(\vec{r}'\right)\right]d\vec{r}', \qquad (5)
\end{aligned}
$$

concerning the density of the surface electrical current

$$
\vec{J}^e(\vec{r}) = \frac{c}{4\pi}\left[\vec{n},\,\vec{H}(\vec{r})\right], \qquad (6)
$$

where $c \approx 2.998 \cdot 10^{10}\,\text{cm/sec}$ is the velocity of light in vacuum.

Thus the problem about scattering (radiation) of the electromagnetic waves by the impedance body of finite sizes is formulated as a rigorous boundary problem of macroscopic electrodynamics, and it is reduced to the integral equation for the current, the solution of which represents itself as an independent problem, connected with substantial mathematical difficulties. If the body's characteristic sizes exceed the wavelength many times (the high-frequency region), then the solution of the set problem is usually searched in the form of expansions in ascending power series on inverse degrees of wave number. When the body has the sizes less than the wavelength in the low-frequency (quasi-static) region, representation of the unknown functions in the form of series in terms of the wave number reduces the problem to the solution of the electrostatic problems sequence. The resonant region, in which at least one of the body's sizes is commensurate with the wavelength, unlike the asymptotic cases, is the most complex for analysis, because one needs the rigorous solution of the field equations. Let us note that from the practical point of view, it is in the very region that the bodies in the form of thin impedance vibrators, to which the paper is devoted, represent special interest.

2.2. Green's Function as the Kernel of the Integral Equation

The dominant problem of analysis of scattering (radiation) of the electromagnetic waves by the material bodies of finite sizes for the linear mediums is the definition of the field, excited by means of the point source. This is a classical problem about searching of the Green's function, which gives the opportunity for analytical representation of the boundary problems solution of macroscopic electrodynamics. As known, the Green's function is the tensor (the symmetrical tensor of the second rank — affinor) function of mutual location of two points: the observation point, having the radius-vector $\vec{r}$, and the source point with the radius-vector $\vec{r}'$, in the case of the vector fields. The Green's tensor functions have first been considered in [24] in electrodynamics, and they were investigated by many authors (see, for example, [25–30]) later on.

The Green's functions of the inhomogeneous vector equations, to which the Maxwells equations are reduced, are defined by the solution of one of the following tensor equations:

$$\mathrm{rotrot}\,\hat{G}_E\left(\vec{r},\vec{r}'\right) - k_1^2\hat{G}_E\left(\vec{r},\vec{r}'\right) = 4\pi\hat{I}\delta\left(\left|\vec{r}-\vec{r}'\right|\right), \tag{7}$$

$$\Delta\hat{G}_A\left(\vec{r},\vec{r}'\right) + k_1^2\hat{G}_A\left(\vec{r},\vec{r}'\right) = -4\pi\hat{I}\delta\left(\left|\vec{r}-\vec{r}'\right|\right) \tag{8}$$

and they satisfy all boundary conditions of the concrete boundary problem. Here, in the case of the rectangular coordinate system $\delta(|\vec{r}-\vec{r}'|) = \delta(x-x')\delta(y-y')\delta(z-z')$ is the Dirac's three-dimensional delta-function; $\hat{I} = (\vec{e}_x \otimes \vec{e}_{x'}) + (\vec{e}_y \otimes \vec{e}_{y'}) + (\vec{e}_z \otimes \vec{e}_{z'})$ is the unit affinor; $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are the orts of the Decart's coordinate system; Δ is the Laplace's operator; $\otimes$ is the sign of tensor multiplication.

The Green's functions allows to obtain the expressions for the electromagnetic fields of the kind of (1) of the arbitrary vector source in any point of space in a closed form. If some volume distribution of the $\vec{J}^e(\vec{r})$ electrical current density is set as a source, then the electrical field is represented by one of the following expressions:

$$\vec{E}\left(\vec{r}\right) = \frac{k_1^2}{i\omega}\int\limits_V \hat{G}_E\left(\vec{r},\vec{r}'\right)\vec{J}^e\left(\vec{r}'\right)d\vec{r}', \tag{9}$$

$$\vec{E}\left(\vec{r}\right) = \frac{1}{i\omega}\left(\mathrm{grad}\ \mathrm{div} + k_1^2\right)\int\limits_V \hat{G}_A\left(\vec{r},\vec{r}'\right)\vec{J}^e\left(\vec{r}'\right)d\vec{r}', \tag{10}$$

where the symbol $\hat{G}^e(\vec{r},\vec{r}') = \hat{G}_A(\vec{r},\vec{r}')$ is introduced, and $\vec{E}_0(\vec{r}) = 0$ is accepted for simplicity. The availability of these two representations may formally be connected with different calibrations of the electromagnetic field potentials. However, $\hat{G}_E$ is called the Green's function for the field, and $\hat{G}_A$ — the Green's function for the vector potential in accordance with the used variant of the equation for the field or the potential, to which the initial Maxwell's equations are reduced in the concrete problem solution. It is not difficult to make ratio between them

$$\hat{G}_E = \left(\hat{I} + \frac{1}{k_1^2}\{\mathrm{grad} \otimes \mathrm{grad}\}\right)\hat{G}_A, \tag{11}$$

considering the tensor product of two symbolic vectors under the operation $\{\mathrm{grad} \otimes \mathrm{grad}\}$.

The only boundary condition, imposed on the Green's function, for infinite space is the Zommerfeld's radiation condition, and it is represented in the form

$$\hat{G}_A\left(\vec{r}, \vec{r}'\right) = \hat{I}\,\frac{e^{-ik_1|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}. \tag{12}$$

If $\vec{r} = \vec{r}'$, then the functions (12) and (11) become the infinity, and it is impossible to define the integrals as the limit of the integral sum in a general sense, because it does not exist. So, strictly speaking, the formulas (9), (10) are just for those points of space, where the sources are absent. In the case, when the observation point $\vec{r}$ coincides with one of the points $\vec{r}'$ of the source, the volume integrals in (9), (10) must be considered as improper ones, that is, $\int\limits_V = \lim\limits_{\rho\to 0}\int\limits_{V-v}$, where v is the eliminated volume, contained in the sphere of the infinitely small radius ρ with the centre in the point $\vec{r}'$. The main difference of the two kinds of integrals consists in that the improper integral in (10) can be modified into the absolutely converging one [28], when, principally, the integral in (9) is the conditionally converging one [28], and its value depends upon the kind of the eliminated region, containing the special point $\vec{r} = \vec{r}'$. It is necessary to consider the main value of the integral in (9), which makes the main contribution to the result of integration, for the field physically correct definition in the source region. One may consider the integrals in a general sense, expanding the singular part in the Green's functions. It is rather difficult to investigate the Green's tensor function behavior (11) in the neighborhood of the points of the field source. Rigorous consideration of these questions requires application of the mathematical apparatus of the generalized functions [29]. One would note the advantage of the use of the expression for the field (10), based on the Green's function for the vector potential in comparison with (9) and (11), and, namely, the field (10) has the integrated peculiarity, and one manages to avoid calculation of the main integral value; though, certainly, it is necessary to make differentiation in (10), taking into account the dependencies of the integral from the parameter.

This ground also refers to the bounded regions completely, because the Green's function of the bounded region has the same peculiarity at the coincidence of arguments as in the case of unbounded space [26].

The general solution of Equation (8) can be represented in this case in the form [21]:

$$\hat{G}\left(\vec{r},\vec{r}'\right)=\hat{I}G\left(\vec{r},\vec{r}'\right)+\hat{G}_0^{V_1}\left(\vec{r},\vec{r}'\right), \tag{13}$$

where $G(\vec{r},\vec{r}')=\frac{e^{-ik_1|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$, and $\hat{G}_0^{V_1}(\vec{r},\vec{r}')$ is the everywhere regular function, satisfying the homogeneous equation

$$\Delta\hat{G}_0^{V_1}\left(\vec{r},\vec{r}'\right)+k_1^2\hat{G}_0^{V_1}\left(\vec{r},\vec{r}'\right)=0 \tag{14}$$

and providing the fulfillment of the boundary conditions on the S_1 surface of the V_1 volume for the point source field, located in the point $\vec{r}'$, together with $\hat{I}G(\vec{r},\vec{r}')$.

2.3. Integral Equations for the Current in Thin Impedance Vibrators

The known mathematical difficulties take place at the direct solution of Equation (5) for the material body V with a complex shape of the surface. However, it is sufficiently simplified for the impedance cylinders, the perimeter of cross section of which is small in comparison with their lengths and the wavelength in the environment (thin vibrators). Besides one manages to extend the boundary condition (4) on the cylindrical surfaces with arbitrary distribution of the complex impedance, independent of the structure of the exciting field and the electrophysical characteristics of the material, of which the vibrator is made, in this case.

Let us transform the integral Equation (5), which is applied to the thin vibrator, representing itself as a bounded circular cylindrical wire of the radius r and length $2L$ (of a curvilinear axial configuration in a general case), for which the following ratios are performed:

$$\frac{r}{2L}\ll 1,\quad \left|\frac{r}{\lambda_1}\right|\ll 1,\quad \frac{r}{\tilde{r}}\ll 1, \tag{15}$$

where λ_1 is the wavelength in the environment, and $\tilde{r}$ is the curvature radius of the vibrator axial line. These inequalities permit to consider that the density of the induced current has only a longitudinal component (we omit the index "e")

$$\vec{J}(\vec{r})=\vec{e}_sJ(s)\psi(\rho,\varphi), \tag{16}$$

and it is distributed on the cross section of the vibrator as in a quasi-stationary case [31], what is more,

$$\int\limits_{\perp}\psi(\rho,\varphi)\,\rho\,d\rho\,d\varphi=1. \tag{17}$$

In expressions (16), (17), $\vec{e}_s$ is the unit vector along the tangent to the $\{0s\}$ axis, coupled with the vibrator; $\psi(\rho,\varphi)$ is the function of the transverse ($\perp$) polar coordinates ρ and φ; $J(s)$ is the searched current, obeying the boundary conditions on the ends of the vibrator:

$$J(-L) = J(L) = 0. \tag{18}$$

Taking all these into consideration and projecting Equation (5) on the vibrator axis when taking into account that $[\vec{n}, \vec{J}^e(\vec{r})] \ll 1$ due to (15), we obtain the equation concerning the current in the thin impedance vibrator, located in the homogeneous isotropic infinitely extended medium:

$$z_i(s)J(s) = E_{0s}(s) + \frac{1}{i\omega\varepsilon_1}\int_{-L}^{L}\left[\frac{\partial}{\partial s}\frac{\partial J(s')}{\partial s'} + k_1^2(\vec{e}_s\vec{e}_{s'})J(s')\right]G_s(s,s')ds'. \tag{19}$$

Here $E_{0s}(s)$ is the projection of the impressed field, parallel to the $\vec{e}_s$ vector, and $z_i(s)$ is the internal impedance per unit length ([Ohm/m]) of the vibrator ($Z_S(\vec{r}) = 2\pi r z_i(\vec{r})$). $\vec{e}_{s'}$ is the ort of the $\{0s'\}$ axis, coupled with the vibrator surface,

$$G_s(s,s') = \int_{-\pi}^{\pi} \frac{e^{-ik_1\sqrt{(s-s')^2+[2r\sin(\varphi/2)]^2}}}{\sqrt{(s-s')^2+[2r\sin(\varphi/2)]^2}}\,\psi(r,\varphi)\,r d\varphi \tag{20}$$

is the exact kernel of the integral equation.

There are rather great difficulties at the solution of equation (19) with the kernel in the form of (20), so the "thin-wire" approximation is used in the theory of vibrators [31]

$$G_s(s,s') = \frac{e^{-ik_1 R(s,s')}}{R(s,s')}, \quad R(s,s') = \sqrt{(s-s')^2 + r^2}, \tag{21}$$

which supposes the source points location on the vibrator geometrical axis and the observation points — on its physical surface. The $G_s(s,s')$ function is continuous everywhere in this case, and the equation for the current is sufficiently simplified without noticeable degradation of preciseness [32]. Applying further integration in parts with taking into account the condition (18) in the Equation (19), we have for the rectilinear conductor ($(\vec{e}_s\vec{e}_{s'}) = 1$):

$$\left(\frac{d^2}{ds^2} + k_1^2\right)\int_{-L}^{L} J(s')G_s(s,s')\,ds' = -i\omega\varepsilon_1 E_{0s}(s) + i\omega\varepsilon_1 z_i(s)J(s). \tag{22}$$

If the vibrator is located in the bounded electrodynamic volume (a waveguide, a resonator), we obtain due to (5) and (13):

$$\begin{aligned}
&\int_{-L}^{L}\left\{\left[\frac{\partial}{\partial s}\frac{\partial J\left(s'\right)}{\partial s'}+k_1^2\left(\vec{e}_s\vec{e}_{s'}\right)J\left(s'\right)\right]G_s\left(s,s'\right)\right.\\
&\left.\quad+J\left(s'\right)\left[\frac{\partial^2}{\partial s^2}+k_1^2\right]\vec{e}_s\left(\hat{G}_{0s}^{V_1}\left(s,s'\right)\vec{e}_{s'}\right)\right\}ds'\\
=&\ -i\omega\varepsilon_1E_{0s}(s)+i\omega\varepsilon_1z_i(s)J(s),
\end{aligned}\tag{23}$$

where $G_{0s}^{V_1}(s,s')=\int\limits_{\perp}G_{0s}^{V_1}(s,\rho,\phi;\,s',\rho',\phi')\psi(\rho',\phi')\rho'd\rho'd\phi'$. Then the equation for the current has the form for the rectilinear impedance vibrator, located in the V_1 volume:

$$\begin{aligned}
&\left(\frac{d^2}{ds^2}+k_1^2\right)\int_{-L}^{L}J\left(s'\right)G_s\left(s,s'\right)ds'\\
=&\ -i\omega\varepsilon_1E_{0s}(s)+i\omega\varepsilon_1z_i(s)J(s)-F_0[s,J(s)],
\end{aligned}\tag{24}$$

where

$$F_0[s,J(s)]=\left(\frac{d^2}{ds^2}+k_1^2\right)\int_{-L}^{L}J\left(s'\right)G_{0s}^{V_1}(s,s')ds'.\tag{25}$$

Let us note that Equation (22) is called the Pocklington's integral one [33,34] in the case, when the volume V_1 is free space, $z_i=0$, and Equation (19) — the Mei's equation [35] after integration of its left and right parts along s in general accepted terminology of the thin vibrators theory.

2.4. Approximate Analytical Methods of the Integral Equations Solutions for the Current

We cannot manage to obtain the rigorous solution of the equations given above for the electrical current in the impedance vibrator in a closed form. However, it does not result from this that it is impossible to approximate the current true distribution by the approximate solution rather precisely. It is natural to use the known methods, developed for perfectly conducting vibrators earlier, for this. So, we can extract the methods for rectilinear vibrators in free space: consistent iterations [31,36], expansion of the searched function in a series on small parameter [37], variational [31], of the searching of the "key" equation [38]. Let us obtain Equation (24) solution by the method of expansion of the searched function in a series on small parameter (briefly, the small parameter method further) and the method of consistent iterations (the iterations method further) for the case of the electromagnetic wave, incident on the perfectly conducting vibrator ($z_i = 0$), located in some volume V_1 at $\varepsilon_1 = \mu_1 = 1$ in order to define advantages and disadvantages of one or another approximate analytical method.

2.4.1. Method of Expansion of the Searched Function in a Series on Small Parameter

The following equation is original for analysis:

$$\left(\frac{d^2}{ds^2}+k^2\right)\int\limits_{-L}^{L} J\left(s'\right)\frac{e^{-ikR(s,s')}}{R\left(s,s'\right)}ds' = -i\omega E_{0s}(s) - f_0[s, J(s)], \quad (26)$$

where $R(s,s') = \sqrt{(s-s')^2+r^2}$,

$$f_0[s, J(s)] = \left(\frac{d^2}{ds^2}+k^2\right)\int\limits_{-L}^{L} J\left(s'\right)G_{0s}^{V_1}\left(s,s'\right)ds' \quad (27)$$

is the regular part of the vibrator own field, defined by the volume V_1 geometry.

Exchanging the differential ds' on dR in (26) and taking into account, that

$$\left.\begin{array}{ll} s' = s - \sqrt{R^2-r^2}, & \text{if} \quad s' \le s \\ s' = s + \sqrt{R^2-r^2}, & \text{if} \quad s' \ge s \end{array}\right\},$$

we transform Equation (26) into the form

$$\begin{aligned}\left(\frac{d^2}{ds^2}+k^2\right)&\left\{-\int\limits_{-L}^{s} J\left(s'\right)e^{-ikR}d\ln\left[C\left(R+\sqrt{R^2-r^2}\right)\right]\right.\\ &\left.+\int\limits_{s}^{L} J(s')e^{-ikR}d\ln\left[C\left(R+\sqrt{R^2-r^2}\right)\right]\right\}\\ =\ & -i\omega E_{0s}(s) - f_0[s, J(s)], \quad (28)\end{aligned}$$

where C is the arbitrary constant. Making integration in parts with the use of the boundary conditions for the current (18) in (28), we obtain

$$\left(\frac{d^2}{ds^2}+k^2\right)\left\{J(s)e^{-ikr}\ln Cr + \int\limits_{-L}^{s}\ln\left[C\left(R+\sqrt{R^2-r^2}\right)\right]\right.$$

$$X d\left[J\left(s'\right)e^{-ikR}\right]-\int\limits_{s}^{L}\ln\left[C\left(R+\sqrt{R^2-r^2}\right)\right]d\left[J\left(s'\right)e^{-ikR}\right]\Bigg\}$$
$$= i\omega E_{0s}(s)+f_0[s,J(s)]. \tag{29}$$

Putting in, with taking into consideration (15), that $e^{-ikr}=1$ and choosing $C=1/2L$ (unlike from [37], where $C=k$), we reduce Equation (26) to the following integral-differential equation for the current with the small parameter

$$\frac{d^2J(s)}{ds^2}+k^2J(s)=\alpha\left\{i\omega E_{0s}(s)+f[s,J(s)]+f_0[s,J(s)]\right\}. \tag{30}$$

Here $\alpha=\frac{1}{2\ln[r/(2L)]}$ is the small parameter,

$$f[s,J(s)]=-\left(\frac{d^2}{ds^2}+k^2\right)\int\limits_{-L}^{L}\mathrm{sign}\left(s-s'\right)\ln\frac{R+(s-s')}{2L}\frac{d}{ds'}\left[J(s')e^{-ikR}\right]ds' \tag{31}$$

is the vibrator own field in free space.

Let us represent $J(s)$ in the form of the power series on small parameter $|\alpha|\ll 1$:

$$J(s)=J_0(s)+\alpha J_1(s)+\alpha^2J_2(s)+\dots \tag{32}$$

The substitution of (32) into (27) and (31) permits to expand into analogous series

$$f_\Sigma[s,J(s)]=f_\Sigma[s,J_0(s)]+\alpha\, f_\Sigma[s,J_1(s)]+\alpha^2 f_\Sigma[s,J_2(s)]+\dots, \tag{33}$$

where $f_\Sigma[s,J(s)]=f[s,J(s)]+f_0[s,J(s)]$ is the vibrator sum own field. Now, having substituted (32) and (33) into Equation (30) and equated the multipliers at equal degrees α between each other in the right and left parts of the equation, we obtain the following system of differential equations:

$$\begin{aligned}
&\frac{d^2J_0(s)}{ds^2}+k^2J_0(s)=0,\\
&\frac{d^2J_1(s)}{ds^2}+k^2J_1(s)=i\omega E_{0s}(s)+f_\Sigma[s,J_0(s)],\\
&\frac{d^2J_2(s)}{ds^2}+k^2J_2(s)=f_\Sigma[s,J_1(s)],\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{d^2J_n(s)}{ds^2}+k^2J_n(s)=f_\Sigma[s,J_{n-1}(s)],
\end{aligned} \tag{34}$$

which can be solved by the method of successive approximations. At this each equation is solved at the boundary conditions of the form of (18), and, namely $J_0(\pm L) = 0$, $J_1(\pm L) = 0$, $J_2(\pm L) = 0$, $\ldots$, $J_n(\pm L) = 0$.

The first equation of system (34) has the solution, independent of the exciting field $E_{0s}(s)$:

$$J_0(s) = C_1 \cos ks + C_2 \sin ks, \tag{35}$$

which satisfies the boundary conditions only at ratios fulfillment

$$C_1 = 0 \text{ at } 2L = m\lambda; \quad C_2 = 0 \text{ at } 2L = (2n+1)\frac{\lambda}{2}, \tag{36}$$

where m and n are the integers. If the vibrator length 2L does not satisfy the conditions (36), then $J_0 \equiv 0$, $f_\Sigma[s, J_0(s)] \equiv 0$ the current in the first approximation is equal to:

$$J(s) = \alpha J_1(s) = -\alpha \frac{i\omega/k}{\sin 2kL} \left\{ \sin k(L-s) \int_{-L}^{s} E_{0s}(s') \sin k(L+s')\, ds' + \sin k(L+s) \int_{s}^{L} E_{0s}(s') \sin k(L-s')\, ds' \right\}. \tag{37}$$

As seen, the functions of the own field of the vibrator $f[s, J(s)]$ and $f_0[s, J(s)]$, which, mainly, define the vibrator resonant and energetic characteristics, are not included into the expressions for the current. Obviously, it is necessary to obtain the following approximations in order to take into account $f_\Sigma[s, J(s)]$, what, however, meets essential mathematical difficulties, and at present only $J_2(0)$ is known for the vibrator in free space, excited in the centre by the point source of voltage [37].

As an example, let us consider the problem about scattering of the dominant wave H_{10} by the vibrator, located in the plain of crossed section of a standard rectangular waveguide, parallel to its narrow wall. The impressed field equals in this case:

$$E_{0s}(s) = E_0 \sin \frac{\pi x_0}{a}. \tag{38}$$

Here E_0 is the amplitude of the dominant wave H_{10}, incident from the region $z = -\infty$, and x_0 is the distance from the waveguide narrow wall up to the vibrator axial line. Then the current, induced in the vibrator, equals due to (37)

$$J(s) = -\alpha E_0 \sin \frac{\pi x_0}{a} \frac{i\omega}{k^2} \frac{(\cos ks - \cos kL)}{\cos kL}. \tag{39}$$

And, finally, the solution of a classical problem about normal incidence of a plane electromagnetic wave on the vibrator in free space by the method of the small parameter in the first approximation gives the result ($E_{0s}(s) = E_0$):

$$J(s) = -\alpha E_0 \frac{i\omega}{k^2} \frac{(\cos ks - \cos kL)}{\cos kL}. \tag{40}$$

Let us note that the condition of formulas (39) and (40) application is the inequality [37]

$$\left| kL - n\frac{\pi}{2} \right| \gg |\alpha|, \tag{41}$$

which limits possibilities of the use of the obtained solution in practice together with the ratios (36) to essential extent.

The solution of Equation (22) for the vibrator, located in free space ($\varepsilon_1 = \mu_1 = 1$), with the constant ($z_i(s) = \text{const}$) distributed impedance, by the method of a small parameter, has been obtained in [39]. The zeroth and first approximations for the current have the following form in this case:

a) for tuned vibrator ($\tilde{\tilde{k}}L = n\frac{\pi}{2}$, where n is the integer):

$$J_0(s) = C_1 \cos \tilde{\tilde{k}}s + C_2 \sin \tilde{\tilde{k}}s, \tag{42}$$

b) for untuned vibrator ($\tilde{\tilde{k}}L \neq n\frac{\pi}{2}$):

$$J(s) = \alpha J_1(s) = -\alpha \frac{i\omega/\tilde{\tilde{k}}}{\sin 2\tilde{\tilde{k}}L} \left\{ \sin \tilde{\tilde{k}}(L-s) \int\limits_{-L}^{s} E_{0s}(s') \sin \tilde{\tilde{k}}(L+s')\, ds' \right.$$
$$\left. + \sin \tilde{\tilde{k}}(L+s) \int\limits_{s}^{L} E_{0s}(s') \sin \tilde{\tilde{k}}(L-s')\, ds' \right\}, \tag{43}$$

where $\tilde{\tilde{k}} = k\sqrt{1 + i\alpha\omega z_i/k^2}$.

2.4.2. Method of the Consistent Iterations

Let us use the method of consistent iterations, suggested by Hallen [36] and developed by King [31] for the investigation of vibrators characteristics in free space in order to eliminate the disadvantages, mentioned above, of the integral equation solution for the current by the method of the small parameter.

Converting the differential operator in the left part of (26), we obtain the following integral equation

$$\int\limits_{-L}^{L} J(s')\, G_s^{\Sigma}(s, s')\, ds' = C_1 \cos ks + C_2 \sin ks$$

$$-\frac{i\omega}{k}\int\limits_{-L}^{s} E_{0s}\left(s'\right)\sin k\left(s-s'\right)\,ds', \tag{44}$$

in which it is taken into account that $G_s^{\Sigma}(s,s') = G_s(s,s') + G_{0s}^{V_1}(s,s')$. It is necessary to use the conditions of symmetry [31], which are definitely connected with the method of vibrator excitation to obtain one of the arbitrary constants C_1 and C_2. In other words, one needs to concretize the field of the impressed sources $E_{0s}(s)$ on this stage of solution of the initial equation by the method of iterations yet. Let us put in $E_{0s}(s) = E_0$ due to (38), which corresponds to the vibrator excitation by the dominant wave in a rectangular waveguide in the case, when the vibrator axis is located at $x_0 = a/2$. Then

$$\int\limits_{-L}^{L} J\left(s'\right) G_s^{\Sigma}(s,s')\,ds' = C_1\cos ks + \frac{i\omega}{k^2}E_0\left(\cos ks\cos kL - 1\right). \tag{45}$$

We note that Equation (45) is analogous to Hallen's linearized integral equation [31, 36], which is the ground of many publications in the theory of thin vibrator antennas, for $G_s^{\Sigma}(s,s') = e^{-ikR}/R$.

The kernel of the integral Equation (45) has peculiarity of quasi-stationary kind on the vibrator surface. Let us extract it, using smallness of the vibrator cross size in comparison with its length and wavelength. For this we rewrite the left part of (45) in the following way:

$$\int\limits_{-L}^{L} J\left(s'\right) G_s^{\Sigma}\left(s,s'\right)\,ds' = \int\limits_{-L}^{L} J\left(s'\right)\frac{e^{-ikR\left(s,s'\right)}}{R\left(s,s'\right)}\,ds' + \int\limits_{-L}^{L} J\left(s'\right) G_{0s}^{V_1}\left(s,s'\right)\,ds'. \tag{46}$$

Then

$$\int\limits_{-L}^{L} J\left(s'\right)\frac{e^{-ikR\left(s,s'\right)}}{R\left(s,s'\right)}ds' = \Omega(s)J(s) + \int\limits_{-L}^{L}\left[J\left(s'\right)\frac{e^{-ikR\left(s,s'\right)}}{R\left(s,s'\right)} - \frac{J(s)}{R\left(s,s'\right)}\right]ds', \tag{47}$$

where

$$\Omega(s) = \int\limits_{-L}^{L}\frac{ds'}{\sqrt{\left(s-s'\right)^2 + r^2}}. \tag{48}$$

The first addendum in the right part of the expression (47) is logarithmically large in comparison with the second regular term, and the $\Omega(s)$ function differs from its average value $\bar{\Omega}(s) = 2\ln(2L/r) -$

0.614 only on the vibrator ends, where the current vanishes: $J(\pm L) = 0$. Taking this into account, Equation (45) is transformed to the form:

$$\begin{aligned} J(s) \ &= \ -\alpha\left[C_1\cos ks + \frac{i\omega}{k^2}E_0(\cos ks\ \cos kL - 1)\right] \\ &\quad +\alpha\int\limits_{-L}^{L}\left[J\left(s'\right)G_s^{\Sigma}\left(s,s'\right) - \frac{J(s)}{R\left(s,s'\right)}\right]ds', \end{aligned} \tag{49}$$

where $\alpha = \frac{1}{2\ln[r/(2L)]}$ is the small parameter, coinciding with the one obtained above in Subsection 2.4.1 at the choice of the constant of integration $C = 1/2L$.

Following the method described in [31, 36] further, let us put in $s = L$ in (49) and subtract the obtained expression from (49) (in fact, we subtract 0, because $J(L) \equiv 0$). At this Equation (49) is transformed in the following way:

$$\begin{aligned} J(s) &= -\alpha\left[C_1(\cos ks - \cos kL) + \frac{i\omega}{k^2}E_0\cos kL(\cos ks - \cos kL)\right] \\ &\quad +\alpha\left\{\int\limits_{-L}^{L}\left[J(s')G_s^{\Sigma}\left(s,s'\right) - \frac{J(s)}{R\left(s,s'\right)}\right]ds' - \int\limits_{-L}^{L}J\left(s'\right)G_s^{\Sigma}\left(L,s'\right)ds'\right\}. \end{aligned} \tag{50}$$

Choosing the addendums in the right part of the first line in Equation (50) as the zeroth-order approximation for the current $J_0(s)$ and using condition (18) to define the C_1 constant, we obtain:

$$J_0(s) = -\alpha E_0\frac{i\omega}{k^2}\frac{(\cos ks - \cos kL)}{\cos kL}, \tag{51}$$

that coincides identically with the expression (39) at $x_0 = a/2$, obtained by means of the small parameter method in the first approximation. Substituting (51) into (50) now, we obtain the first approximation for the current with accuracy to the terms of the α^2 order:

$$J_1(s) = -\alpha E_0\frac{i\omega}{k^2}\frac{(\cos ks - \cos kL)}{\cos kL + \alpha F(kr, kL)}, \tag{52}$$

where

$$F(kr, kL) = \int\limits_{-L}^{L}\left[\left(\cos ks' - \cos kL\right)G_s^{\Sigma}\left(L, s'\right)\right]ds' \tag{53}$$

is the vibrator own field function, giving opportunity to analyze both tuned ($\cos kL = 0$) and unturned ($\cos kL \neq 0$) vibrators already in the first approximation along α (unlike the small parameter method), what is more, the integral in (53) is taken analytically with the help of the generalized integral functions [31] in the case, when $G_s^{\Sigma}(s, s') = e^{-ikR}/R$ (the vibrator in free space).

Let us note that if one solves Equation (24) at $z_i(s) = \text{const}$, $\varepsilon_1 = \mu_1 = 1$ and $E_{0s}(s) = E_0$ by the method of iterations, then one obtains the following expression for the current in the impedance vibrator in the first approximation as a result:

$$J_1(s) = -\alpha E_0 \frac{i\omega}{k^2} \frac{(\cos ks - \cos kL)}{\cos kL + \alpha F(kr, kL, z_i)}, \tag{54}$$

that is, availability of the distributed impedance in the vibrator operates (unlike the solution by the method of a small parameter) only in the function of the vibrator own field, but not in the function of the current distribution.

2.5. Solution of the Integral Equation for the Current in the Thin Impedance Vibrator by the Asymptotic Averaging Method

Let us choose Equation (22) as the original one to analyze (at $z_i(s) = \text{const}$, $\varepsilon_1 = \mu_1 = 1$) with the approximate kernel (21), being quasi-one-dimensional analogue of the exact integral equation with the kernel (20):

$$\left(\frac{d^2}{ds^2} + k^2\right) \int_{-L}^{L} J\left(s'\right) \frac{e^{-ikR(s,s')}}{R\left(s, s'\right)} ds' = -i\omega E_{0s}(s) + i\omega z_i J(s), \tag{55}$$

where $R(s, s') = \sqrt{(s - s')^2 + r^2}$. It is obvious that $F_0[s, J(s)] \equiv 0$ here. We extract the kernel logarithmic peculiarity of Equation (55) analogically with (47):

$$\int_{-L}^{L} J(s') \frac{e^{-ikR(s,s')}}{R(s, s')} ds' = \Omega(s)J(s) + \int_{-L}^{L} \frac{J(s')e^{-ikR(s,s')} - J(s)}{R(s, s')} ds'. \tag{56}$$

Here

$$\Omega(s) = \int_{-L}^{L} \frac{ds'}{\sqrt{(s - s')^2 + r^2}} = \Omega + \gamma(s), \tag{57}$$

$\gamma(s) = \ln \frac{[(L+s)+\sqrt{(L+s)^2+r^2}]\,[(L-s)+\sqrt{(L-s)^2+r^2}]}{4L^2}$ — some function, equal to zero in the vibrator centre and reaching its largest value on the vibrator ends, where the current equals zero according to the boundary conditions (18), and $\Omega = 2\ln\frac{2L}{r}$ is the large parameter. Then, taking into account (57), Equation (55) is transformed into the following integral-differential equation with the small parameter:

$$\frac{d^2J(s)}{ds^2} + k^2J(s) = \alpha\left\{i\omega E_{0s}(s) + F[s, J(s)] - i\omega z_i J(s)\right\}, \tag{58}$$

where $\alpha = \frac{1}{2\ln[r/(2L)]}$ is the problem natural small parameter ($|\alpha| \ll 1$),

$$\begin{aligned} F[s, J(s)] = & -\frac{dJ(s')}{ds'}\frac{e^{-ikR(s,s')}}{R(s,s')}\Bigg|_{-L}^{L} + \left[\frac{d^2J(s)}{ds^2} + k^2J(s)\right]\gamma(s) \\ & + \int\limits_{-L}^{L} \frac{\left[\frac{d^2J(s')}{ds'^2} + k^2J(s')\right]e^{-ikR(s,s')} - \left[\frac{d^2J(s)}{ds^2} + k^2J(s)\right]}{R(s,s')}\,ds' \end{aligned} \tag{59}$$

is the vibrator own field in free space.

Let us use the asymptotic averaging method. The main grounds and the principles of which are represented in [40, 41], in order to obtain the approximate analytical solution of Equation (58). We change the variables, following the method of variation of arbitrary constants, in order to reduce Equation (58) to the equations system of a standard kind [40, 41] with the small parameter:

$$\begin{aligned} & J(s) = A(s)\cos ks + B(s)\sin ks, \\ & \frac{dJ(s)}{ds} = -A(s)k\sin ks + B(s)k\cos ks, \\ & \frac{d^2J(s)}{ds^2} + k^2J(s) = -\frac{dA(s)}{ds}\sin ks + \frac{dB(s)}{ds}\cos ks, \end{aligned} \tag{60}$$

where $A(s)$ and $B(s)$ are the new unknown functions. Then Equation (58) transits into the following system of the integral-differential equations:

$$\begin{aligned} \frac{dA(s)}{ds} &= -\frac{\alpha}{k}\left\{\begin{matrix} i\omega E_{0s}(s) + F\left[s, A(s), \frac{dA(s)}{ds}, B(s), \frac{dB(s)}{ds}\right] \\ -i\omega z_i[A(s)\cos ks + B(s)\sin ks] \end{matrix}\right\}\sin ks, \\ \frac{dB(s)}{ds} &= +\frac{\alpha}{k}\left\{\begin{matrix} i\omega E_{0s}(s) + F\left[s, A(s), \frac{dA(s)}{ds}, B(s), \frac{dB(s)}{ds}\right] \\ -i\omega z_i[A(s)\cos ks + B(s)\sin ks] \end{matrix}\right\}\cos ks. \end{aligned} \tag{61}$$

The obtained equations are completely equivalent to Equation (58) and are the system of the integral-differential equations of a standard kind, unsolved relatively to the derivative. The right parts of these equations are proportional to the α small parameter, so the functions $A(s)$ and $B(s)$ in the right parts of Equation (61) can be considered as slowly changing functions, and the averaging asymptotic method can be used to solve the equations system (61). Then putting the simplified system, in which $\frac{dA(s)}{ds}=0$ and $\frac{dB(s)}{ds}=0$ in the right parts of the equations, in accordance to equations system (61) and making partial averaging along s explicitly input variable in it (the term "partial" designates the effect by the averaging operator on all addendums except those ones which contain $E_{0s}(s)$, what is possible [41] for the system of the kind (61) in this case), we obtain the equations of the first approximation:

$$\begin{aligned}\frac{d\bar{A}(s)}{ds}&=-\alpha\left\{\frac{i\omega}{k}E_{0s}(s)+\bar{F}[s,\bar{A}(s),\bar{B}(s)]\right\}\sin ks+\chi\bar{B}(s),\\ \frac{d\bar{B}(s)}{ds}&=+\alpha\left\{\frac{i\omega}{k}E_{0s}(s)+\bar{F}[s,\bar{A}(s),\bar{B}(s)]\right\}\cos ks-\chi\bar{A}(s),\end{aligned}\tag{62}$$

in which $\chi=\alpha\frac{i\omega}{2k}z_i$,

$$\bar{F}\left[s,\bar{A}(s),\bar{B}(s)\right]=\left[\bar{A}\left(s'\right)\sin ks'-\bar{B}\left(s'\right)\cos ks'\right]\left.\frac{e^{-ikR(s,s')}}{R\left(s,s'\right)}\right|_{-L}^{L}\tag{63}$$

is the vibrator own field (59), averaged along its length.

We shall obtain equations system (62) solution in the following form [42]:

$$\begin{aligned}\bar{A}(s)&=C_1(s)\cos\chi s+C_2(s)\sin\chi s,\\ \bar{B}(s)&=-C_1(s)\sin\chi s+C_2(s)\cos\chi s.\end{aligned}\tag{64}$$

Then we have after transformations instead of (62):

$$\begin{aligned}\frac{dC_1(s)}{ds}&=-\alpha\left\{\frac{i\omega}{k}E_{0s}(s)+\bar{F}[s,C_1,C_2]\right\}\sin(k+\chi)s,\\ \frac{dC_2(s)}{ds}&=+\alpha\left\{\frac{i\omega}{k}E_{0s}(s)+\bar{F}[s,C_1,C_2]\right\}\cos(k+\chi)s.\end{aligned}\tag{65}$$

We obtain $C_1(s)$ and $C_2(s)$ from (65) and also $\bar{A}(s)$ and $\bar{B}(s)$ from (64) further, using them as the approximating functions for the current in (60). As a result, we obtain the general asymptotic (along the α parameter) expression for the current in the thin impedance vibrator at its arbitrary excitation:

$$\begin{aligned}J(s)\;=\;&\bar{A}(-L)\cos\left(\tilde{k}s+\chi L\right)+\bar{B}(-L)\sin\left(\tilde{k}s+\chi L\right)\\ &+\alpha\int_{-L}^{s}\left\{\frac{i\omega}{k}E_{0s}(s')+\bar{F}\left[s',\bar{A},\bar{B}\right]\right\}\sin\tilde{k}\left(s-s'\right)ds',\end{aligned}\tag{66}$$

where $\tilde{k} = k + \chi = k + i(\alpha/r)\bar{Z}_S$.

It is necessary to use the boundary conditions (18) and the conditions of symmetry [31], which are definitely coupled with the vibrator excitation method: if $E_{0s}(s) = E_{0s}^s(s)$, then $J(s) = J(-s) = J^s(s)$ and $\bar{A}(-L) = \bar{A}(+L)$, $\bar{B}(-L) = -\bar{B}(+L)$; if $E_{0s}(s) = E_{0s}^a(s)$, then $J(s) = -J(-s) = J^a(s)$ and $\bar{A}(-L) = -\bar{A}(+L)$, $\bar{B}(-L) = \bar{B}(+L)$ to define the constants $\bar{A}(\pm L)$ and $\bar{B}(\pm L)$. Then taking into consideration symmetrical (the "s" index) and antisymmetrical (the "a" index) current components, we, finally, obtain $E_{0s}(s) = E_{0s}^s(s) + E_{0s}^a(s)$ at arbitrary excitation of the vibrator:

$$\begin{aligned} J(s) = J^s(s) + J^a(s) = \alpha\frac{i\omega}{k}\Bigg\{ &\int_{-L}^{s} E_{0s}(s')\sin\tilde{k}(s-s')ds' \\ &- \frac{\sin\tilde{k}(L+s) + \alpha P^s\left[kr, \tilde{k}(L+s)\right]}{\sin 2\tilde{k}L + \alpha P^s\left(kr, 2\tilde{k}L\right)}\int_{-L}^{L} E_{0s}^s(s')\sin\tilde{k}(L-s')ds' \\ &- \frac{\sin\tilde{k}(L+s) + \alpha P^a\left[kr, \tilde{k}(L+s)\right]}{\sin 2\tilde{k}L + \alpha P^a\left(kr, 2\tilde{k}L\right)}\int_{-L}^{L} E_{0s}^a(s')\sin\tilde{k}(L-s')ds' \Bigg\}, \end{aligned} \tag{67}$$

where P^s and P^a are the vibrator own field functions, equal to, correspondingly

$$\begin{aligned} P^s\left[kr, \tilde{k}(L+s)\right] &= \int_{-L}^{s} \left[\frac{e^{-ikR(s',-L)}}{R(s',-L)} + \frac{e^{-ikR(s',L)}}{R(s',L)}\right] \sin\tilde{k}(s-s')ds' \Bigg|_{s=L} \\ &= P^s(kr, 2\tilde{k}L), \end{aligned} \tag{68a}$$

$$\begin{aligned} P^a\left[kr, \tilde{k}(L+s)\right] &= \int_{-L}^{s} \left[\frac{e^{-ikR(s',-L)}}{R(s',-L)} - \frac{e^{-ikR(s',L)}}{R(s',L)}\right] \sin\tilde{k}(s-s')ds' \Bigg|_{s=L} \\ &= P^a(kr, 2\tilde{k}L). \end{aligned} \tag{68b}$$

The field of induced sources equals $E_{0s}(s) = E_{0s}^s(s) = E_0$ in the problem about normal incident of the plane electromagnetic wave on the impedance vibrator in free space. Then the expression for the current in the vibrator has the form (with the accuracy to the terms of the α^2 order) at substitution of $E_{0s}(s)$ into (67):

$$J(s) = -\alpha E_0 \frac{i\omega}{k\tilde{k}} \frac{\left(\cos\tilde{k}s - \cos\tilde{k}L\right)}{\cos\tilde{k}L + \alpha P_L^s\left(kr, \tilde{k}L\right)}, \tag{69}$$

where due to (68a) $P_L^s(kr, \tilde{k}L) = \int\limits_{-L}^{L} \frac{e^{-ikR(s,L)}}{R(s,L)} \cos\tilde{k}s ds$. It follows from the analysis of formula (69) that the vibrator distributed surface impedance value is included not only into the function of its own field, but also into the function of the current distribution

$$f(s) = \cos\tilde{k}s - \cos\tilde{k}L, \tag{70}$$

what differs the solution of the integral equation for the current by the averaging method from the solution (54) by the method of iterations considerably.

3. VIBRATOR EXCITATION IN THE CENTRE BY THE CONCENTRATED EMF

Let us consider the classical problem about excitation of the vibrator in its geometrical centre of the concentrated EMF with the V_0 amplitude in order to ground rightness and the ranges of application of the obtained solution (67). The mathematical model of excitation is represented in this case as:

$$E_{0s}(s) = E_{0s}^s(s) = V_0\delta(s - 0), \tag{71}$$

where $\delta(s-0) = \delta(s)$ is the delta-function. Then the expression for the current has the form:

$$J(s) = -\alpha V_0 \left(\frac{i\omega}{2\tilde{k}}\right) \frac{\sin\tilde{k}(L-|s|) + \alpha P_\delta^s\left(kr, \tilde{k}s\right)}{\cos\tilde{k}L + \alpha P_L^s\left(kr, \tilde{k}L\right)}. \tag{72}$$

Here $P_\delta^s(kr, \tilde{k}s) = P^s[kr, \tilde{k}(L+s)] - (\sin\tilde{k}s + \sin\tilde{k}|s|)\, P_L^s(kr, \tilde{k}L)$, $P^s[kr, \tilde{k}(L+s)]$ is defined by formula (68a), and $P_L^s(kr, \tilde{k}L) = \int\limits_{-L}^{L} \frac{e^{-ikR(s,L)}}{R(s,L)} \cos\tilde{k}s ds$.

Let us note that using the apparatus of the generalized integral functions [31], it is possible to obtain $P_\delta^s(kr, \tilde{k}s)$ and $P_L^s(kr, \tilde{k}L)$ in an explicit form [11].

Knowledge of real current distribution (72) allows to calculate the electrodynamic characteristics of the impedance vibrator. So, we obtain the following expression for the vibrator input impedance in the feed point $Z_{in} = R_{in} + iX_{in}$ (or the input admittance $Y_{in} = G_{in} + iB_{in} = 1/Z_{in}$):

$$Z_{in}[\mathrm{Ohm}] = \frac{V_0}{J(0)} = \left(\frac{60i\tilde{k}}{\alpha k}\right) \frac{\cos\tilde{k}L + \alpha P_L^s\left(kr, \tilde{k}L\right)}{\sin\tilde{k}L + \alpha P_{\delta L}\left(kr, \tilde{k}L\right)}, \tag{73}$$

in which $P_{\delta L}(kr, \tilde{k}L) = \int\limits_{-L}^{L} \frac{e^{-ikR(s,L)}}{R(s,L)} \sin \tilde{k}|s| ds$.

Then the voltage standing wave ratio (VSWR) in the line antenna feeder with the W wave impedance equals:

$$\text{VSWR} = \frac{1 + |S_{11}|}{1 - |S_{11}|}, \qquad S_{11} = \frac{Z_{in} - W}{Z_{in} + W}, \tag{74}$$

where S_{11} is the reflection coefficient in the feeder.

Let us give some numerical results. Figure 2 represents the amplitude-phase distributions of the current $J(s) = |J(s)|e^{i \arg J(s)}$ in the thin ($r/\lambda = 0.007022$) perfectly conducting vibrators of different electrical lengths, calculated by formula (72), in comparison with the experimental data from [43]. As seen from the plots, the trend of theoretical curves reproduces the trend of the experimental ones rather satisfactory, though some differences are observed in the absolute

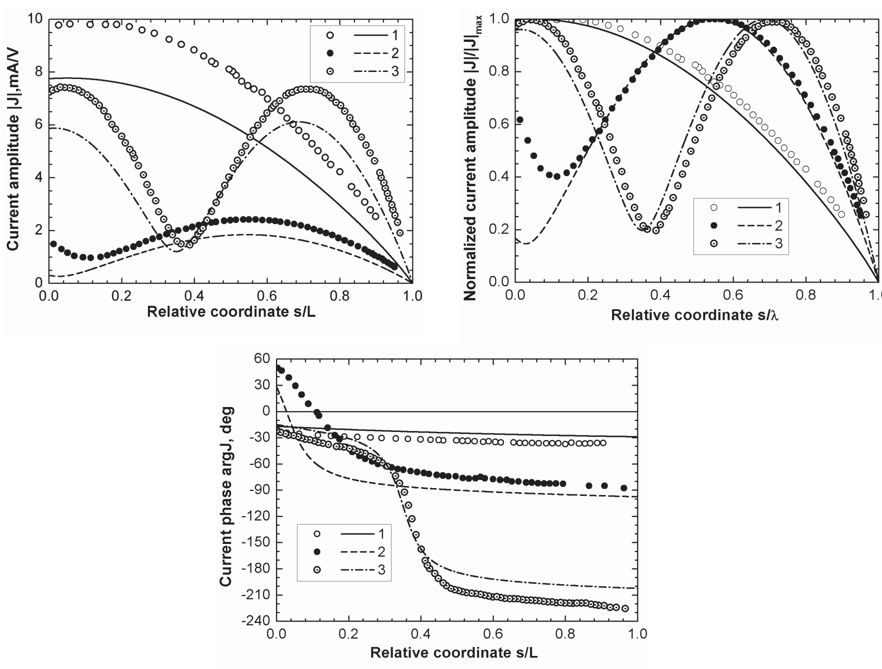

Figure 2. The current amplitude-phase distributions along the perfectly conducting vibrators with variable electric length in free space at $r/\lambda = 0.007022$, $f = 663\,\text{MHz}$: $1 - 2L/\lambda = 0.5$, $2 - 2L/\lambda = 1.0$, $3 - 2L/\lambda = 1.5$; the curves — the calculation (formula (72)), the circles — the experimental data [43].

values. They also take place in the input characteristics of the vibrators $Y_{in} = f(2L/\lambda)$, calculated by formulas (73) and given in Figure 3.

Analogous conformities to natural laws are observed at the calculation of the impedance vibrators input characteristics. The plots of the input admittance for two cases of realization of the surface impedance are represented in Figures 4, 5: 1) the metallic conductor of the radius $r_i = 0.3175\,\mathrm{cm}$, covered by the dielectrical ($\varepsilon = 9.0$) shell of the radius $r = 0.635\,\mathrm{cm}$ (Figure 4, the experimental data from [44]); 2) the metallic conductor of the radius $r_i = 0.5175\,\mathrm{cm}$, covered by the ferrite ($\mu = 4.7$) shell of the radius $r = 0.6\,\mathrm{cm}$ (Figure 5, the experimental data from [45]).

If the vibrator, excited by the δ-generator in the centre, is located

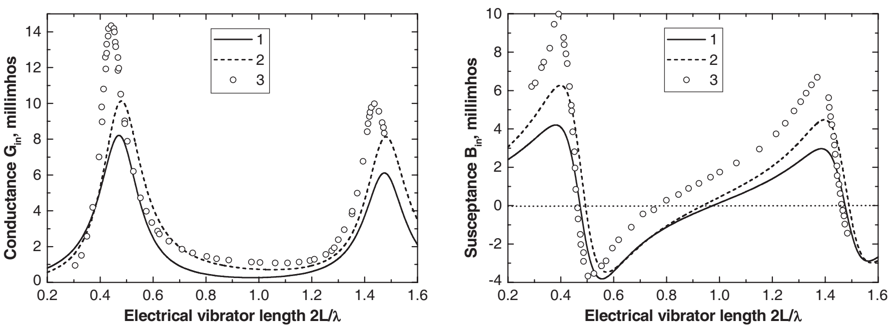

Figure 3. The input admittance of the perfectly conducting vibrator in dependence of its electrical length at $r/\lambda = 0.007022$: 1 — the calculation (73), 2 — the calculation (76), 3 — the experimental data [43].

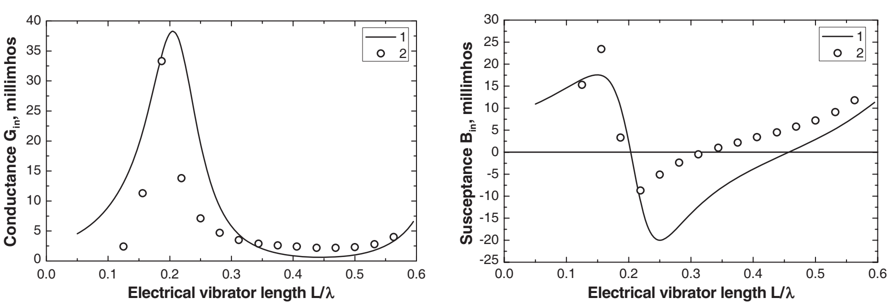

Figure 4. The input admittance of the metallic conductor of the radius $r_i = 0.3175\,\mathrm{cm}$, covered by the dielectrical ($\varepsilon = 9.0$) shell of the radius $r = 0.635\,\mathrm{cm}$ in dependence of its electrical length at $f = 600\,\mathrm{MHz}$: 1 — the calculation (73), 2 — the experimental data [44].

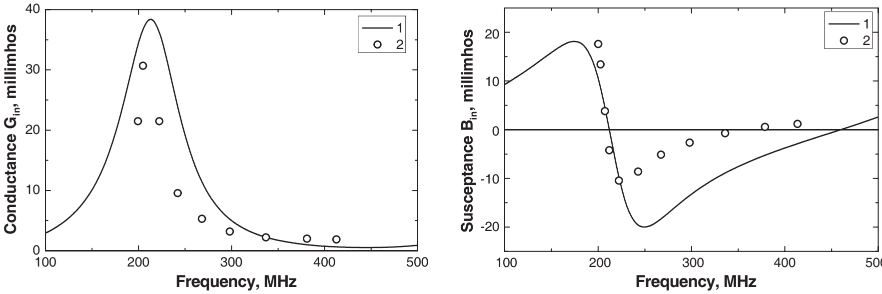

Figure 5. The input admittance of the metallic conductor of the radius $r_i = 0.5175\,\text{cm}$, covered by the ferrite ($\mu = 4.7$) shell of the radius $r = 0.6\,\text{cm}$ in dependence of the frequency at $2L = 30.0\,\text{cm}$: 1 — the calculation (73), 2 — the experimental data [45].

in the material medium with the parameters ε_1 and μ_1, then the expression for the current has the form [11]:

$$J(s) = -\alpha V_0 \left(\frac{i\omega\varepsilon_1}{2\tilde{k}_1}\right) \frac{\sin\tilde{k}_1 (L - |s|) + \alpha P_\delta^s \left(k_1 r, \tilde{k}_1 s\right)}{\cos\tilde{k}_1 L + \alpha P_L^s \left(k_1 r, \tilde{k}_1 L\right)}. \tag{75}$$

Here $k_1 = k\sqrt{\varepsilon_1\mu_1} = k_1' - ik_1''$, $\tilde{k}_1 = k_1 + i(\alpha/r)\bar{Z}_S\sqrt{\varepsilon_1/\mu_1}$, $P_L^s(k_1 r, \tilde{k}_1 L) = \int\limits_{-L}^{L} G(s, L)\cos\tilde{k}_1 s ds$, $P_\delta^s(k_1 r, \tilde{k}_1 s) = P^s[k_1 r, \tilde{k}_1(L+s)] - (\sin\tilde{k}_1 s + \sin\tilde{k}_1|s|)\, P_L^s(k_1 r, \tilde{k}_1 L)$, and $P^s[k_1 r, \tilde{k}_1(L + s)]$ is defined by formula (68a). The plots of the current amplitude-phase distribution in the perfectly conducting vibrator for the medium (salt water) with the parameters $\varepsilon_1 = 83.5 - i55.3$, $\mu_1 = 1$ ($k_1''/k_1' = 0.301$, λ_1 is the wavelength in the medium) are represented in Figure 6 in comparison with the experimental data from [43], too.

From our point of view, the differences of the theoretical curves, obtained on the basis of the integral equation solution for the current by means of the averaging method, from the experimental values are explained by that solving the vibrator own field (59) undergoes averaging, and thus the current amplitude is with some error. However, as seen from the comparison of the given calculated and experimental results, the vibrators resonant characteristics ($(2L/\lambda)_{res}$ at $B_{in} = 0$) are defined rather precisely, and the calculated curves of the currents normalized amplitudes ($|J(s)|/|J|_{\max}$) agree with the experimental data in permissible limits. Thus the formulas for the current, obtained in the format of the first approximation of the averaging method, are suitable for the vibrators integral characteristics calculation, such as

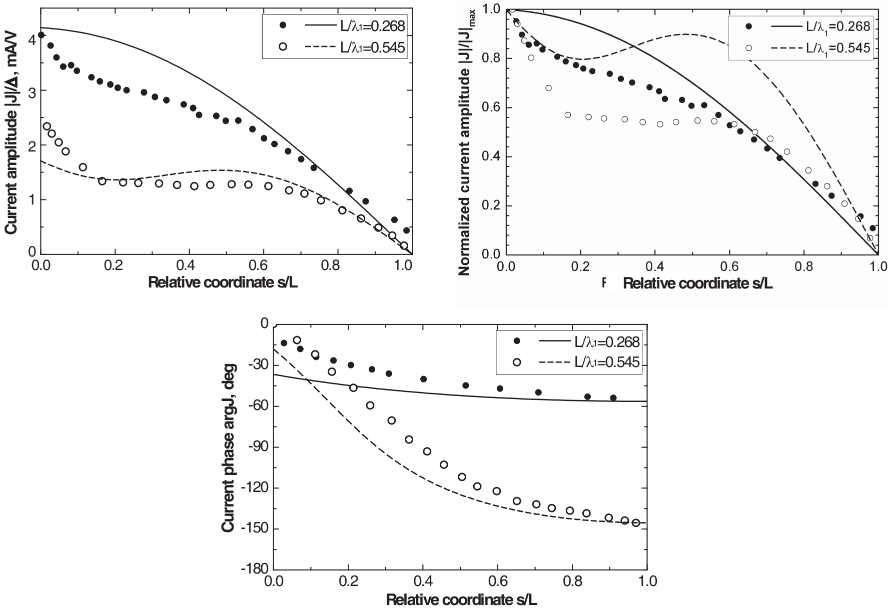

Figure 6. The current amplitude-phase distributions along the perfectly conducting vibrators with variable electric length in salt water at $k_1''/k_1' = 0.301$, $r/\lambda_1 = 0.0028$, $f = 28\,\mathrm{MHz}$, $\Delta = \lambda/\lambda_1 = 9.58$: the curves — the calculation (formula (75)), the circles — the experimental data [43].

the radiated (scattered) electromagnetic field in all zones of observation and also at investigation of the vibrators resonant properties.

As indicated in [40],Equation (61) solution can be obtained in the format of the improved first approximation. It means, in our case, that the transition from Equation (61) to Equation (62) is made with the help of substitution of $-\frac{d\bar{A}(s)}{ds}\sin ks + \frac{d\bar{B}(s)}{ds}\cos ks = \alpha i\omega E_{0s}(s)$, and, finally, it leads to the following expression for the vibrator input impedance:

$$Z_{in}^{imp} = \left(\frac{60i\tilde{k}}{\alpha k}\right) \frac{\cos\tilde{k}L + \alpha P_L^s\left(kr, \tilde{k}L\right)}{\sin\tilde{k}L + \alpha P_{\delta L}\left(kr, \tilde{k}L\right) + \left[\sin\tilde{k}L + \alpha P_{\delta 1}\left(kr, \tilde{k}L\right) + \alpha^2 P_{\delta 2}\left(kr, \tilde{k}L\right)\right]}. \quad (76)$$

Here $P_{\delta 1}\left(kr,\tilde{k}L\right)=P_{\delta L}\left(kr,\tilde{k}L\right)+\sin\tilde{k}LP_0^s\left(kr,\tilde{k}L\right)-\cos\tilde{k}LP_{\delta 0}\left(kr,\tilde{k}L\right)$,

$$P_{\delta 2}\left(kr,\tilde{k}L\right)=P_{\delta L}\left(kr,\tilde{k}L\right)P_0^s\left(kr,\tilde{k}L\right)-P_L^s\left(kr,\tilde{k}L\right)P_{\delta 0}\left(kr,\tilde{k}L\right),$$

$$P_0^s\left(kr,\tilde{k}L\right)=\int\limits_{-L}^{L}\frac{e^{-ikR(s,0)}}{R(s,0)}\cos\tilde{k}sds,$$

$$P_{\delta 0}\left(kr,\tilde{k}L\right)=\int\limits_{-L}^{L}\frac{e^{-ikR(s,0)}}{R(s,0)}\sin\tilde{k}|s|ds.$$

The calculated curves, corresponding to formula (76), which correlate with the experimental data very closely, are given in Figure 3 (the dotted lines). If one solves Equation (61) by the averaging method in the format of the second approximation further, then, as a result, it is possible to obtain much more bulk formulas, which, of course, increase preciseness of the calculated results, but they turn out to be suitable very little for practical use.

4. CONCLUSION

Thus the solution of the quasi-one-dimensional integral equation for the electrical current in thin vibrators by the small parameter method leads to different expressions for the current in the case of a tuned vibrator (the impressed field frequency differs a bit from the vibrator own frequency) and an untuned one (when this condition is not performed), though the solution can be obtained for the untuned vibrator at its arbitrary excitation in the first approximation. The solution of the integral equation for the current by the iterations method is given in the form of one formula, suitable both for tuned and untuned vibrators. However, application of this method is possible only at concretization of the impressed sources field at the initial stage of analysis. The solution of the equation for the current by the averaging method combines in itself the advantages of the solutions both by the small parameter method and iteration method, namely, the analytical expression for the current is obtained in the form of one formula, suitable both for tuned and untuned vibrators without concretization of the impressed sources field and electrodynamic volumes in which they are located. To our mind, it is expedient to use the functions of the current distribution, obtained by means of the averaging method as basic ones at realization of the numerical-analytical methods (for examples, of the method of induced EMF) in order to increase preciseness of the calculated results at the solution of the problems about scattering (radiation) of electromagnetic waves by thin impedance vibrators.

ACKNOWLEDGMENT

The author considers his pleasant duty to express gratitude to Professor Katrich V. A. for his valuable consultations during the process of this work.

REFERENCES

1. King, R. W. P. and T. T. Wu, "The imperfectly conducting cylindrical transmitting antenna," *IEEE Trans. Antennas and Propagat.*, Vol. 14, 524–534, 1966.
2. Taylor, C. D., "Cylindrical transmitting antenna: Tapered resistivity and multiple impedance loadings," *IEEE Trans. Antennas and Propagat.*, Vol. 16, 176–179, 1968.
3. Rao, B. L. J., J. E. Ferris, and W. E. Zimmerman, "Broadband characteristics of cylindrical antennas with exponentially tapered capacitive loading," *IEEE Trans. Antennas and Propagat.*, Vol. 17, 145–151, 1969.
4. Inagaki, N., O. Kukino, and T. Sekiguchi, "Integral equation analysis of cylindrical antennas characterized by arbitrary surface impedance," *IEICE Trans. Commun.*, Vol. 55-B, 683–690, 1972.
5. Gorobets, N. N., M. V. Nesterenko, and V. A. Petlenko, "Resonance characteristics of thin impedance dipoles in a cutoff rectangular waveguide," *Telecommunications and Radio Engineering*, Vol. 45, No. 4, 110–112, 1990.
6. Andersen, L. S., O. Breinbjerg, and J. T. Moore, "The standard impedance boundary condition model for coated conductors with edges: A numerical investigation of the accuracy for transverse magnetic polarization," *Journal of Electromagnetic Waves and Applications*, Vol. 12, No. 4, 415–446, 1998.
7. Galdi, V. and I. M. Pinto, "SDRA approach for higher-order impedance boundary conditions for complex multi-layer coatings on curved conducting bodies," *Journal of Electromagnetic Waves and Applications*, Vol. 13, No. 12, 1629–1630, 1999.
8. Ikiz, T., S. Koshikawa, K. Kobayashi, E. I. Veliev, and A. H. Serbest, "Solution of the plane wave diffraction problem by an impedance strip using a numerical-analytical method: E-polarized case," *Journal of Electromagnetic Waves and Applications*, Vol. 15, No. 3, 315–340, 2001.

9. Papakanellos, P. J. and C. N. Capsalis, "Numerical analysis of cylindrical dipole antennas using an auxiliary sources model," *Journal of Electromagnetic Waves and Applications*, Vol. 17, No. 3, 389–407, 2003.
10. Mäkinen, R. M., "An efficient surface-impedance boundary condition for thin wires of finite conductivity," *IEEE Trans. Antennas and Propagat.*, Vol. 52, 3364–3372, 2004.
11. Nesterenko, M. V., "The electomagnetic wave radiation from a thin impedance dipole in a lossy homogeneous isotropic medium," *Telecommunications and Radio Engineering*, Vol. 61, 840–853, 2004.
12. Hanson, G. W., "Fundamental transmitting properties of carbon nanotube antennas," *IEEE Trans. Antennas and Propagat.*, Vol. 53, 3426–3435, 2005.
13. Arnold, M. D., "An efficient solution for scattering by a perfectly conducting strip grating," *Journal of Electromagnetic Waves and Applications*, Vol. 20, No. 7, 891–900, 2006.
14. Collard, B., M. B. Fares, and B. Souny, "A new formulation for scattering by impedant 3D bodies," *Journal of Electromagnetic Waves and Applications*, Vol. 20, No. 10, 1291–1298, 2006.
15. Ruppin, R., "Scattering of electromagnetic radiation by a perfect electromagnetic conductor cylinder," *Journal of Electromagnetic Waves and Applications*, Vol. 20, No. 13, 1853–1860, 2006.
16. Hady, L. K. and A. A. Kishk, "Electromagnetic scattering from conducting circular cylinder coated by meta-materials and loaded with helical strips under oblique incidence," *Progress In Electromagnetics Research B*, Vol. 3, 189–206, 2008.
17. Nesterenko, M. V., V. A. Katrich, V. M. Dakhov, and S. L. Berdnik, "Impedance vibrator with arbitrary point of excitation," *Progress In Electromagnetics Research B*, Vol. 5, 275–290, 2008.
18. Hatamzadeh-Varmazyar, S. and M. Naser-Moghadasi, "New numerical method for determining the scattered electromagnetic fields from thin wires," *Progress In Electromagnetics Research B*, Vol. 3, 207–218, 2008.
19. Wu, J.-J. and T. J. Yang, "Subwavelength microwave guiding by a periodically corrugated metal wire," *Journal of Electromagnetic Waves and Applications*, Vol. 23, No. 10, 11–19, 2009.
20. Nesterenko, M. V., D. Yu. Penkin, V. A. Katrich, and V. M. Dakhov, "Equation solution for the current in radial impedance monopole on the perfectly conducting sphere,"

Progress In Electromagnetics Research B, Vol. 19, 95–114, 2010.
21. Khizhnyak, N. A., *Integral Equations of Macroscopical Electrodynamics*, Naukova dumka, Kiev, 1986 (in Russian).
22. Nesterenko, M. V., V. A. Katrich, Y. M. Penkin, and S. L. Berdnik, "Analytical methods in theory of slot-hole coupling of electrodynamic volumes," *Progress In Electromagnetics Research*, PIER 70, 79–174, 2007.
23. Leontovich, M. A., "On the approximate boundary conditions for the electromagnetic field on surfaces of good conductive bodies," *Investigations of Radiowave Propagation*, Printing House of the Academy of Sciences of the USSR, Moscow-Leningrad, 1948 (in Russian).
24. Levin, H. and J. Schwinger, "On the theory of electromagnetic wave diffraction by an aperture in an infinite plane conducting screen," *Commun. Pure Appl. Math.*, 355–391, 1950.
25. Collin, R. E., *Field Theory of Guided Waves*, McGraw-Hill, New York, 1960.
26. Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953.
27. Tai, C. T., *Dyadic Green's Function in Electromagnetic Theory*, Intex Educ. Publ., Scranton, 1971.
28. Tikhonov, A. N. and A. A. Samarsky, *Equations of Mathematical Physics*, Nauka, Moscow, 1977 (in Russian).
29. Van Bladel, J., "Some remarks on Green's dyadic for infinite space," *IRE Trans. Antennas and Propagat.*, Vol. 9, 563–566, 1961.
30. Felsen, L. B. and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice-Hall, Inc., New Jersey, 1973.
31. King, R. W. P., *The Theory of Linear Antennas*, Harv. Univ. Press, Cambr., MA, 1956.
32. King, R. W. P., E. A. Aronson, and C. W. Harrison, "Determination of the admittance and effective length of cylindrical antennas," *Radio Science*, Vol. 1, No. 7, 835–850, 1966.
33. Pocklington, H. C., "Electrical oscillations in wires," *Proc. Cambr. Phil. Soc.*, Vol. 9, Pt. VII, 324–332, 1897.
34. Brillouin, L., "The antenna problem," *Quart. Appl. Math.*, 201–214, 1943.
35. Mei, K. K., "On the integral equation of thin wire antennas," *IEEE Trans. Antennas and Propagat.*, Vol. 13, 374–378, 1965.
36. Hallen, E., "Theoretical investigations into the transmitting and receiving qualities of antenna," *Nova Acta Reg. Soc. Sci. Ups.*,

Ser. IV, Vol. 11, 1–44, 1938.

37. Leontovich, M. and M. Levin, "To the theory of oscillations excitation in antennas vibrators," *Journal of Technical Physics*, Vol. 14, 481–506, 1944 (in Russian).
38. Vainshtein, L. A., "The current waves in a thin cylindrical conductor," *Journal of Technical Physics*, Vol. 29, 65–91, 1959 (in Russian).
39. Glushkovskiy, E. A., B. M. Levin, and E. Y. Rabinovich, "The integral equation for the current in the thin impedance vibrator," *Radiotechnika*, Vol. 22, 18–23, 1967 (in Russian).
40. Bogolyubov, N. N. and U. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Fluctuations*, Nauka, Moskow, 1974 (in Russian).
41. Philatov, A. N., *Asymptotic Methods in the Theory of Differential and Integral-Differential Equations*, PHAN, Tashkent, 1974 (in Russian).
42. Kamke, E., *Differentialgleichungen Lösungsmethoden und Lösungen I. Gewöhnliche Differentialgleichungen*, 6, Verbesserte Auflage, Leipzig, 1959 (in German).
43. King, R. W. P. and L. D. Scott, "The cylindrical antenna as a probe for studying the electrical properties of media," *IEEE Trans. Antennas and Propagat.*, Vol. 19, 406–416, 1971.
44. Lamensdorf, D., "An experimental investigation of dielectric-coated antennas," *IEEE Trans. Antennas and Propagat.*, Vol. 15, 767–771, 1967.
45. Bretones, A. R., R. G. Martin, and I. S. García, "Time-domain analysis of magnetic-coated wire antennas," *IEEE Trans. Antennas and Propagat.*, Vol. 43, 591–596, 1995.