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EVOLUTION OF TRANSIENT ELECTROMAGNETIC FIELDS IN RADIALLY INHOMOGENEOUS NONSTATIONARY MEDIUM

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Abstract—To solve radiation problems in time domain directly the modal representation of transient electromagnetic fields is considered. Using evolutionary approach the initial nonstationary three-dimensional electrodynamic problem is transformed into the problem for one-dimensional evolutionary equations by the construction of the modal basis for electromagnetic fields with arbitrary time dependence in spherical coordinate system. Elimination of the radial components of electrical and magnetic field from Maxwell equation system permits to form the four-dimensional differential operators. It is proved that the operators are self- adjoint ones. The eigen-functions of the operators form the basis. The completeness of the basis is proved by means of Weyl Theorem about orthogonal detachments of Hilbert space. The expansion coefficients of arbitrary electromagnetic field are found from

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the set of evolutionary equations. The transient electromagnetic field can be found directly without Fourier transform application by means of one-dimensional FDTD method for the medium with dependence on longitudinal coordinate and time or using Laplace transform and wave splitting for the case of homogeneous stationary medium. The above mentioned methods are compared with the three-dimensional FDTD method for the case of the problem of small loop excitation by transient current.

1. INTRODUCTION

Modern computer systems for direct calculation of transient electrodynamic problems in bounded domains permit to calculate characteristics of complicated structures that can have dielectric filling, non-coordinate boundaries, and sources with arbitrary time dependence [1]. But it is difficult to construct and optimize radiating systems by the same means because of the significant increase of amount of required main memory and computer time. One of solutions of this problem consists in a restriction of an electrodynamic volume by absorbing boundaries [2–4] but it does not simplify the problem noticeably if the electrical size of calculated electrodynamic structure is significant. Usually, after carrying out the calculation in the bounded volume, the characteristics of radiation systems in far zone are estimated with the use of equivalent sources of transient electrical and magnetic currents on boundaries. The suggested evolutionary approach in the present work allows the finding of the all components of radiated fields in near and far zone by means of the solution of nonstationary one-dimensional equations in partial derivatives. The decreasing of dimension of a problem simplifies its solving significantly irrespective of a method [5].

Solving of a radiation problem in time domain directly assists to clarify electromagnetic phenomena in different structures [6]. To solve inner problems in time domain directly the Modal Basis Method was proposed by Tretyakov [7]. Using the method there was obtained the set of evolutionary equations that contain dependence of electrical and magnetic parameters of medium inside resonator on time in an explicit form rather than its Fourier Transform [8–10].

The method also known as Evolutionary Approach in Electromagnetics was applied to waveguides filled with nonstationary nonlinear inhomogeneous in longitudinal direction medium [11, 12]. The initial transient three-dimensional electrodynamic problem was transformed into the problem for one-dimensional evolutionary equations by the construction of the modal basis for electromagnetic fields with arbitrary time dependence. There were obtained self-adjoint operators in transverse plane of waveguide with arbitrary contour of cross-section. The eigen-functions of the operators form the modal basis. Using Weyl Theorem about orthogonal detachments of Hilbert space [13] the completeness of the basis of an arbitrary regular waveguide was proved for the first time. Projection of the set of Maxwell equations onto the Modal Basis permits to get the set of Evolutionary Equations. Actually, the MBM is one of varieties of the method of incomplete separation of variables [2, 14] for more general statements of problems. MBM allows to exclude the dependence of functions on transversal coordinates from Maxwell Equations and leave the dependence of functions on longitudinal coordinate and time that permits to satisfy the causality principle in solution of the problem. To solve a problem in waveguide filled with inhomogeneous medium not only in longitudinal but in transversal plane as well for the problems like in [15] the MBM was successfully applied later [16].

The same Evolutionary Approach was found suitable for radiation problems too. Firstly it was applied to construct Modal Basis in cylindrical coordinate system [17, 18]. The set of Evolutionary Equations for more complicated case of inhomogeneous medium in transversal plane was obtained in [19, 20] as well as it was received in [16] for waveguides. But the spherical coordinate system is more widely used in radiation problems than the cylindrical coordinate system because of the spherical coordinate system reflects physical essence of radiation processes conveniently. The analysis of different broadband antennas in time domain is more simple in spherical coordinate system [21–23].

The present paper is devoted to the construction of the Modal Basis and obtaining of Evolutionary Equation Set in spherical coordinate system to solve radiation problems in time domain. The same approach was applied in [24] but the obtained field expansion wasn't complete. The drawback was eliminated in [25] that permitted to solve radiation problems [26–29]. The comparison of the analytical and numerical solutions of the problem of propagation of spherical electromagnetic wave with arbitrary time dependence from a radiator with the results of direct three-dimensional numerical simulation is carried out. The operator of transient field transformation from an initial spherical boundary to another one is constructed. The problem of propagation of spherical electromagnetic wave with arbitrary time dependence through radially inhomogeneous medium is solved by numerical calculations of nonstationary one-dimensional problem for Evolutionary Equation in partial derivatives. The obtained results is compared with the direct solution of the problem by three-dimensional

FDTD method.

2. STATEMENT OF THE PROBLEM

Considering electromagnetic wave propagation in the space filled by spherical inhomogeneous nonstationary medium the material equations and the continuity equations are of the following form

$$\vec{\mathbf{D}}(\vec{r},t)=\varepsilon_0\varepsilon(r,t)\,\vec{\mathbf{E}}(\vec{r},t), \quad \vec{\mathbf{B}}(\vec{r},t)=\mu_0\mu(r,t)\,\vec{\mathbf{H}}(\vec{r},t),$$
$$\frac{\partial}{\partial t}\rho=-\mathrm{div}\vec{\mathbf{J}}, \quad \frac{\partial}{\partial t}\rho^m=-\mathrm{div}\vec{\mathbf{J}}^m,$$

where $\vec{r}\equiv\vec{r}(r,\varphi,\theta)$. Write down Maxwell's equations in any space region

$$\begin{aligned}&\mathrm{rot}\vec{\mathbf{H}}=\frac{\partial}{\partial t}\varepsilon_0\varepsilon\vec{\mathbf{E}}+\vec{\mathbf{J}}; \quad \mathrm{rot}\vec{\mathbf{E}}=-\frac{\partial}{\partial t}\mu_0\mu\vec{\mathbf{H}}-\vec{\mathbf{J}}^m;\\ &\mathrm{div}\left(\varepsilon_0\varepsilon\vec{\mathbf{E}}\right)=\rho; \quad \mathrm{div}\left(\mu_0\mu\vec{\mathbf{H}}\right)=\rho^m.\end{aligned} \tag{1}$$

The problem is completed by initial and boundary conditions.

According to procedure firstly proposed in [7], we should construct basis in transversal plane. It will permits to transform original three-dimensional nonstationary problem into one-dimensional nonstationary problem as it was done for the case of transient wave propagation in waveguides [11].

3. ELIMINATION OF RADIAL COMPONENTS OF FIELD

Writing arbitrary three-dimensional vector as a sum of two-dimensional angular and one-dimensional radial vectors we introduce the following notations

$$\vec{\mathbf{E}}=\vec{E}+\vec{r}_0E_r; \quad \vec{\mathbf{H}}=\vec{H}+\vec{r}_0H_r; \quad \vec{\mathbf{J}}=\vec{J}+\vec{r}_0J_r; \quad \vec{\mathbf{J}}^m=\vec{J}^m+\vec{r}_0J_r^m,$$

where $\vec{r}_0$ is unit vector.

One should keep in mind that $\nabla=\vec{\theta}_0\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{\varphi}_0\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}+\vec{r}_0\frac{\partial}{\partial r}=\frac{1}{r}\nabla_t+\vec{r}_0\frac{\partial}{\partial r}$, where ∇_t is angular Hamilton's operator. After projecting (1) to longitudinal axis and sphere we obtain expressions

$$\begin{aligned}&\frac{\partial}{\partial r}\left(r^2\mu_0\mu H_r\right)=-r\nabla_t\cdot\left(\mu_0\mu\vec{H}\right)+r^2\rho^m;\\ &\frac{\partial}{\partial r}\left(r^2\varepsilon_0\varepsilon E_r\right)=-r\nabla_t\cdot\left(\varepsilon_0\varepsilon\vec{E}\right)+r^2\rho;\end{aligned} \tag{2}$$

$$\begin{aligned}
&r\frac{\partial}{\partial t}\left(\mu_0\mu H_r\right) = -\vec{r}_0\cdot\left[\nabla_t\times\vec{E}\right] - rJ_r^m;\\
&r\frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon E_r\right) = \vec{r}_0\cdot\left[\nabla_t\times\vec{H}\right] - rJ_r;
\end{aligned}\tag{3}$$

$$\begin{aligned}
&\left[\nabla_t\times\vec{r}_0\right]H_r = r\left(\frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon\vec{E}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(r\vec{H}\right)\times\vec{r}_0\right] + \vec{J}\right);\\
&\left[\nabla_t\times\vec{r}_0\right]E_r = r\left(-\frac{\partial}{\partial t}\left(\mu_0\mu\vec{H}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(r\vec{E}\right)\times\vec{r}_0\right] - \vec{J}^m\right).
\end{aligned}\tag{4}$$

Substituting in (4) the expressions (2) and (3) we eliminate E_r and H_r from the Equations (2)–(4) and obtain

$$\begin{aligned}
&\left[\vec{r}_0\times\nabla_t\right]\nabla_t\cdot\vec{H} = \frac{1}{r\mu}\frac{\partial}{\partial r}\left(\mu r^3\left\{\frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon\vec{E}\right)\right.\right.\\
&\quad\left.\left.+\frac{1}{r}\frac{\partial}{\partial r}\left[\left(r\vec{H}\right)\times\vec{r}_0\right] + \vec{J}\right\}\right) - \frac{r}{\mu_0\mu}\left[\nabla_t\times\vec{r}_0\right]\rho^m;\\
&\nabla_t\left[\vec{r}_0\times\nabla_t\right]\cdot\vec{E} = -r^2\frac{\partial}{\partial t}\left(\mu_0\mu\left[\vec{r}_0\times\left\{\frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon\vec{E}\right)\right.\right.\right.\\
&\quad\left.\left.\left.+\frac{1}{r}\frac{\partial}{\partial r}\left[\left(r\vec{H}\right)\times\vec{r}_0\right] + \vec{J}\right\}\right]\right) - r\nabla_t J_r^m;
\end{aligned}\tag{5}$$

$$\begin{aligned}
&\left[\nabla_t\times\vec{r}_0\right]\nabla_t\cdot\vec{E} = \frac{1}{r\varepsilon}\frac{\partial}{\partial r}\left(\varepsilon r^3\left\{\frac{\partial}{\partial t}\left(\mu_0\mu\vec{H}\right)\right.\right.\\
&\quad\left.\left.-\frac{1}{r}\frac{\partial}{\partial r}\left[\left(r\vec{E}\right)\times\vec{r}_0\right] + \vec{J}^m\right\}\right) + \frac{r}{\varepsilon_0\varepsilon}\left[\nabla_t\times\vec{r}_0\right]\rho;\\
&\nabla_t\left[\nabla_t\times\vec{r}_0\right]\cdot\vec{H} = r^2\frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon\left[\vec{r}_0\times\left\{\frac{\partial}{\partial t}\left(\mu_0\mu\vec{H}\right)\right.\right.\right.\\
&\quad\left.\left.\left.-\frac{1}{r}\frac{\partial}{\partial r}\left[\left(r\vec{E}\right)\times\vec{r}_0\right] + \vec{J}^m\right\}\right]\right) - r\nabla_t J_r.
\end{aligned}\tag{6}$$

So, we state the problem in the form (5), (6), adding initial and boundary conditions. Radial vectors of electromagnetic field are calculated by integration of formulas (2), (3).

4. FOUR-DIMENSIONAL ELECTROMAGNETIC FIELD VECTORS

The vectors $\vec{E}$ and $\vec{H}$ form the four-dimensional electromagnetic field vector $X(\varphi,\theta) = \begin{pmatrix}\vec{E}(\varphi,\theta)\\ \vec{H}(\varphi,\theta)\end{pmatrix}$. Let's introduce the Hilbert functional space $L_2^4(S)$ of vector-functions $X(\varphi,\theta)$ with energy measure

$$\langle X_1, X_2 \rangle = \frac{1}{4\pi}\int\limits_S dS \left(\vec{E}_1 \cdot \vec{E}_2^* + \vec{H}_1 \cdot \vec{H}_2^*\right),$$

where symbol * notes complex conjugation, S is the unit sphere with the center in the origin.

Let us now consider two matrix differential operations in the space $L_2^4(S)$

$$\tilde{W}_H X = \begin{pmatrix} 0 & [\vec{r}_0 \times \nabla_t]\,\nabla_t \cdot \\ \nabla_t\,[\vec{r}_0 \times \nabla_t] \cdot & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} [\vec{r}_0 \times \nabla_t]\,\nabla_t \cdot \vec{H} \\ \nabla_t\,[\vec{r}_0 \times \nabla_t] \cdot \vec{E} \end{pmatrix};$$

$$\tilde{W}_E X = \begin{pmatrix} 0 & \nabla_t\,[\nabla_t \times \vec{r}_0] \cdot \\ [\nabla_t \times \vec{r}_0]\,\nabla_t \cdot & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} \nabla_t\,[\nabla_t \times \vec{r}_0] \cdot \vec{H} \\ [\nabla_t \times \vec{r}_0]\,\nabla_t \cdot \vec{E} \end{pmatrix},$$

where W_H and W_E are linear operators in $L_2^4(S)$ that are given by $\tilde{W}_H$ and $\tilde{W}_E$ correspondingly and included proper boundary conditions. Operators W_H and W_E have orthonormal sets of eigenfunctions $Y_{\pm mk} = \begin{pmatrix} \nabla_t \Psi_{mk} \times \vec{r}_0 \\ \pm\nabla_t \Psi_{mk} \end{pmatrix}$, $Z_{\pm nl} = \begin{pmatrix} \nabla_t \Psi_{nl} \\ \pm\vec{r}_0 \times \nabla_t \Psi_{nl} \end{pmatrix}$ that correspond to real eigenvalues $p_{\pm m} = \pm m(m+1)$, $q_{\pm n} = \pm n(n+1)$, where $\Psi_{mk} = \sqrt{\frac{2m+1}{2}\frac{(m-|k|)!}{(m+|k|)!}} P_m^{|k|}(\cos\theta)\, e^{ik\varphi}$ with $P_m^k(x)$ being the associate Legendre functions, $n, m = 1, 2, \ldots$, $k = \overline{-m, m}$, $l = \overline{-n, n}$. Vector-functions Y_{mk}, Z_{nl} form the orthonormal basis in Hilbert functional space $L_2^4(S)$ according Weyl's theorem about orthogonal splitting [13].

5. EVOLUTIONARY EQUATIONS

Let's unknown coefficients of the expansion of electromagnetic field on modal basis are found from the solution of evolutionary equations. Set of evolutionary equations are found by projection of initial Maxwell equations on the basis obtained. So the problem is reduced to the following equations obtained from evolutionary equation set supplemented with relations between evolutionary coefficients and field components [25]:

5.1. TE-wave

$$\left\{\frac{1}{c^2}\frac{\partial}{\partial t}\varepsilon\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\frac{1}{\mu}\frac{\partial}{\partial r} + \frac{p_{mk}}{r^2\mu}\right\}\left(r^2 \mu B_{mk}\right) = -\frac{1}{2\pi\mu_0}\left\{\frac{\partial}{\partial r}\left(\frac{r^2}{\mu}\int\limits_S \rho^m \Psi_{mk}^* dS\right)\right.$$

$$-r\mu_0\int\limits_S \vec{J}\cdot\left[\nabla_t\times\vec{r}_0\right]\Psi^*_{mk}dS+\frac{r^2}{c^2}\frac{\partial}{\partial t}\left(\varepsilon\int\limits_S J^m_r\Psi^*_{mk}dS\right)\Bigg\};$$

$$\left\{\frac{1}{c^2}\frac{\partial}{\partial t}\varepsilon\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\frac{1}{\mu}\frac{\partial}{\partial r}\right\}\left(r^2\mu B_0\right)$$

$$=-\frac{1}{4\pi\mu_0}\frac{\partial}{\partial r}\left(\frac{r^2}{\mu}\int\limits_S\rho^m dS\right)-\frac{r^2\varepsilon_0}{4\pi}\frac{\partial}{\partial t}\left(\varepsilon\int\limits_S J^m_r dS\right);$$

$$\vec{E}=\sum_{m=1}^{\infty}\sum_{k=-m}^{m}\left[\nabla_t\Psi_{mk}\times\vec{r}_0\right]\left\{-r\mu_0\frac{\partial}{\partial t}\left(\mu B_{mk}\right)-\frac{r}{2\pi}\int\limits_S J^m_r\Psi^*_{mk}dS\right\};$$

$$\vec{H}=\sum_{m=1}^{\infty}\sum_{k=-m}^{m}\nabla_t\Psi_{mk}\left\{\frac{1}{\mu r}\frac{\partial}{\partial r}\left(r^2\mu B_{mk}\right)-\frac{r}{2\pi\mu_0\mu}\int\limits_S\rho^m\Psi^*_{mk}dS\right\};$$

$$H_r=\sum_{m=1}^{\infty}\sum_{k=-m}^{m}B_{mk}p_m\Psi_{mk}+B_0,$$

5.2. TM-wave

$$\left\{\frac{1}{c^2}\frac{\partial}{\partial t}\mu\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\frac{1}{\varepsilon}\frac{\partial}{\partial r}+\frac{q_{nl}}{r^2\varepsilon}\right\}\left(r^2\varepsilon A_{nl}\right)=-\frac{1}{2\pi\varepsilon_0}\left\{\frac{\partial}{\partial r}\left(\frac{r^2}{\varepsilon}\int\limits_S\rho\Psi^*_{nl}dS\right)\right.$$

$$\left.+r\varepsilon_0\int\limits_S\vec{J}^m\cdot\left[\nabla_t\times\vec{r}_0\right]\Psi^*_{nl}dS+\frac{r^2}{c^2}\frac{\partial}{\partial t}\left(\mu\int\limits_S J_r\Psi^*_{nl}dS\right)\right\};$$

$$\left\{\frac{1}{c^2}\frac{\partial}{\partial t}\mu\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\frac{1}{\varepsilon}\frac{\partial}{\partial r}\right\}\left(r^2\varepsilon A_0\right)$$

$$=-\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{r^2}{\varepsilon}\int\limits_S\rho dS\right)-\frac{r^2\mu_0}{4\pi}\frac{\partial}{\partial t}\left(\mu\int\limits_S J_r dS\right);$$

$$\vec{E}=\sum_{n=1}^{\infty}\sum_{l=-n}^{n}\nabla_t\Psi_{nl}\left\{\frac{1}{\varepsilon r}\frac{\partial}{\partial r}\left(r^2\varepsilon A_{nl}\right)-\frac{r}{2\pi\varepsilon_0\varepsilon}\int\limits_S\rho\Psi^*_{nl}dS\right\};$$

$$\vec{H}=\sum_{n=1}^{\infty}\sum_{l=-n}^{n}\left[\vec{r}_0\times\nabla_t\Psi_{nl}\right]\left\{-r\varepsilon_0\frac{\partial}{\partial t}\left(\varepsilon A_{nl}\right)-\frac{r}{2\pi}\int\limits_S J_r\Psi^*_{nl}dS\right\};$$

$$E_r = \sum_{n=1}^{\infty} \sum_{l=-n}^{n} A_{nl} q_n \Psi_{nl} + A_0.$$

To complete the statement of the problem the initial and (or) boundary conditions for the evolutionary equations are necessary. The same set of equations was obtained in [24] but without coefficients A_0 and B_0.

6. ANALYTICAL SOLUTIONS OF EVOLUTIONARY EQUATIONS

Let us consider propagation of TE-wave in free space ($\varepsilon \equiv 1$, $\mu \equiv 1$) without sources of given electrical and magnetic current. Initial conditions for B_{mk} are homogeneous. The source of a field is given on a sphere of the radius r_0: $B_{mk}(r_0, t) = F_{mk}(t)$, where $F_{mk}(t)$ are arbitrary functions of time, m and k describe angular distribution sources on the sphere. We will take into account the outgoing waves only.

Let's apply Laplace Transform to the written above equation for evolutionary coefficient $B_{mk}(r, t)$. We obtain

$$\left\{ r^2 \frac{\partial^2}{\partial r^2} - \left(\frac{r^2 s^2}{c^2} + m(m+1) \right) \right\} \left(r^2 B_{mk}(r, s) \right) = 0,$$

where s is a parameter of Laplace Transform. The general solution of this equation [30] is

$$B_{mk}(r, s) = r^{-3/2} \left(C_1(s) J_{m+1/2}\left(i\frac{s}{c} r \right) + C_2(s) J_{-m-1/2}\left(i\frac{s}{c} r \right) \right),$$

where $J_\nu(z)$ is a Bessel function. Using the known representation of Bessel function of half-integer order one can convert the solution to the form

$$B_{mk}(r, s) = r^{m-1} \left(\frac{1}{r} \frac{d}{dr} \right)^m \left(\frac{1}{r} \left(\tilde{C}_1 e^{-\frac{s}{c} r} + \tilde{C}_2 e^{\frac{s}{c} r} \right) \right).$$

Taking into account outgoing waves only we assume that $\tilde{C}_2 \equiv 0$. $\tilde{C}_1$ is found from boundary condition. As a result one can get

$$B_{mk}(r, s) = F_{mk}(s) \left(\frac{r}{r_0} \right)^{m-1} \frac{\left(\frac{1}{r} \frac{d}{dr} \right)^m \left(\frac{1}{r} e^{-\frac{s}{c} r} \right)}{\left(\frac{1}{r} \frac{d}{dr} \right)^m \left(\frac{1}{r} e^{-\frac{s}{c} r} \right) \Big|_{r=r_0}}.$$

Performing the differentiation the expression for $B_{mk}(r, s)$ is reduced to

$$B_{mk}(r, s) = \left(\frac{r_0}{r} \right)^2 F_{mk}(s) e^{-\frac{s}{c}(r-r_0)} + \left(\frac{r_0}{r} \right)^{m+2} F_{mk}(s) e^{-\frac{s}{c}(r-r_0)} \frac{Q_{m-1}\left(\frac{sr}{c} \right)}{P_m\left(\frac{sr_0}{c} \right)},$$

where $Q_{m-1}(x)$ and $P_m(x)$ are the polynomials of degrees $m-1$ and m correspondingly,

$$P_m\left(\frac{sr}{c}\right) = (-1)^m e^{\frac{s}{c}r} r^{2m+1} \left(\frac{1}{r}\frac{d}{dr}\right)^m \left(\frac{1}{r}e^{-\frac{s}{c}r}\right);$$
$$Q_{m-1}\left(\frac{sr}{c}\right) = P_m\left(\frac{sr}{c}\right) - \left(\frac{r}{r_0}\right)^m P_m\left(\frac{sr_0}{c}\right).$$

After returning to the originals the amplitude of longitudinal component of magnetic field can be written in form of the operator of wave propagation [27]

$$B_{mk}(r,t) = \left(\frac{r_0}{r}\right)^2 F_{mk}\left(t-\frac{r-r_0}{c}\right) + \left(\frac{r_0}{r}\right)^{m+2} F_{mk}\left(t-\frac{r-r_0}{c}\right) * G_m(r,t),$$

where * denotes the operation of conjunction, $G_m(r,t) = \sum_{l=1}^{m} \frac{Q_{m-1}\left(\frac{r\alpha_l}{r_0}\right)}{P'(\alpha_l)} e^{\frac{c}{r_0}\alpha_l t}$, α_l are the simple roots of the polynomial $P_m(x)$. For example,

$$G_1(r,t) = \frac{c}{r_0^2}(r_0-r)\, e^{-\frac{c}{r_0}t},$$
$$G_2(r,t) = 2\sqrt{3}\frac{c}{r_0^3}(r_0-r)e^{-\frac{3}{2}\frac{c}{r_0}t}\left(r\sin\left(\frac{\sqrt{3}}{2}\frac{ct}{r_0}-\frac{\pi}{6}\right) - r_0\sin\left(\frac{\sqrt{3}}{2}\frac{ct}{r_0}\right)\right).$$

Easy to see that the first term in the expression for $B_{mk}(r,t)$ coincides with the solution of the problem for spherically symmetrical source $B_0(r,t)$, other terms describe retarding part of nonstationary wave.

As for the problem for Neumann boundary condition one can get [28]

$$B_{mk}(r,t) = r\left(\frac{r_0}{r}\right)^{m+3} G_{mk}\left(t-\frac{r-r_0}{c}\right) * \hat{G}_m(r,t),$$

where $\hat{G}_1(r,t) = 2\frac{c}{\sqrt{3}r_0^2}e^{-\frac{3}{2}\frac{c}{r_0}t}\left(\sqrt{3}r\sin\left(\frac{\sqrt{3}}{2}\frac{ct}{r_0}-\frac{\pi}{6}\right) - r_0\sin\left(\frac{\sqrt{3}}{2}\frac{ct}{r_0}\right)\right)$.

7. NUMERICAL SIMULATION

As an example, let's consider the small loop of radius $r_0 = 0.01\,\mathrm{m}$ with infinitesimal radius of wire which is excited by the transient current with time dependence in form of short pulse

$$f(t) = \frac{1}{2}\left(1-\cos\left(\frac{2\pi t}{T}\right)\right)(H(t)-H(t-T)),$$

where $H(t)$ is Heaviside step function, $T = 1\,\mathrm{ns}$.

It is easily seen that the chosen source radiates TE-waves only, and does not generate zero mode. The radiation problem can be stated either in form of problem for inhomogeneous partial differential equation with homogeneous initial and boundary conditions or in form of problem for homogeneous partial differential equation with homogeneous initial and inhomogeneous Neumann boundary conditions. Let solve the problem in the second form transforming the given currents of the loop into equivalent magnetic field on the sphere of radius r_0.

Excitation of loop by point source with above mentioned time dependence of current is accompanied by appearance of significant induced currents that cause the change of time dependence of total current. The change of current time dependence can be calculated by three-dimensional FDTD method. Normalized initial current time dependence and total current time dependence obtained by 3D FDTD method are depicted on Fig. 1. Really, neither the shape of curve 1, nor the form of curve 2 is not the time dependence of surface currents because of nonsynchronous excitation by point source, but we will neglect the effect in our further simulations.

One dimensional evolutionary equations can be solved numerically as well by well-known change of operations of differentiation by time and radial coordinate with finite differences (one-dimensional FDTD method) [31]. As a result on the Fig. 2 and Fig. 3 time dependences of

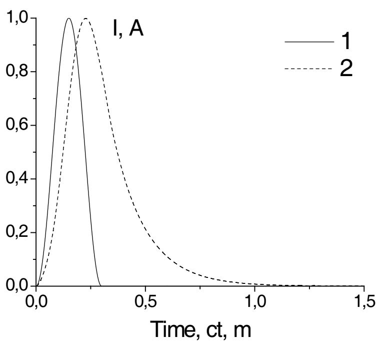

Figure 1. Normalized time dependences of given current of external source (curve 1) and current with accounting of induced surface currents (curve 2).

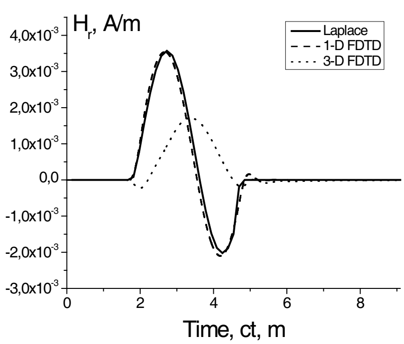

Figure 2. Time dependences of radial component of magnetic field for $r = 0.167\,\mathrm{m}$, $\theta = 0^\circ$ (1 — Laplace transform method, 2 — 1D FDTD, 3 — 3D FDTD).

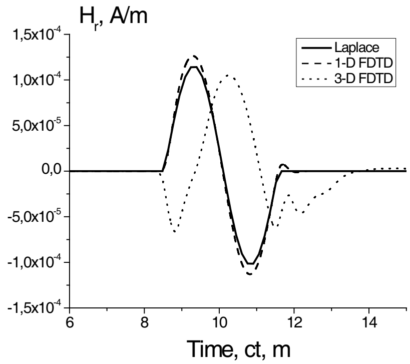

Figure 3. Time dependences of radial component of magnetic field for $r = 0.83\,\mathrm{m}$, $\theta = 0°$ (1 — Laplace transform method, 2 — 1D FDTD, 3 — 3D FDTD).

the amplitude of radial component of magnetic field are presented for the points of observation $r = 0.167\,\mathrm{m}$ and $r = 0.83\,\mathrm{m}$ correspondingly at $\theta = 0°$. Curves 1 correspond to the time shape of excitation current obtained by the analytical approach, curve 2 received by one-dimensional FDTD method for the same form of current, curve 3 is obtained by three-dimensional simulation with taking into account the distortion of time shape of current and asynchronous excitation of the loop. It is seen that curves 1 and 2 show change the time form of pulse from the form of current in near-field region (Fig. 2) to the form of its first derivative in far-field region (Fig. 3).

Let consider the case when the radiator is surrounded by the spherical layer of radially inhomogeneous medium with permittivity $\varepsilon(r) = 1 + \varepsilon_1 e^{\frac{-4(r-r_1)^2}{h^2}}$, where $\varepsilon_1 = 4$, effective radius of sphere $r_1 = 0.6\,\mathrm{m}$ and effective thickness of layer $h = 0.01\,\mathrm{m}$. The problem can be solved easily by one-dimensional FDTD method. The sampling step is chosen reasoning from the quickest change of medium characteristics on r and time dependence of exciting current to avoid numerical instability [31, 32].

Time dependences of the amplitude of radial component of magnetic field are presented on the Fig. 4 and Fig. 5 for the points of observation $r = 0.167\,\mathrm{m}$ and $r = 0.83\,\mathrm{m}$, inside and outside of sphere correspondingly. Curves 1 correspond to the initial time shape of excitation current from Fig. 1 (curve 1), curves 3 correspond to the corrected shape from the same figure (curve 2), curve 2 is obtained by three-dimensional simulation with taking into account asynchronous excitation of the loop. Curve 2 is calculated in approximation of

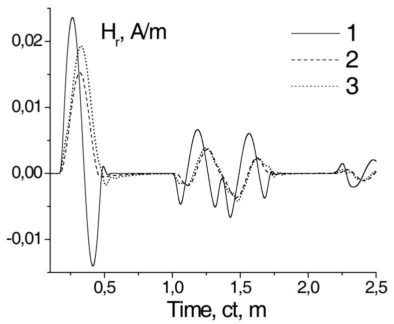

Figure 4. Time dependences of radial component of magnetic field for $r = 0.167\,\mathrm{m}$, $\theta = 0°$ (1 — Numerical solution for given current of external source (1D FDTD), 2 — Numerical solution with calculation of induced currents (3D FDTD), 3 — Numerical solution for given current with accounting of induced currents (1D FDTD).

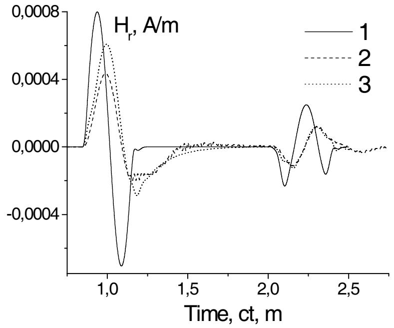

Figure 5. Time dependences of radial component of magnetic field for $r = 0.83\,\mathrm{m}$, $\theta = 0°$ (1 — Numerical solution for given current of external source (1D FDTD), 2 — Numerical solution with calculation of induced currents (3D FDTD), 3 — Numerical solution for given current with accounting of induced currents (1D FDTD).

homogeneous dielectric spherical layer ($\varepsilon = 5$) with thickness of the layer $h = 0.01\,\mathrm{m}$. The waves reflected from spherical layer once ($ct = 1\,\mathrm{m}$) and twice ($ct = 2.2\,\mathrm{m}$) are seen on the Fig. 4. The travelling through spherical layer waves are depicted on Fig. 5 where one of pulses is reflected from the other side of sphere ($ct = 2.1\,\mathrm{m}$). One can see that the correction of the time shape of exciting current obtained by 3D FDTD method for the use of 1D FDTD method give us satisfactory precision of result with the significant decrease of calculation time. The decreasing of amplitude of pulses of curves 2 and 3 is explained by widening of exciting pulse shown on the Fig. 1 and asynchronous excitation of the loop (curves 2).

According to the idea of modal expansion, to obtain the amplitudes of fields we must calculate the amplitudes of fields of plenty of modes. But even for the case of points situated close to the loop one can calculate the amplitudes of fields of several modes. It is illustrated on the Fig. 6 where dependence of ratio between values of amplitudes of radial component of magnetic field and its precise value on number of accounted modes for the close to loop distances of observation r is depicted.

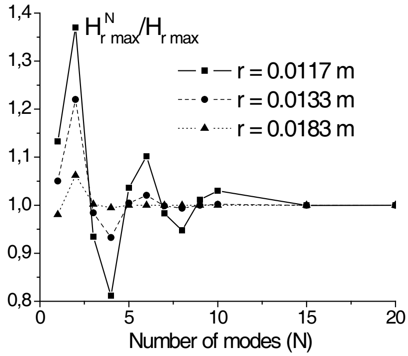

Figure 6. Dependence of ratio between values of amplitudes of radial component of magnetic field accounting first N modes and its precise value for different distances of observation r.

One-dimensional numerical simulation have been carried out for maximum frequency of signal spectrum $F_{\max} = 400\,\text{GHz}$, and the process consumed 2 Mb RAM. Three-dimensional simulation was performed for $F_{\max} = 3\,\text{GHz}$, and needed 200 times more RAM and 40 times more CPU time. It means that to reach the same accuracy in numerical calculation by 3D FDTD method we need in 27000 times more RAM than using evolutionary equation technique and 1D FDTD method for the considered problem.

8. CONCLUSIONS

The nonstationary one-dimensional equation set that describes evolution of transient electromagnetic waves in spherically inhomogeneous transient medium is obtained. Using Laplace Transform the operator of transformation of transient electromagnetic field on the sphere of smaller radius to the field on the sphere of bigger radius is constructed.

Combination of three-dimensional numerical simulation for obtaining the space-time distribution of a source of field and modal basis method with the use of the numerical or analytical solving of evolutionary equations permits to decrease the calculation time and main memory significantly.

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