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FREQUENCY DISPERSION LIMITS RESOLUTION IN VESELAGO LENS

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Abstract—The properties of a lossless Veselago lens is examined when the material parameters epsilon and mu are frequency dispersive. A complete solution is presented that is based on the use of Fourier transforms in the frequency domain and is obtained in terms of the residues at the poles and branch cut integrals. It is shown that for an incident field with a finite frequency spectrum the excited evanescent field consists of resonant even and odd surface wave modes that do not grow exponentially within the slab. For a lossless slab and a sinusoidal signal of finite duration, at a single frequency corresponding to that where the relative values of epsilon and mu equal -1, Pendry's solution is obtained along with excited surface wave modes and other interfering waves that makes it impossible to obtain a coherent reconstruction of the spatial spectrum of the object field at the image plane. If the slab material is lossy the excited interfering surface wave modes will decay away in a relatively short time interval, but as shown by other investigators the resolution of the lens will be reduced in a very substantial way if the losses are moderate to large.

1. INTRODUCTION

In 1968 Veselago introduced the concept of a material having simultaneous negative values of epsilon and mu. He discussed a number of properties of such media, such as negative refraction, left handed wave solutions, a flat lens configuration, among other features [1]. Veselago pointed out that any medium with negative mu or epsilon would have to be frequency dispersive in order for the field energy to be positive. Many years later Pendry considered a flat slab lens made from material with the relative values of both epsilon and mu equal to -1, [2]. He showed that a propagating plane wave incident upon this lens would be perfectly matched at the interface, i.e., the reflection

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coefficient would be zero and the transmission coefficient through the slab would equal 1 whenever the relative values of epsilon and mu were both equal to -1. In the case of an incident evanescent wave the reflection and transmission coefficients become infinite when the index of refraction becomes equal to -1 so the standard method of solving for the reflected and transmitted waves can not be used. In order to overcome this difficulty Pendry began with relative values of epsilon and mu different from -1 and expressed the solution as a series of multiple reflected waves within the slab. This series is a geometric series which was summed and then the limit was taken as the relative values of epsilon and mu approached -1. The result showed that the overall transmission through the slab for the evanescent waves was in the form of a single exponentially growing wave within the slab. This was a somewhat surprising result but was generally considered to be a correct result by many people carrying out research on negative index of refraction media. It is interesting to note that in Pendry's original work he assumed the presence of a decaying wave in the slab but after summing the multiple reflected wave series and letting the relative values of epsilon and mu become equal to -1 this part of the solution was cancelled out. At the frequency for which the index of refraction is -1 in the slab and 1 outside the slab the evanescent wave attenuation constants are the same in both media. Consequently the exponential growth of the waves within the slab compensates for the exponential decay outside the slab and results in the amplitude of all evanescent waves to be restored to their original values at the image plane as illustrated in Fig. 1. From this result Pendry concludes that such a lens would reproduce the original scene on the object plane with infinite resolution at the image plane.

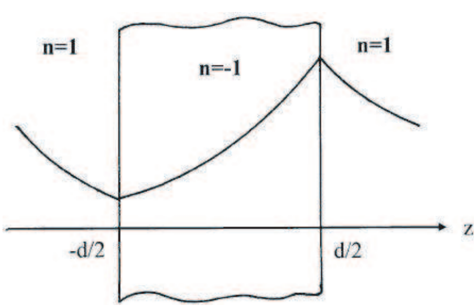

Figure 1. Electric field distribution for Pendrys solution for an evanescent incident wave on a slab with an index of refraction equal to -1.

There is a flaw in Pendry's method that was overlooked, which is that as the relative values of epsilon and mu become quite close to -1 in value the magnitude of the ratio of successive terms in Pendry's multiple reflected wave series become greater than 1 and the geometric series does not converge. However, there is a mathematical solution that satisfies the boundary conditions and which is consistent with Pendry's solution for a lossless lens. The final result he obtained may be easily demonstrated in the following way. Assume that the relative values of epsilon and mu are equal to -1 and let an exponentially decaying evanescent wave be incident on the first interface between free space and the negative index of refraction medium. It is then easy to show that the boundary conditions on the tangential electric and magnetic fields at the first interface can be satisfied by assuming that in the negative index of refraction slab the field consists of a single exponentially growing wave. Likewise it can be shown that at the output interface the boundary conditions can be satisfied by assuming an exponentially decaying wave on the output side, which confirms Pendry's result. These three waves together will satisfy the boundary conditions that require the tangential electric and magnetic fields to be continuous across the two interfaces. This solution is a solution of the source free Maxwell's equations and must be a resonant mode even though the evanescent wave on the input side was the incident evanescent wave in Pendry's solution.

Pendry's solution requires the field to be a steady state time harmonic oscillation of infinite duration, at the frequency for which the index of refraction is exactly equal to -1. If we view this solution as a resonant mode then it is not a proper physical solution since the field grows exponentially away from both sides of the first interface. For a single interface a surface wave that decays away from both sides of the interface can also be supported by the surface. This is a proper physical mode solution. A proper physical solution must vary in a continuous manner when the physical parameters that characterize the problem change. Pendry's solution fails this test as can be seen by noting that if the relative values of epsilon, mu, or the index of refraction change from the exact value of -1 the boundary conditions are no longer satisfied because the tangential magnetic field will no longer be continuous across each interface, the latter requiring the attenuation constants to be the same in the slab as in the surrounding medium, and the relative value of mu to equal -1. Thus any change in epsilon and mu, even the addition of small loss, will cause a change in the solution such that the tangential magnetic field is no longer continuous across the interfaces. If Pendry's solution is viewed as a resonant mode then it also fails the uniqueness test for field solutions since the field becomes infinite at infinity.

When the relative values of epsilon and mu do not equal -1 then the transfer function obtained by Pendry before the limit of setting the relative values of epsilon and mu equal to -1 must be used. This transfer function, which describes the electric field at the output interface of the lens in terms of the field at the input interface, has the following form for an incident evanescent wave [2]

$$\tau_r(k_x, \omega) = \frac{tt' e^{-\alpha d}}{1 - r'^2 e^{-2\alpha d}} = \frac{4\mu\alpha_0 \alpha e^{-\alpha d}}{(\mu\alpha_0 + \alpha)^2 - (\mu\alpha_0 - \alpha)^2 e^{-2\alpha d}}$$

where μ and ε are the relative values of the permeability and permittivity of the lens slab material. The attenuation constants in the free space region and within the lens are given by $\alpha_0 = \sqrt{k_x^2 - k_0^2}$, $\alpha = \sqrt{k_x^2 - \mu\varepsilon k_0^2}$ where k_x is the transverse wave number of the evanescent wave, and $k_0 = \omega/c$ is the free space wave number. When the relative values of epsilon and mu become equal to -1 then $\tau_r = e^{\alpha d}$ which cancels the corresponding decay $e^{-\alpha_0 d}$ in the field between the object plane and the first interface of the lens and that from the output interface of the lens to the image plane. This results in the amplitudes of all evanescent waves to be restored at the image plane and results in a lens with perfect resolution. However, the field within the slab becomes divergent for large values of the transverse wave number. This divergent behavior raises some questions as to whether or not Pendry's solution is a valid physical solution. In our analysis we arrive at the conclusion that it is not a complete solution by itself.

Several authors have derived modifications to Pendry's transfer function given above by introducing small losses in epsilon and mu or by considering a small perturbation in epsilon or mu away from the value of -1 for large values of the transverse wave number [3–14]. This has the effect of eliminating the field divergence problem but reduces the resolving power of the lens and thus limiting the resolution that can be obtained.

Various authors have recognized that the negative index slab can support resonant surface wave modes and that these modes play an important role in the behavior of the lens [3–13]. The excitation of these resonant surface wave modes do not appear in the strict steady state solutions. Gómez-Santos considered an input sinusoidal signal that was turned on at $t = 0$ and turned off at $t = \tau$ [9]. He proposed modeling the lens as two coupled mechanical resonators, with resonant frequencies corresponding to those of the even and odd surface wave modes that can exist on the negative index slab. The resonant frequencies are well separated for small values of the wave attenuation constants which occur for small values of k_x, but merge together as k_x approaches infinity. From the solution to the coupled oscillator problem Gómez-Santos showed that the length of time for the oscillations to build up to the steady state value was proportional to the reciprocal of the resonant frequency separation $\Delta\omega$ of the two modes. In the limit the amplitude of the response of the second oscillator, corresponding to the output interface of the lens, was found to be proportional to $(\Delta\omega t)^2 e^{-2\alpha d}$ in the initial phase of the build up of the oscillations. Thus in the limit of infinitely large transverse wave numbers the oscillations would never build up to infinite values in any finite time interval. This mechanism was proposed by Gómez-Santos to eliminate the singularity in Pendry's solution. On this basis he concluded that Pendry's solution was acceptable.

Grbic also considered a time domain solution and included frequency dispersion in epsilon and mu [10]. He chose a cosinusoidal input signal of semi-infinite duration which had a frequency spectrum proportional to $1/(\omega^2 - \omega_0^2)$ where ω_0 is the frequency at which the relative values of epsilon and mu equal -1. He obtained a result similar to that obtained by Gómez-Santos. Neither Grbic or Gómez-Santos include the branch cut integrals that occur in the inverse Fourier or Laplace transform evaluations. A similar input signal, but with a finite duration, was considered by Yagjian and Hansen [11]. They also analyzed the effect of losses on the resolving power of the lens. For the case of a sinusoidal signal turned on at $t = -t_0$ and turned off at $t = t_0$ their spectral function given by Eq. (36) in their paper should have been expressed as a spectral function that is applicable for $(L+2d)/c - t_0 < t < t_0 + (L+2d)/c$ and a spectral function applicable for $t > (L+2d)/c + t_0$, i.e., causality requires the output to be delayed by the propagation time delay. In the first time interval the spectral function exhibits poles at $\omega = \pm\omega_0$ and would produce a dominant wave at the frequency ω_0, corresponding to Pendry's solution, plus the excitation of the even and odd surface waves that would interfere with the dominant wave. After the signal is turned off their spectral function as given by their Eq. (36) is applicable. We will also analyze the problem using a sinusoidal signal of finite duration but evaluate the inverse Fourier transform using the residues at the poles plus branch cut integrals, and thus obtain a more complete solution.

An approximate solution to the lens problem when the dispersion obeys the Smith-Kroll model was developed by de Wolf [12, 13]. He found approximate expressions for the resonant surface wave modes but did not identify these as resonant surface wave modes. He considered the input signal spectrum to be a narrow band of frequencies with finite density and obtained an expression for the slab transmission coefficient by integrating the transmission coefficient over a narrow band of frequencies, but did not evaluate the solution as an inverse Fourier transform which would have included a $e^{j\omega t}$ time factor. Thus his solution is not in the form of resonant surface wave modes. However, he did find from his approximate solution that for a narrow band signal spectrum the field in a slab lens did not grow exponentially for slabs with a thickness greater than some minimum value, which typically was very small.Chew has also examined this problem but he uses a strict steady state solution and thus encounters wave solutions that diverge [14]. However, by introducing loss along the lines other earlier investigators followed the field divergences can be eliminated. There have been a number of studies of other lens configurations that depart from the specific configuration considered by Pendry but these will not be reviewed here.

Several researchers have built periodic structures to simulate a negative index of refraction medium and constructed flat slab lenses and attempted to demonstrate super resolution, but with only limited success [15–18].

Even though a number of authors have expressed the view that the resonant surface wave modes will play an important role in the behavior of Veselago's lens it does not appear that anyone has expressed the complete field solution for Veselago's lens, and explicitly including the excitation of the even and odd resonant surface wave modes within the slab as well as exterior to the slab, probably because they do not appear directly in a steady state solution when epsilon and mu are not frequency dependent. The fact that for Pendry's solution the boundary conditions at the first interface can be satisfied by a single exponentially growing wave on each of the two sides of the interface is a troubling result because we are at liberty to assume that for the slab of negative index of refraction media the thickness of the slab can be made arbitrarily large and we then have the capability to create an electric field with enormous intensity. This can hardly be accepted as being a valid physical result. The original solution for the transmission factor or transfer function from the object plane through the negative index of refraction slab and to the image plane was dependent on the relative values of epsilon and mu being exactly equal to -1. Furthermore, the incident field was assumed to be a steady state single frequency oscillation. The operation of the lens depended critically on the excitation of a resonant mode that was intimately tied to the two interfaces of the lens. The assumption of a strict steady state solution is the primary cause of the divergence associated with Pendry's solution when the losses are set equal to zero. Clearly an input signal of semi-infinite or infinite duration is non-physical.

In view of the above considerations we were led us to consider the effect of frequency dispersion, which as Veselago stated, is necessary. We therefore examined a simple model were both epsilon and mu were considered to have a simple dependence on frequency, as was used for an example in Veselago's paper [1]. In the analysis given below we first consider the problem of a single interface which is found to support a resonant surface wave mode for which the field decays in an exponential manner away from both sides of the interface. The results of this analysis showed that at the frequency ω_e where the relative values of epsilon and mu become equal to -1 a pole in the frequency response of a single interface occurs. This leads to a discrete mode with frequency ω_e and which undergoes exponential decay, not growth, in the negative index of refraction medium. If the incident field consists of a sinusoidal signal at the frequency ω_e for which the relative values of epsilon and mu are equal to -1, and of finite duration, then a double pole occurs in the response function for which the time response is proportional to $te^{j\omega_e t}$. Thus this example suggests that if frequency dispersion is taken into account the dilemma of a single exponentially growing wave on the output side of the first interface in Veselago's lens will be eliminated.

We next consider the two interface problem on which a pair of resonant coupled surface wave modes are excited. When the incident field has a continuous frequency spectrum with finite density our solution results in a set of proper surface waves or resonant modes on the slab and do not require the introduction of loss or small perturbations in epsilon or mu in order to avoid the short transverse wavelength divergence. This leads to a proper solution for the Veselago lens, but this solution does not support the exponential growth of the evanescent waves within the slab and thus for this type of incident field no super resolution is possible. We also consider an incident field consisting of a single discrete frequency sinusoidal oscillation of finite duration. For this case the field at the image plane is found to consist of a driven mode, corresponding to Pendry's solution, at the frequency of the incident field along with the resonant even and odd surface wave modes, plus fields with continuous frequency spectra that arise from branch cut integrals. For the lossless slab the unavoidable excitation of the even and odd surface wave modes at their resonant frequencies produces interference at the image plane that makes it difficult, if at all even possible, to coherently reconstruct the amplitudes of the evanescent waves at the image plane. We also found that a continuous spectrum of interfering propagating waves would also be produced at the image plane and which will cause blurring of the image. This solution is based on the evaluation of the excitation of the resonant surface wave modes in terms of the residues at the poles and sheds new insight into the operation of the flat lens. When the excited surface waves are included as part of the solution the divergence of the field for large transverse wave numbers is eliminated.

The existence of surface wave poles can be accounted for by using standard Fourier or Laplace transform techniques and thus obtaining finite responses in terms of the residues at the poles. There is a very large body of literature dealing with radiation from various kinds of sources over layered media that dates back to the 1909 classical work by Sommerfeld for dipole radiation over a lossy earth [19–21]. The many techniques developed in that research can be applied equally well to the Veselago's lens problem. This is a very realistic approach since many scenes that one might want to image are best represented by a stochastic process which has a broad frequency spectrum. A quantity of interest is the optical coherence function and this is best dealt with using Fourier transforms [22]. In a recent paper by Gralac and Tip the complete solution for the electromagnetic field excited in a layered structure containing negative index media with permittivity and permeability described by Lorentz type dispersion is developed [23]. Their solution is based on the use of Laplace transforms in the time domain and is formulated in terms of the convolution of the impulse response of the polarization in the media with an appropriate Green's function. Both p and s polarized waves are included. One important result derived is that the excited electromagnetic field will always be square integrable if the input source or initial field is square integrable. These are required physical conditions and ensures that the solutions do not have unlimited spatial exponential growth. Our solution is not as general as the one provided by these authors but we do note that an input sinusoidal signal of finite duration is square integrable. These authors give a detailed solution for the single interface problem and obtain results that support our solution for this case. They do not give a detailed solution for the two interface problem that involves coupled surface wave modes.

2. ANALYSIS-SINGLE INTERFACE

Consider the interface at $z = 0$ between free space and a negative index of refraction medium as shown in Fig. 1. The medium parameters of the output half-space will be assumed to be given by $\varepsilon\varepsilon_0$ and $\mu\mu_0$ where

$$\varepsilon = 1 - \frac{2\omega_e^2}{\omega^2} \tag{1a}$$

$$\mu = 1 - \frac{2\omega_e^2}{\omega^2} \tag{1b}$$

For this medium the relative values of both epsilon and mu become equal to -1 at the frequency ω_e and remain negative for $\omega < \sqrt{2}\omega_e$. In the analysis given below we will make use of Gabor's concept of an analytic signal [24]. Consider a real time signal $f(t)$ and its Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

The original signal can be recovered using the inverse Fourier transform, thus

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega = \frac{1}{\pi}\mathrm{Re}\int_{0}^{\infty} F(\omega)e^{j\omega t}d\omega$$

because $F(-\omega) = F^*(\omega)$, which is the complex conjugate function. The integral without the real part designation is called the analytic signal. The advantage gained by using the analytic signal is that we only need to include positive frequencies. The real part of the analytic signal gives the physical solution.

We will assume that the source illumination comes from a finite width aperture that is located at $z = -a$ and extends from minus to plus infinity along the y-direction. The electric field will be assumed to be in the y-direction and its intensity is a function of x only. The Fourier transform of the spatial intensity will be represented by $A(k_x, \omega)$ and its frequency spectral density by $S(\omega, k_x)$. In general the aperture field will be a function of the radian frequency ω and the frequency spectrum of each spatial component may be different. Thus we show the spatial spectral density and the frequency spectral density as a function of both ω and k_x. The incident field will be assumed to be that of a S polarized wave (TE wave) with a y-directed electric field and a magnetic field with x and z components. Let the incident electric field of an evanescent wave be given by the Fourier transform

$$E_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k_x, \omega)S(\omega, k_x)e^{-jk_x x-\alpha_0(\omega)(z+a)}e^{j\omega t}d\omega \tag{2a}$$

The corresponding Fourier transform representation of the output signal will be

$$E_o = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k_x, \omega)S(\omega, k_x)T(\omega)e^{-jk_x x-\alpha_0 a-\alpha(\omega)z}e^{j\omega t}d\omega \tag{2b}$$

where the transmission coefficient across the interface for an S polarized wave is $T(\omega)$. On the input side there will be a reflected evanescent decaying wave with a reflection coefficient $R(\omega)$. The reflection and transmission coefficients are given by [2]

$$R(\omega) = \frac{\mu\alpha_0 - \alpha}{\mu\alpha_0 + \alpha} \tag{3a}$$

$$T(\omega) = \frac{2\mu\alpha_0}{\mu\alpha_0 + \alpha} = 1 + R(\omega) \tag{3b}$$

where $\alpha_0 = \sqrt{k_x^2 - k_0^2}$, $\alpha = \sqrt{k_x^2 - \mu\varepsilon k_0^2}$, and $k_0 = \frac{\omega}{c}$. (Note that Pendry uses the $e^{-i\omega t}$ time dependence). The transmission coefficient can be expressed in the following form

$$T(\omega) = \frac{2\mu\alpha_0(\mu\alpha_0 - \alpha)}{\mu^2\alpha_0^2 - \alpha^2} = \frac{2\mu\alpha_0(\mu\alpha_0 - \alpha)}{\left(1 - \frac{2\omega_e^2}{\omega^2}\right)^2 (k_x^2 - k_0^2) - k_x^2 + \left(1 - \frac{2\omega_e^2}{\omega^2}\right)^2 k_0^2}$$

which can be simplified to the for

$$T(\omega) = -\frac{2\mu\alpha_0(\mu\alpha_0 - \alpha)\omega^4}{4\omega_e^2 k_x^2(\omega^2 - \omega_e^2)} \tag{4}$$

We see that $T(\omega)$ has poles at $\omega = \pm\omega_e$. When ω becomes very large both α_0 and α become equal to $j\omega/c$. Hence for $t < (a+z)/c$ the contour of integration can be closed in the lower half of the complex ω plane and there will be no contribution to the output field before $t = (a+z)/c$, which is required by causality. We will assume that the spectral density function $S(\omega, k_x)$ vanishes sufficiently fast for large values of ω so that the integral converges in the upper half of the ω plane and we choose a contour of integration that runs below the poles at $\pm\omega_e$. The pole contribution to the integral may be evaluated by residue theory and gives

$$\begin{aligned} E_o &= -2\pi j A(k_x, \omega_e) \frac{\mu\alpha_0(\mu\alpha_0 - \alpha)\omega_e}{8\pi k_x^2} S(\omega_e, k_x) e^{j\omega_e t} e^{-jk_x x - \alpha_0 a - \alpha z} + CC \\ &= -jA(k_x, \omega_e) \frac{k_x^2 - k_0^2}{2k_x^2} \omega_e S(\omega_e, k_x) e^{j\omega_e t} e^{-jk_x x - \alpha_0 a - \alpha z} + CC \\ & \qquad t > (a+z)/c \end{aligned} \tag{5a}$$

where all terms are evaluated for $\omega = \omega_e$ and CC represents the complex conjugate term. The first term by itself represents the analytic signal which we designate as $\tilde{E}_o$, thus

$$\tilde{E}_o = -jA(k_x, \omega_e) \frac{k_x^2 - k_0^2}{2k_x^2} \omega_e S(\omega_e, k_x) e^{j\omega_e t} e^{-jk_x x - \alpha_0 a - \alpha z} \quad t > (a+z)/c \tag{5b}$$

This solution is a discrete exponentially decaying wave at the frequency ω_e. The discrete wave represents some type of resonance, which could be said to be due to surface plasmons [2].

From the relation (3b) it is clear that the tangential electric field will be continuous across the interface. The tangential magnetic field on the input side is proportional to

$$[1-R(\omega)]\frac{j\alpha_0}{\omega\mu_0}A(k_x,\omega)=\frac{j2\alpha_0\alpha(\mu\alpha_0-\alpha)}{\omega\mu_0[(\mu\alpha_0)^2-\alpha^2]}A(k_x,\omega) \tag{6a}$$

while on the output side it is proportional to

$$[1+R(\omega)]\frac{j\alpha}{\omega\mu\mu_0}A(k_x,\omega)=\frac{j2\mu\alpha_0\alpha(\mu\alpha_0-\alpha)}{\omega\mu\mu_0[(\mu\alpha)^2-\alpha^2]}A(k_x,\omega) \tag{6b}$$

These two expressions are equal and have the same residues at the poles. However, it should be noted that there is no pole associated with the incident field. The reflection coefficient and the transmission coefficient have the same residues. The tangential electric and magnetic fields of this resonant mode are continuous across the interface and the field has exponential decay away from the interface on both sides. Thus this discrete frequency mode that is excited is clearly a resonance effect. Once this mode is excited by the incident field it will continue to oscillate, eventually decaying to zero because of losses that usually will be present. The mode may also be viewed as a surface wave that is bound to the interface and its electric field decays in an exponential manner away from both sides of the interface.

An interesting variation of the above results are obtained if we consider a sinusoidal signal of finite duration and with a frequency equal to the resonant frequency of the surface wave. Thus consider the input signal consisting of a sinusoidal oscillation $\sin\omega_e t$ at the frequency ω_e, which is turned on at $t=0$ and turned off at $t=\tau$. For this signal the spectral function is given by

$$\begin{aligned} S(\omega)&=\frac{e^{j(\omega_e-\omega)\tau}-1}{2(\omega-\omega_e)}-\frac{e^{-j(\omega+\omega_e)\tau}-1}{2(\omega+\omega_e)} \\ &=je^{j(\omega_e-\omega)\tau/2}\frac{\tau}{2}\frac{\sin(\omega-\omega_e)\tau/2}{(\omega-\omega_e)\tau/2}-je^{-j(\omega+\omega_e)\tau/2}\frac{\tau}{2}\frac{\sin(\omega+\omega_e)\tau/2}{(\omega+\omega_e)\tau/2} \end{aligned} \tag{7}$$

The part of $S(\omega)$ that involves the terms depending on τ, which corresponds to when the signal is turned off, does not contribute to the output at $a+z$ for $t<\tau+(a+z)/c$. Prior to that time the spectral function has a pole at the resonant frequency of the surface wave. Thus the system has a double pole at $\omega=\omega_e$. The time response for a double pole is $-te^{j\omega_e t}$ and thus the response for the single interface is given by

$$E_0 = jA\left(k_x, \omega_e\right) \frac{\left(k_x^2 - k_0^2\right)\omega_e}{4k_x^2} e^{-jk_x x - \alpha_0 a - \alpha z}\left(te^{j\omega_e t}\right) + CC$$

$$\frac{a+z}{c} < t < \tau + \frac{a+z}{c} \quad (8)$$

This type of response is well known for a lossless resonant circuit when excited by a sinusoidal signal of finite duration and at the resonant frequency of the circuit.

Later on we will show that for a lossless slab there will be two surface wave modes whose frequencies coalesce at $\omega = \omega_e$, when the transverse wave number k_x approaches infinity, to produce a triple pole with a time response proportional to $t^2 e^{j\omega_e t}$. This result was obtained by Gómez-Santos from a model of the slab as a pair of coupled mechanical resonators [9]. The signal output does not become infinite since the signal is turned off at $t = \tau$. When the signal input is turned off the frequency spectrum no longer has a pole and is of the form that makes the output become small because of a rapidly oscillating spectral function.

The integrand also has branch points at $\omega = \pm k_x c$, and $\omega = \pm\sqrt{k_x^2 c^2 + 2\omega_e^2}$ where c is the velocity of light in free space. Thus in addition to the pole contributions there are additional contributions to the field from branch cut integrals. The branch cut integrals give rise to a continuous spectrum of waves and these fields will satisfy the boundary conditions at the interface. The branch cut integrals will not give rise to exponentially growing fields because the branches for which the real part of α_0 and α are greater than zero must be chosen in order to ensure that the fields remain finite as z approaches infinity. The analysis to include the branch cut integrals would be similar to that described by Stratton for propagation in a dispersive medium but will not be pursued in this paper for the single interface problem [25].

3. SOLUTION FOR THE TWO INTERFACE PROBLEM

The Veselago's flat lens involves two interfaces, one at $z = -d/2$ and one at $z = d/2$ as shown in Fig. 2. The slab is characterized by the same dispersion relations used for the single interface and given earlier by Eq. (1). Each interface by itself can support a resonant surface plasmon mode, i.e., surface wave. When the spacing between the two interfaces is finite these two modes will interact and the result will be two new perturbed modes with resonant frequencies that lie above and below that of the mode supported by a single interface. An analogy with a similar waveguide problem will help to clarify the phenomenon involved. Consider a rectangular waveguide that is operated below cutoff. Let a short section of this waveguide be filled with a high dielectric constant material so that it forms a resonant cavity. Since a dielectric slab can support even and odd surface wave modes at discrete frequencies these represent the resonant modes of the cavity. The response function of this cavity will have poles at the resonant frequencies of the cavity. If a spectrum of evanescent waves with a continuous frequency spectrum is incident on this cavity this incident field will excite a finite response in the resonant cavity at its resonant frequencies. If we assumed that a steady state sinusoidal field at the resonant frequency of one of the cavity modes was incident upon it the response would be infinite. But with a field having a finite frequency spectral density the response of the cavity is also finite and is determined by the residues at the poles that lie in the frequency range of the incident field spectrum. This same phenomenon occurs with a slab of negative index material sandwiched between two regions with a positive index of refraction. When we include frequency dispersion the overall transmission factor through the slab of negative index material exhibits poles corresponding to the new frequencies for the two coupled resonators. Thus an incident field with a continuous but finite frequency spectrum will excite these resonances in addition to a continuous spectrum of transmitted waves. These resonant responses do not exhibit growing exponential waves. If the Veselago lens was to be used to image the aperture field distribution at $z = -d$ at an image plane located at $d/2$ beyond the second interface then the excitation of the surface plasmons would represent an artifact that should not be present in the image since the aperture field does not contain discrete

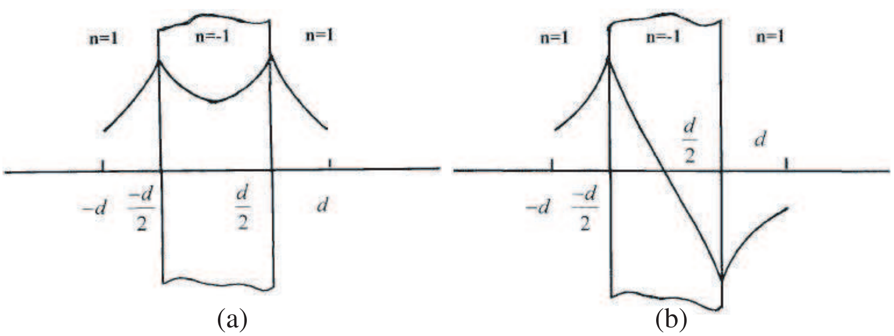

Figure 2. (a) The even surface wave mode field distribution in a flat slab lens. (b) The odd surface wave mode field distribution in a flat slab lens.

frequency components (by assumption) [3]. When frequency dispersion is taken into account and the analysis is carried out in a more complete manner, rather than assuming a steady state sinusoidal incident field of infinite duration, the results are different from what Pendry obtained.

Our solution has a number of features similar to what Gómez-Santos included but differ in some important respects. If we assume an incident field with a continuous frequency spectrum that overlaps both resonant frequencies, and with a finite spectral density, the response of the lens can be evaluated in terms of the residues at the two poles corresponding to the surface wave mode resonances. This results in a field distribution within the lens that does not grow exponentially and is quite different from what Pendry had found. For a lossless slab and a sinusoidal input signal of finite duration, and at the frequency for which the relative values of epsilon and mu equal -1, the response does not become exponentially large when the transverse wave number approaches infinity because of the excitation of the surface wave modes. The interference produced by the excited surface wave modes destroys the super resolution properties of the lens. We also show that an interfering signal that blurs the output at the image plane for the propagating waves also occurs. This interfering signal arises from branch cut integrals that are part of the complete solution.

The analysis given below will provide the details that support the above description and conclusions. We will take advantage of the symmetry inherent in the problem and construct the even and odd mode solutions in separate steps. We can superimpose the two solutions so as to obtain the solution for a field incident from one side of the slab only. This has the advantage that it provides simpler expressions to evaluate for the residues at the poles. We will express the even solution, for one component of the spatial spectrum $A(k_x, \omega)$ of the electric field, in the form

$$E_e = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_e(\omega) e^{-jk_x x} e^{j\omega t} d\omega \tag{9}$$

where

$$\begin{aligned} \Psi_e(\omega) &= C_1 e^{-\alpha_0(z+d)} + C_2 e^{\alpha_0(z+d/2)} && -d < z < -d/2 \\ &= C_3 \cosh \alpha z && -d/2 < z < d/2 \\ &= C_1 e^{\alpha_0(z-d)} + C_2 e^{-\alpha_0(z-d/2)} && d/2 < z \end{aligned}$$

The corresponding spectral function for the magnetic field is (we omit a factor $1/j\omega\mu_0$)

$$\begin{aligned}
\Psi_{he}(\omega) &= -\alpha_0 C_1 e^{-\alpha_0(z+d)} + \alpha_0 C_2 e^{\alpha_0(z+d/2)} && -d < z < -d/2 \\
&= \frac{\alpha}{\mu} C_3 \sinh \alpha z && -d/2 < z < d/2 \\
&= C_1 \alpha_0 e^{\alpha_0(z-d)} - \alpha_0 C_2 e^{-\alpha_0(z-d/2)} && d/2 < z
\end{aligned}$$

We now match the fields at the interfaces and solve for the amplitude constants to obtain

$$C_2 = C_1 \frac{\mu\alpha_0 \cosh(\alpha d/2) - \alpha \sinh(\alpha d/2)}{\mu\alpha_0 \cosh(\alpha d/2) + \alpha \sinh(\alpha d/2)} e^{-\alpha_0 d/2} \tag{10a}$$

$$C_3 = C_1 \frac{2\mu\alpha_0 e^{-\alpha_0 d/2}}{\mu\alpha_0 \cosh(\alpha d/2) + \alpha \sinh(\alpha d/2)} \tag{10b}$$

where $\alpha_0 = \sqrt{k_x^2 - k_0^2}$, $\alpha = \sqrt{k_x^2 - \mu^2 k_0^2}$. For the odd mode solution the electric field spectral functions are chosen as

$$\begin{aligned}
\Psi_o(\omega) &= C_1 e^{-\alpha_0(z+d)} + D_2 e^{\alpha_0(z+d/2)} && -d < z < -d/2 \\
&= -D_3 \sinh \alpha z && -d/2 < z < d/2 \\
&= -C_1 e^{\alpha_0(z-d)} - D_2 e^{-\alpha_0(z-d/2)} && d/2 < z
\end{aligned}$$

The magnetic field spectral functions are

$$\begin{aligned}
\Psi_{ho}(\omega) &= -\alpha_0 C_1 e^{-\alpha_0(z+d)} + \alpha_0 D_2 e^{\alpha_0(z+d/2)} && -d < z < -d/2 \\
&= -\frac{\alpha}{\mu} D_3 \cosh \alpha z && -d/2 < z < d/2 \\
&= -C_1 \alpha_0 e^{\alpha_0(z-d)} + \alpha_0 D_2 e^{-\alpha_0(z-d/2)} && d/2 < z
\end{aligned}$$

The solutions for the amplitude constants are

$$D_2 = C_1 \frac{\mu\alpha_0 \sinh(\alpha d/2) - \alpha \cosh(\alpha d/2)}{\mu\alpha_0 \sinh(\alpha d/2) + \alpha \cosh(\alpha d/2)} e^{-\alpha_0 d/2} \tag{11a}$$

$$D_3 = C_1 \frac{2\mu\alpha_0 e^{-\alpha_0 d/2}}{\mu\alpha_0 \sinh(\alpha d/2) + \alpha \cosh(\alpha d/2)} \tag{11b}$$

where all amplitude constants are functions of k_x and ω. The superposition of the two solutions gives the expressions for the electric field spectral functions in the three regions. However, we will leave the solutions in the form of the even and odd modes since the expressions we need to evaluate to obtain the residues at the poles are simpler. In the slab the superposition of the even and odd modes reduces to the results obtained by Pendry when the frequency is set equal to ω_e and $D_3 = -C_3$, i.e., a strict steady state solution is assumed to exist. Since we still have to carry out the inversion of the Fourier transform the final solution will be determined by the residues at the poles along with contributions from integration along the branch cuts.

The expressions for the pole locations are obtained by equating the denominators in (10) and (11) to zero. These equations are, for the even and odd modes respectively [3, 8, 9],

$$\coth \frac{\alpha d}{2} = -\frac{\alpha}{\mu \alpha_0} \tag{12a}$$

$$\tanh \frac{\alpha d}{2} = -\frac{\alpha}{\mu \alpha_0} \tag{12b}$$

An alternative form of these equations that are convenient to use for numerical calculations are

$$e^{-\alpha d} = \frac{\alpha + \mu\alpha_0}{\alpha - \mu\alpha_0} \tag{13a}$$

$$e^{-\alpha d} = \frac{\mu\alpha_0 + \alpha}{\mu\alpha_0 - \alpha} \tag{13b}$$

The first equation has a zero when $\omega = \omega_e^+ > \omega_e$ while the second equation has a zero when $\omega = \omega_e^- < \omega_e$. The negatives of these two frequencies are also zeros. It is easy to interpret the form of the equations given above. The two resonant modes correspond to a mode with an even electric field distribution about the mid-plane of the slab and a second mode with an electric field distribution that is odd about the mid-plane of the slab. Eq. (12a) is a statement of the equality of the field impedance seen when looking into the slab with an open-circuit at the mid-plane to that seen looking outward from the interface. Similarly Eq. (12b) corresponds to setting the impedance seen looking into the slab with a short-circuit at the mid-plane and equating this to the impedance looking out from the interface. This is an application of the well known transverse resonance method that is used to solve for the surface waves on many microwave structures. For each value of the transverse wave number k_x there is a pair of resonant modes with resonant frequencies that depend on k_x. For large values of k_x the two interfaces are electrically far apart so the interaction between the two modes is small and the two resonant frequencies will be close to ω_e. The electric field in the slab for the even mode is described by the function $\cosh \alpha z$ and by $-\sinh \alpha z$ for the odd mode, as illustrated in Fig. 2. Both of these modes will be excited when the input signal has a spectral width that extends from at least ω_e^- to ω_e^+. These modal solutions can be evaluated in terms of the residues at the poles.

We now superimpose the even and odd solutions to obtain the final solution with a field incident only from the object plane at $z = -d/2$.

On the input side the electric field is given by

$$E=\frac{1}{2\pi}\int_{-\infty}^{\infty}[2C_1e^{-\alpha_0(z+d)}+(C_2+D_2)e^{\alpha_0(z+d/2)}]e^{-jk_xx+j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}A(k_x,\omega)S(\omega)\left[2e^{-\alpha_0(z+d)}+\frac{\mu\alpha_0\cosh(\alpha d/2)-\alpha\sinh(\alpha d/2)}{\mu\alpha_0\cosh(\alpha d/2)+\alpha\sinh(\alpha d/2)}e^{\alpha_0 z}\right.$$

$$\left.+\frac{\mu\alpha_0\sinh(\alpha d/2)-\alpha\cosh(\alpha d/2)}{\mu\alpha_0\sinh(\alpha d/2)+\alpha\cosh(\alpha d/2)}e^{\alpha_0 z}\right]e^{-jk_xx+j\omega t}d\omega \quad -d<z<-d/2 \tag{14a}$$

which is an equation that shows that the constant $C_1=A(k_x,\omega)S(\omega)$. Within the slab the total electric field is

$$E=\frac{1}{2\pi}\int_{-\infty}^{\infty}A(k_x,\omega)S(\omega)\left[\frac{2\mu\alpha_0e^{-\alpha_0d/2}\cosh(\alpha z)}{\mu\alpha_0\cosh(\alpha d/2)+\alpha\sinh(\alpha d/2)}\right.$$

$$\left.-\frac{2\mu\alpha_0e^{-\alpha_0d/2}\sinh\alpha z}{\mu\alpha_0\sinh(\alpha d/2)+\alpha\cosh(\alpha d/2)}\right]e^{-jk_xx+j\omega t}d\omega \quad -d/2<z<d/2 \tag{14b}$$

while at the image plane at $z=d$ the solution is given by

$$E_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}A(k_x,\omega)S(\omega,k_x)\left[\frac{\mu\alpha_0\cosh(\alpha d/2)-\alpha\sinh(\alpha d/2)}{\mu\alpha_0\cosh(\alpha d/2)+\alpha\sinh(\alpha d/2)}e^{-\alpha_0 d}\right.$$

$$\left.-\frac{\mu\alpha_0\sinh(\alpha d/2)-\alpha\cosh(\alpha d/2)}{\mu\alpha_0\sinh(\alpha d/2)+\alpha\cosh(\alpha d/2)}e^{-\alpha_0 d}\right]e^{-jk_xx+j\omega t}d\omega \quad z=d \tag{14c}$$

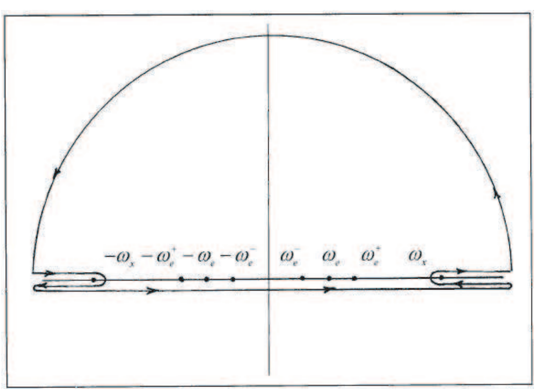

Figure 3. The modified integration contour showing the contours around the branch cuts.

The residues can be found by the usual procedure of evaluating the frequency derivative of each denominator at the corresponding zero. For the first factor the denominator vanishes when $\omega = \omega_e^+$ where $\mu\alpha_0 \cosh(\alpha d/2) = -\alpha \sinh(\alpha d/2)$. The residue associated with this factor is

$$\begin{aligned}
\text{Res}\left(\omega_e^+\right) &= \frac{-2\alpha \sinh\left(\alpha d/2\right) e^{-\alpha_0 d + j\omega t}}{\frac{\partial}{\partial\omega}\left[\mu\alpha_0 \cosh(\alpha d/2) + \alpha \sinh(\alpha d/2)\right]}\Big|_{\omega_e^+} \\
&= \frac{-2\omega_e \alpha \left(1 - e^{-\alpha d}\right) e^{-\alpha_0 d} e^{j\omega_e^+ t}}{(A + B + C + D) + e^{-\alpha d}(E + F - GH)}\Big|_{\omega_e^+} \\
&= \hat{\text{Res}}(\omega_e^+) e^{-\alpha_0(\omega_e^+)d + j\omega_e^+ t}
\end{aligned} \tag{15a}$$

Similarly the residue at the pole $\omega = \omega_e^-$ is given by

$$\begin{aligned}
\text{Res}(\omega_e^-) &= \frac{2\alpha \cosh(\alpha d/2) e^{-\alpha_0 d + j\omega t}}{\frac{\partial}{\partial\omega}\left[\mu\alpha_0 \sinh(\alpha d/2) + \alpha \cosh(\alpha d/2)\right]}\Big|_{\omega_e^-} \\
&= \frac{2\omega_e \alpha \left(1 - e^{-\alpha d}\right) e^{-\alpha_0 d} e^{j\omega_e^- t}}{(A + B + C + D) - e^{-\alpha d}(E + F - GH)}\Big|_{\omega_e^-} \\
&= \hat{\text{Res}}(\omega_e^-) e^{-\alpha_0(\omega_e^-)d + j\omega_e^- t}
\end{aligned} \tag{15b}$$

where $A = \frac{4k_e^3\alpha_0}{k_0^3}$, $B = -\frac{\mu k_0 k_e}{\alpha_0}$, $C = -\frac{4\mu k_e^3}{\alpha k_0}$, $D = -\frac{\mu^2 k_0 k_e}{\alpha}$, $E = -\frac{\mu k_0 k_e}{\alpha_0}$, $F = \frac{4\alpha_0 k_e^3}{k_0^3}$, $G = (\alpha - \mu\alpha_0)d - 1$, $H = \frac{4\mu k_e^3}{\alpha k_0} + \frac{\mu^2 k_0 k_e}{\alpha}$, $k_e = \frac{\omega_e}{c}$. We note that A, B, and C are positive constants and that the sum $A + B + C + D$ is never zero. Thus the residues at the image plane will be proportional to $e^{-\alpha_0 d}$. The form of the solution obtained from Eq. (14c) is thus (we only need to evaluate the expressions for the poles at $\omega_e^\pm$ to obtain the analytic signal representation)

$$\begin{aligned}
\tilde{E}_o = \frac{2\pi j}{2\pi}&\Big[A\left(k_x, \omega_e^+\right) S\left(\omega_e^+\right) e^{-jk_x x} \hat{\text{Res}}\left(\omega_e^+\right) e^{-\alpha_0\left(\omega_e^+\right)d} e^{j\omega_e^+ t} \\
&+ A\left(k_x, \omega_e^-\right) S\left(\omega_e^-\right) e^{-jk_x x} \hat{\text{Res}}\left(\omega_e^-\right) e^{-\alpha_0\left(\omega_e^-\right)d} e^{j\omega_e^- t}\Big] \quad z = d,\ t > 2d/c
\end{aligned} \tag{16}$$

Inside the slab the solution is given by Eq. (14b) and the contribution from the surface waves when evaluated in terms of the residues is given by

$$\begin{aligned}
\tilde{E} = j\Big[&A(k_x, \omega_e^+) S(\omega_e^+) e^{-jk_x x} \hat{\text{Res}}(\omega_e^+) e^{-\alpha_0(\omega_e^+)d/2 + j\omega_e^+ t} \frac{\cosh(\alpha z)}{\cosh(\alpha d/2)}\Big|_{\omega_e^+} \\
&+ A(k_x, \omega_e^-) S(\omega_e^-) e^{-jk_x x} \hat{\text{Res}}(\omega_e^-) e^{-\alpha_0(\omega_e^-)d/2 + j\omega_e^- t} \frac{\sinh \alpha z}{\sinh(\alpha d/2)}\Big|_{\omega_e^-}\Big] \\
&-d/2 < z < d/2,\ t > (z + d)/c
\end{aligned} \tag{17}$$

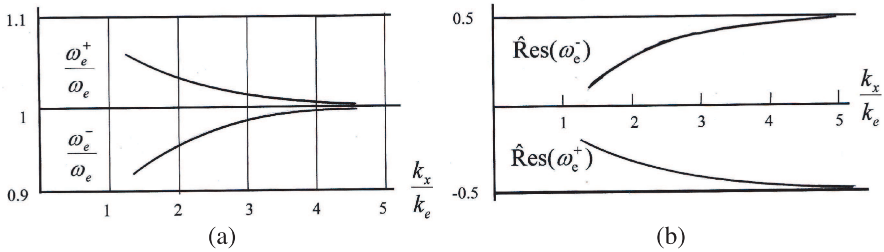

Figure 4. A plot of the resonant frequencies for the even and odd surface wave modes, and their residues, as a function of $k_x/k_e = k_x c/\omega_e$.

The residues are given by Eq. (15) and the extra factors compensate for the difference in the numerators in Eqs. (14b) and (14c). These resonant surface wave modes that are excited are proportional to $e^{-\alpha_0 d/2}$ at $z = d/2$ and do not exhibit exponential growth. In Fig. 4 we show the resonant frequencies for the even and odd surface wave modes and their residues as a function of k_x/k_e. Note that the residues remain bounded as k_x becomes large. The above solutions are also valid for $k_x < k_0$ with α_0 and α replaced by $j\beta_0 = \sqrt{k_0^2 - k_x^2}$ and $j\beta = \sqrt{\mu\varepsilon k_0^2 - k_x^2}$.

Let us, at this point, assume that the field in the aperture plane consists of a sinusoidal oscillation $\sin\omega_e t$ at the frequency ω_e, which is turned on at $t = 0$ and turned of at $t = \tau$. For this signal the spectral function is given by Eq. (7) and is repeated below:

$$S(\omega)=\frac{e^{j(\omega_e-\omega)\tau}-1}{2(\omega-\omega_e)}-\frac{e^{-j(\omega+\omega_e)\tau}-1}{2(\omega+\omega_e)}$$
$$=je^{j(\omega_e-\omega)\tau/2}\frac{\tau}{2}\frac{\sin(\omega-\omega_e)\tau/2}{(\omega-\omega_e)\tau/2}-je^{-j(\omega+\omega_e)\tau/2}\frac{\tau}{2}\frac{\sin(\omega+\omega_e)\tau/2}{(\omega+\omega_e)\tau/2} \quad (18)$$

For notational convenience we will let the function in the integrand in Eq. (14c) be denoted by $F(\alpha_0, \omega)$ where

$$F(\alpha_0,\omega) = \left[\frac{\mu\alpha_0\cosh(\alpha d/2)-\alpha\sinh(\alpha d/2)}{\mu\alpha_0\cosh(\alpha d/2)+\alpha\sinh(\alpha d/2)} - \frac{\mu\alpha_0\sinh(\alpha d/2)-\alpha\cosh(\alpha d/2)}{\mu\alpha_0\sinh(\alpha d/2)+\alpha\cosh(\alpha d/2)}\right]e^{-\alpha_0 d}$$
$$= \frac{8\mu\alpha_0\alpha e^{-\alpha_0 d-\alpha d}}{(\mu\alpha_0+\alpha)^2-(\mu\alpha_0-\alpha)^2 e^{-2\alpha d}} \quad (19)$$

which was obtained by combining the even and odd mode solutions in

Eq. (14c). At the frequencies $\omega = \pm\omega_e$ where the relative values of epsilon and mu are equal to -1 this function equals 2. There are no poles at $\pm\omega_e$. However, the function $F(\alpha_0, \omega)$ has poles at $\pm\omega_e^{\pm}$ which are the resonant frequencies of the even and odd surface wave modes.

The inversion contour for the Fourier transform runs parallel to the real ω axis and just below the poles at $\pm\omega_e^+$, $\pm\omega_e^-$, and $\pm\omega_e$. From a consideration of the wave function $e^{j\omega t - j\omega\tau - j2\sqrt{k_0^2 - k_x^2}d}$ we see that for large values of ω that for $t < \tau + 2d/c$ the wave function will become small on the semi-circle at infinity in the lower half of the complex ω plane. Thus we can close the inversion contour in the lower half of the complex plane and since no singularities are enclosed the value of the integral will be zero. For the part of the spectral density function in Eq. (18) that does not depend on τ the inversion contour can also be closed in the lower half of the complex ω plane when $t < 2d/c$ and will not give any contribution to the field. These conditions are simply the requirements of causality. Note that as ω becomes very large the propagation factor $\sqrt{\mu\varepsilon k_0^2 - k_x^2}$ in the slab becomes the same as that in free space because epsilon and mu approach the free space values, and hence there is no problem with time running backwards as far as imposing the causality condition is concerned. The spectral density function of the input signal that depends on τ will give the output field at the image plane after the sinusoidal signal has been turned off. This signal would be of less interest since it is unlikely that measurements of the field at the image plane would be made after the illumination of the object has been turned off.

The evaluation of the fields in terms of the residues require that the Fourier inversion integral be taken over a closed contour enclosing the poles and that the integrand be single valued within the contour. The expression in Eq. (14) is an even function of α and hence has branch points associated only with α_0. Suitable branch cuts are the lines joining the two branch points, corresponding to the zeroes $\omega = \pm k_x c = \pm\omega_x$ of α_0, to plus and minus infinity as shown in Fig. 3. The original inversion contour runs parallel to the real axis, from minus infinity to plus infinity, but below the poles at $\pm\omega_e^{\pm}$ and $\pm\omega_e$. This contour is closed by the contour shown in Fig. 3, which includes a contour running around the branch cut from $\omega_x = -k_x c$ to $-\infty$ and from $k_x c$ to ∞, and closed by a semi-circle contour at infinity in the upper half of the complex $\omega + j\sigma$ plane. There is no contribution to the integrals from the semi-circle contour. The poles are enclosed within the contour. The value of the integral in Eq. (14) thus consists of the terms corresponding to the residues at the poles plus integrals around the branch cuts but traversed in the opposite direction. In the absence of loss the poles lie on the real ω axis. On the bottom side of the branch cut in the left half plane $\alpha_0 = j\sqrt{k_x^2 - k_0^2}$ but on the top side of this branch cut α_0 changes sign. On the branch cut in the right half plane $\alpha_0 = j\sqrt{k_x^2 - k_0^2}$ on the top side of the cut and equals the negative of this on the lower side of the cut. The integration along the top and bottom sides of the branch cuts can be combined and is given by

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{-\infty}^{-\omega_x} A(k_x)\left[F\left(j\sqrt{k_0^2-k_x^2},\omega\right) - F\left(-j\sqrt{k_o^2-k_x^2},\omega\right)\right] \\ &\quad \left[\frac{1}{2(\omega+\omega_e)} - \frac{1}{2(\omega-\omega_e)}\right] e^{-jk_x x + j\omega t} d\omega \\ &\quad + \frac{1}{2\pi}\int_{\omega_x}^{\infty} A(k_x)\left[F\left(-j\sqrt{k_0^2-k_x^2},\omega\right) - F\left(j\sqrt{k_o^2-k_x^2},\omega\right)\right] \\ &\quad \left[\frac{1}{2(\omega+\omega_e)} - \frac{1}{2(\omega-\omega_e)}\right] e^{-jk_x x + j\omega t} d\omega \end{aligned} \tag{20}$$

For the propagating waves the solution at the image plane consists of the residue wave from the poles at $\pm\omega_e$ with a frequency of ω_e, plus the field from the branch cut integrals, which is given by Eq. (20). The desired wave is the pole wave at the frequencies $\pm\omega_e$ which is given by the residues at $\pm\omega_e$ (note that $F(j\sqrt{k_0^2 - k_x^2}, \omega_e) = 2$),

$$E_{0p} = jA(k_x)e^{-jk_x x + j\omega_e t} + CC \quad 2d/c < t < \tau + 2d/c \tag{21a}$$

where CC is the complex conjugate term. The contribution from the branch cut integrals is

$$\begin{aligned} &-\frac{1}{2\pi}\int_{\omega_x}^{\infty} A(k_x)\left[\frac{4\mu\beta_0\beta e^{-j(\beta_0+\beta)d}}{(\mu\beta_0+\beta)^2 - (\mu\beta_0-\beta)^2 e^{-j2\beta d}}\right. \\ &\left. + \frac{4\mu\beta_0\beta e^{-j(\beta-\beta_0)d}}{(\mu\beta_0-\beta)^2 - (\mu\beta_0+\beta)^2 e^{-j2\beta d}}\right] \frac{\omega_e e^{-jk_x x + j\omega t}}{\omega^2 - \omega_e^2} d\omega + CC \\ &\qquad 2d/c < t < \tau + 2d/c \end{aligned} \tag{21b}$$

where $\beta_0 = \sqrt{k_0^2 - k_x^2}$ and $\beta = \sqrt{\mu\varepsilon k_0^2 - k_x^2}$. For large values of k_0 and for $\varepsilon = \mu$ the integrand in the above expression becomes

$$\begin{aligned} &A(k_x)\left\{\frac{-8j\sqrt{\varepsilon\mu}\sin[(\sqrt{\varepsilon\mu}+1)k_0 d]}{(\sqrt{\mu}+\sqrt{\varepsilon})^2}\right\}\frac{\omega_e}{\omega^2}e^{-jk_x x + j\omega t} \\ &= -2jA(k_x)\sin\left[2\left(1-\omega_e^2/\omega^2\right)k_0 d\right]\frac{\omega_e}{\omega^2}e^{-jk_x x + j\omega t} \end{aligned}$$

When the complex conjugate of this expression is added the result shows that the contribution from the branch cut integrals vanish for large values of k_0.

The waves from the branch cut integrals have a continuous frequency spectrum which will cause some blurring of the image at the image plane because the transmission coefficient for these propagating waves depend on the frequency. Only the pole wave at "$\omega = \omega_e$" is transmitted with an overall transmission coefficient equal to one. If the frequency ω_e is greater than ω_x then the point $\omega = \omega_e$ lies on the branch cut but the rest of the integrand vanishes for $\omega = \omega_e$ so there is no pole on the branch cut. There are no surface wave poles associated with the propagating waves.

For the evanescent waves the solution consists solely of the pole waves. Thus for $2d/c < t < \tau + 2d/c$ the sum of the dominant wave given by Eq. (21a) and the excited surface waves are given by

$$E_0 = \frac{j}{2}A\left(k_x\right)e^{-jk_xx}\left[2e^{j\omega_e t} + \frac{\omega_e e^{-\alpha_0\left(\omega_e^+\right)d}}{\omega_e^+ - \omega_e}\hat{\mathrm{R}}\mathrm{es}\left(\omega_e^+\right)e^{j\omega_e^+ t} - \frac{\omega_e e^{-\alpha_0\left(\omega_e^-\right)d}}{\omega_e - \omega_e^-}\hat{\mathrm{R}}\mathrm{es}\left(\omega_e^-\right)e^{j\omega_e^- t}\right] + CC \tag{22}$$

where CC is the complex conjugate terms. The value of the residues are given by Eqs. (15a) and (15b). The first wave at $\pm\omega_e$ corresponds to Pendry's solution for the loss free lens. The other two terms are the resonant surface wave modes at $\pm\omega_e^{\pm}$ whose excitation cannot be avoided when frequency dispersion is included for epsilon and mu. These resonant modes will cause interference with the desired mode with frequency ω_e and thus will make it very difficult to obtain a coherent reconstruction of the evanescent wave amplitudes since the resonant frequencies of the surface wave modes vary with the transverse wave number. One can anticipate that some loss will be present in the slab and this will limit the life time of the resonant surface wave modes (surface plasmons) so the interference will die out. But as noted by a number of investigators the presence of loss will limit the sub-wavelength resolution of the lens. With either scenario the performance of the lens is reduced. In either case the frequency dispersion or loss, or a combination of both, will avoid any field divergence for large values of the transverse wave number. It is also important to keep in mind that for large values of k_x the surface wave resonant frequencies are very close to ω_e so the frequency of the incident field must be carefully controlled. If the frequency of the incident field should shift to either ω_e^+ or ω_e^- this will create a double pole with a time response proportional to $te^{j\omega_0 t}e^{-\alpha_0(\omega_0)d}$ where ω_0 equals ω_e^+ or ω_e^- instead of the desired dominant wave at ω_e.

It can be shown that for k_x greater than 5 the resonant surface wave frequencies are essentially equal, and furthermore to a high degree of accuracy [9],

$$\frac{\omega_e}{\omega_e^+ - \omega_e} e^{-\alpha_0 d} \approx \frac{\omega_e}{\omega_e - \omega_e^-} e^{-\alpha_0 d} \approx 2 \tag{23}$$

Also as reference to Eqs. (15a) and (15b) shows the residues are approximately equal to $-0.5\omega_e e^{-\alpha_0 d}$ and $0.5\omega_e e^{-\alpha_0 d}$ because A is the dominant coefficient in the denominator and very nearly equal to $4k_x$. Thus we can express the solution in the form

$$E_0 = jA(k_x) e^{-jk_x x} \left[e^{j\omega_e t} - e^{j\left(\omega_e^+ + \omega_e^-\right)t/2} \cos \Delta\omega t \right] + CC$$
$$2d/c < t < \tau + 2d/c \tag{24}$$

where $\Delta\omega = \frac{\omega_e^+ - \omega_e^-}{2}$. We can also make the approximation $(\omega_e^+ + \omega_e^-)/2 \approx \omega_e$. This is essentially the result obtained by Goméz-Santos by using a model of two coupled mechanical resonators [9]. For $\Delta\omega t$ small the expansion of the cosine function gives

$$E_0 \approx \frac{j}{2} A(k_x) e^{-jk_x x} (\Delta\omega t)^2 e^{j\omega_e t} = \frac{j}{2} A(k_x) e^{-jk_x x + j\omega_e t} (\omega_e t)^2 e^{-2\alpha_0 d} \tag{25}$$

which shows that it takes a considerable length of time for the field at the image plane to build up to its steady state value when the decaying exponential factor is very small. For example, if $d = \lambda_0/4$ and $k_x = 20k_0$, then

$$t = \frac{1}{\Delta\omega} = \frac{1}{\omega_e} e^{20k_0\lambda_0/4} = 4.4 \times 10^{13} \frac{1}{\omega_e}$$

which for a frequency of 10 GHz. gives t equal to 11.7 minutes. The result shown in Eq. (25) can also be obtained from a different approach. When k_x approaches infinity $\Delta\omega$ approaches zero and ω_e^+ and ω_e^- coalesce to produce a triple pole given by

$$\frac{1}{4} e^{-\alpha_0 d} \left[\frac{\omega_e}{\omega - \omega_e^+} - \frac{\omega_e}{\omega - \omega_e^-} \right] \frac{1}{\omega - \omega_e} \rightarrow \frac{\omega_e}{4} e^{-\alpha_0 d} \frac{\omega_e^+ - \omega_e^-}{(\omega - \omega_e)^3} \rightarrow \frac{\omega_e^2}{4} e^{-2\alpha_0 d} \frac{1}{(\omega - \omega_e)^3}$$

which has the time response $(j/8)(\omega_e t)^2 e^{-2\alpha_0 d} e^{j\omega_e t}$. Since the signal is turned off at a finite time τ the field at the image plane vanishes as k_x approaches infinity. For finite values of k_x and $\Delta\omega t$ that is large the field described by Pendry's solution is slowly modulated by the cosine factor. Since the resonant frequency of the surface wave modes depend on k_x the field at the image plane is not coherent in frequency and thus it would be virtually impossible to achieve a coherent reconstruction of the evanescent wave amplitudes at the image plane. If features as small as one tenth of a wavelength was to be observed values of k_x up to about $10k_e \approx 10k_0$ would have to be retained. The interference from the excited resonant surface waves would make it unlikely that any useful coherent reconstruction of the evanescent waves with these values of k_x could be achieved.

After the signal has been turned off the spectral function $S(\omega)$ that must be used is

$$S(\omega) = je^{j(\omega_e - \omega)\tau/2} \frac{\tau}{2} \frac{\sin(\omega - \omega_e)\tau/2}{(\omega - \omega_e)\tau/2} - je^{-j(\omega + \omega_e)\tau/2} \frac{\tau}{2} \frac{\sin(\omega + \omega_e)\tau/2}{(\omega + \omega_e)\tau/2}$$

For this spectrum there are no poles at $\pm\omega_e$. The rapid oscillations of the spectral function for large values of $|\omega - \omega_e|$ and $|\omega + \omega_e|$ will ensure that the branch cut integrals are small.

Various authors have shown that losses in the negative index slab would also reduce the resolution capability of the lens even if there was no frequency dispersion in epsilon and mu. The excited resonant surface wave modes will decay to zero because of losses which will be present, even though we did not include losses in the above analysis which focused on the limitations of the loss free Veselago lens because of frequency dispersion in epsilon and mu.

When the losses in the lens material are small the new surface wave eigenvalues can be found using a perturbation method based on the Newton-Raphson method. Consider the eigenvalue Equation (13a) and let

$$f(\omega) = \mu\alpha_0 - \alpha + (\mu\alpha_0 + \alpha)e^{\alpha d}$$

We now assume that the loss in the material is the same for epsilon and mu and thus let

$$\varepsilon = 1 - \frac{2\omega_e^2}{\omega(\omega + j\gamma)} \approx 1 - \frac{2\omega_e^2}{\omega^2} + \frac{j2\gamma\omega_e^2}{\omega^3} = \mu$$

where γ is the loss parameter and is considered to be very small relative to ω_e. Since the root for this equation is very close to ω_e the first approximation to the root when loss is included is given by

$$f(\omega) = f(\omega_e) + \frac{\partial f}{\partial \omega}\Big|_{\omega_e} (\omega - \omega_e) = 0 \tag{26a}$$

which gives

$$\omega = \omega_e - \frac{f(\omega_e)}{\partial f(\omega)/\partial \omega\,|_{\omega_e}} \tag{26b}$$

This expression can be evaluated and when only the first order terms in γ are retained and the transverse wave number k_x is assumed to be large it is found that $\omega = \omega_e^+ + j\frac{\gamma}{2}$, and similarly $\omega = \omega_e^- + j\frac{\gamma}{2}$, for the eigenvalues of the even and odd surface wave modes when a small loss is included. This result is the same as what Grbic found [10]. When these values for the eigenvalues are used in Eq. (19) it is found that for large k_x that

$$F(\alpha_0, \omega_e) \approx \frac{2}{1 + (\gamma/\omega_e)^2 e^{2k_x d}} \tag{27}$$

which is now the residue for the dominant wave. The solution for the excited surface waves will now be

$$\begin{aligned} E_0 \;=\; & \frac{j}{2} A(k_x) e^{-jk_x x} \left[\frac{\omega_e e^{-\alpha_0(\omega_e^+)d}}{\omega_e^+ - \omega_e + j\gamma/2} \hat{\mathrm{R}}\mathrm{es}(\omega_e^+) e^{j\omega_e^+ t - \gamma t} \right. \\ & \left. - \frac{\omega_e e^{-\alpha_0(\omega_e^-)d}}{\omega_e - \omega_e^- - j\gamma/2} \hat{\mathrm{R}}\mathrm{es}(\omega_e^-) e^{j\omega_e^- t - \gamma t} \right] \end{aligned}$$

We now make use of Eq. (23) to obtain

$$\frac{\omega_e}{(\omega_e^+ - \omega_e)[1 + j\gamma/2(\omega_e^+ - \omega_e)]} = \frac{2e^{\alpha_0 d}}{1 + j\omega_e/[(\omega_e^+ - \omega_e)Q]} = \frac{2e^{\alpha_0 d}}{1 + je^{\alpha_0 d}/Q}$$

where Q is the quality factor ω_e/γ for the surface wave resonator. A similar expression will hold for the odd surface wave mode. In place of Eq. (22) the solution for the evanescent waves for large values of k_x is now given by

$$\begin{aligned} E_o \approx\, & jA(k_x, \omega_e) e^{-jk_x x} \\ & \left[\frac{e^{j\omega_e t}}{1 + e^{2\alpha_0 d}/Q^2} - \frac{1}{2[1 + je^{\alpha_0 d}/Q]} e^{-\gamma t/2} \left(e^{j\omega_e^+ t} + e^{j\omega_e^- t} \right) \right] + CC \end{aligned} \tag{28}$$

for $2d/c < t < \tau + 2d/c$.

For the propagating waves where $k_x < k_0$ the result given in Eq. (21a) should be replaced by

$$jA(k_x, \omega_e) e^{-jk_x x} \left[\frac{(1 - j/Q) e^{j\omega_e t}}{1 + e^{j2\beta d} k_x^2/(\beta_0^2 Q^2)} \right] + CC \quad 2d/c < t < \tau + 2d/c \tag{29}$$

which is valid for small losses and $|k_x| < k_0$. This term, together with the branch cut integrals, when integrated over $|k_x| < k_0$, gives the total image field arising from the propagating waves at the image plane. As noted earlier, the propagating wave spectrum given by the branch cut integrals produce some blurring of the image, an artifact that was not present in Pendry's ideal lens solution. Eq. (29) shows that losses will also produce some blurring of the image.

The loss reduces the amplitudes of the residues by a substantial amount. In addition it can be seen that the excited surface wave modes will decay quite rapidly. For example, if the quality factor or Q of the surface wave mode resonances equals 10^4 then the surface wave modes become negligible in less than a microsecond if the frequency is equal to 10 GHz. Hence interference from the excited surface wave modes is not likely to be a serious factor in reducing the performance of a lossy Veselago lens. However, the factor multiplying the dominant pole wave will become small whenever the factor $(\gamma/\omega_e)e^{\alpha_0 d} = Q^{-1}e^{\alpha_0 d} \approx Q^{-1}e^{k_x d} > 1$. For the above example this occurs for $k_x > 9.2/d$. If $d = \lambda_0/4$ this corresponds to $\lambda_x > 0.17\lambda_0$. The amplitude of features smaller than this will be reduced by a factor of more than 1/2 at the image plane. This reduction in the amplitudes of the evanescent waves increases exponentially with k_x. Merlin showed that a small perturbation σ in the relative value of ε reduced the resolution of the lens in accordance with a formula like that in Eq. (28) with $j2\gamma/\omega_e$ replacing σ [4]. Thus if the frequency of the incident field drifts away from the value ω_e this will be equivalent to a change in the relative values of epsilon and mu from -1 and can produce a significant reduction in the resolution of the lens. Hence in practice, if a Veselago lens could be constructed, it will be the losses and the frequency stability of the source that illuminates the object that will limit the resolution, not the interference from the excited surface wave modes.

We can now understand what happens when a steady state sinusoidal incident field at the frequency ω_e is assumed. For this case the frequency spectral function can be represented by a delta function, i.e., $S(\omega) = 2\pi\delta(\omega - \omega_e)$ and thus the inverse Fourier transform results in the field solutions being evaluated at the frequency ω_e where $\varepsilon = \mu = -1$, which gives Pendry's solution. When frequency dispersion is neglected then the only pole is a double pole that occurs when k_x becomes infinite. Although Pendry's solution has some of the characteristics of a resonant mode its resonant frequency is not clearly defined except perhaps through the condition that the relative values of epsilon and mu must equal -1. This is the cause for the divergent behavior of Pendry's solution since it corresponds to a steady state sinusoidal signal being applied to a resonant system at its resonant frequency. When frequency dispersion is included and an incident field with finite frequency spectral density is assumed then the response is obtained in terms of the residues at the surface wave poles and this response is finite even when k_x becomes infinite. For the case of a sinusoidal signal turned on at $t = 0$ and later turned off the frequency spectrum contains a pole term at the frequency of the sinusoidal signal instead of a delta function spectral term. The use of a sinusoidal signal of finite duration reveals much richer physical phenomena associated with Veselago's lens that is completely missed in a steady state sinusoidal solution.

4. CONCLUSION

An analysis of transmission through a flat slab lens was carried out. For the first case considered it was assumed that the input field had a continuous frequency spectrum with finite density, and that both epsilon and mu exhibited frequency dispersion. The field excited in the lens was expressed in terms of the even and odd resonant surface wave modes whose amplitudes were evaluated in terms of the residues at the poles. For this case it was found that there were no exponentially growing evanescent waves in the slab. However, when the incident field was chosen as a sinusoidal signal with finite duration a dominant wave at the frequency ω_e, at which the relative values of epsilon and mu where equal to -1, was also excited but due to interference from the excited surface wave modes a coherent reconstruction of the evanescent wave amplitudes was not possible. As a consequence of this result a lossless Veselago flat lens with super resolution is not physically possible. When small loss is included in the material parameters the excited surface wave modes decay away in a very short period of time and their interference effects become negligible. The resolution of the Veselago lens in now limited by the loss, a result previously established by a number of investigators, and/or a signal frequency that deviates from that for which the relative values of epsilon and mu are exactly equal to -1. A new result that others had not found was the existence of a continuous spectrum of propagating waves that arise from branch cut integrals and which will blur the image of these waves at the image plane. It was also concluded that Pendry's solution for a lossless lens was not a continuous function of the physical parameters and hence did not constitute a proper physical solution. The analysis presented in this paper is a classical one and gives a solution that satisfies the required conditions for a proper physical solution.

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