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THE PROPAGATION PROBLEM IN A BI-ISOTROPIC WAVEGUIDE

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Abstract—We investigate the problem of defining propagating constants and modes in metallic waveguides of an arbitrary cross section, filled with a homogeneous bi-isotropic material. The approach follows the guidelines of the classical theory for the isotropic, homogeneous, lossless waveguide: starting with the Maxwell system, we formulate a spectral problem where the square of the propagation constant shows up as the eigenvalue and the corresponding mode as the eigenvector. The difficulty that arises, and this is a feature of chirality, is that the eigenvalue is involved in the boundary conditions. The main result is that the problem is solvable whenever the Dirichlet problem for the Helmholtz equation in the cross section is solvable and two technical hypotheses are fulfilled. Our method, inspired by the null-field method, is quite general and has a good potential to work in various geometries.

1. INTRODUCTION

Wave propagation in bi-isotropic materials is well documented in the literature. The monographs [1] and [2] develop the relevant theory in a systematic way, and they also give an extensive list of references of the accomplishments in the field. As a branch of mathematical physics, bi-isotropic materials have also been studied quite well, see e.g., [3] and the references therein. An investigation of the transmission line problem with bi-isotropic material is given by Olyslager [4].

On the other hand, the theory of a guided wave seems to go back in the history to the years of Lord Rayleigh [5], and a systematic account of it can be found in [6]. We also refer to [7] for some early discussions about the nomenclature and the problem of the mode ordering and

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classification in an arbitrary waveguide. The interested reader should also look up the articles [8–10] for some recent developments concerning waveguide theory and applications.

The electromagnetic propagation in a waveguide is principally characterized by three items:

geometry: the shape of the cross section of the cylinder or waveguide

walls: the properties of the medium from which the cylinder is manufactured

material: the properties of the medium enclosed by the walls of the cylinder

In this paper, we adopt a waveguide with perfectly conducting walls and an arbitrary cross section that encloses a homogeneous bi-isotropic medium. From the mathematical point of view, geometry describes the domain, walls provide the boundary conditions and the material defines the appropriate differential equation.

The circular chiral waveguide, referred to as "the circular chirowaveguide", drew some attention in the 90's. In fact, the topic was already discussed a few years earlier [11]. In [12], a complete solution was provided and a dispersion relation was obtained. The numerical solution of this equation gives the propagation constants. A theory concerning the chirowaveguide with an arbitrary cross section was presented in [13]. For the bi-isotropic waveguide a similar treatment was made in [2]. However, the eigenvalue problem, and the operator it concerns, is not very clear in neither of these two references. Early numerical treatments are published by Svedin [14], who implements a FEM method for the propagation problem in a chirowaveguide.

Nevertheless, we are not aware of a strict mathematical treatment of the propagation problem in a general bi-isotropic waveguide. Thereby, the main aim of this article is to close this gap and we wish to formulate the eigenvalue problem for the metallic homogeneous bi-isotropic waveguide and discuss its solvability, providing a solid mathematical foundation for future reference. The approach follows the guidelines of the classical theory for the metallic isotropic, homogeneous, lossless waveguide, as it is described in [15, Ch. 4], [16, Ch. IX.4], [17, Ch. 8]. Actually, we give two possible forms for the eigenvalue problem; roughly speaking one with a complicated operator and the other with complicated boundary conditions. This reveals clearly that the existence of chirality always leads to a problem of a non-standard form.

Taking these complicated boundary conditions as the starting point, we reformulate the problem as a functional differential or integral equation, which potentially involves the Dirichlet-to-Neumann

operator for the 2D Helmholtz equation in the cross section. The (implicit) dispersion equation can be realized as the condition for the functional equation to admit non-trivial solutions. The solutions of the dispersion equation, which is expected to consist of a complex sequence with no accumulation points, provide the propagation constants. Going back to the functional equation, its corresponding solutions serve as boundary data for the Dirichlet problem of the 2D Helmholtz equation. Thus, if we are able to solve the relevant Dirichlet problem, we can construct the modes of the waveguide.

We propose a fairly general method for solving the aforementioned problem; our approach is inspired by the null-field method [18, 19]. The idea is to expand the boundary data in appropriate bases in the space of square integrable functions. There is a freedom on how to choose this basis but the mathematical details for its existence are quite complicated. The null-field method answers partially this problem: a fixed sequence is chosen but the basis property is not always fulfilled.

As an intermediate result, we show that there exists a natural splitting of the field in left and right circularly polarized components. This fact confirms the Bohren transformation, proposed already in 1974 [20]. Actually, the Bohren transformation has been used extensively in the study of chiral media described by the Drude–Born–Fedorov constitutive relations, see [21] and references therein. We do not a priori assume this splitting, but we confirm that this is an essential step for the solution.

Another special feature of our approach is that we do not make any assumptions for the material matrix apart from a reasonable invertibility condition. This means that the electromagnetic properties of the material in question can have a very wide frequency behavior. Moreover, we provide the proper mathematical framework to treat the inverse problem, which is extremely important in applications. In the inverse problem we infer information about the electromagnetic properties of the material by measurements in the waveguide. See the discussion in [22] and the more general investigation presented in [23].

The paper is essentially based on the technical report [24] and is organized as follows: in Section 2, the notation and the basic equations are introduced. The waveguide geometry and the perfect conductor boundary condition are discussed in Section 3. In Section 4, we formulate the spectral problem in an operator pencil form. Section 5 discusses a transformation to a more standard form and Section 6 develops the relevant theory. A general method for solving the problem is exposed in Section 7. This method is applied in the well known case of circular geometry in Section 8, and a numerical illustration is given in Section 9.

2. PREREQUISITES

In this section, we formally determine the equation that is used in this paper. The starting point is the time-harmonic Maxwell equations without sources

$$\begin{cases} \mathrm{i}\omega\mathbf{D} = -\nabla\times\mathbf{H} \\ \mathrm{i}\omega\mathbf{B} = \nabla\times\mathbf{E} \end{cases} \tag{1}$$

We have here assumed time harmonic waves with angular frequency ω (the time convention is taken to be $\mathrm{e}^{-\mathrm{i}\omega t}$). The bi-isotropic material is modeled by the constitutive relations

$$\begin{cases} \mathbf{D} = \varepsilon\mathbf{E} + \xi\mathbf{H} \\ \mathbf{B} = \zeta\mathbf{E} + \mu\mathbf{H} \end{cases} \tag{2}$$

This is the Lindell et al. model [2], which is widely used as a model for bi-isotropic materials. Other, equivalent formulations are also frequently used, see in the same reference. The parameters ε, ξ, ζ, μ are complex numbers, which, in general, are dependent on the spatial variables and the frequency ω. We pose now two postulates concerning these parameters.

Spatial homogeneity ε, ξ, ζ, μ are independent of the spatial variables $\mathbf{r} := x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \in \Omega$. Ω is an open set in the 3-space.

Invertibility We define the material matrix as the 2×2 matrix

$$M := \begin{pmatrix} \varepsilon & \xi \\ \zeta & \mu \end{pmatrix}$$

and for its determinant we assume $d := \varepsilon\mu - \zeta\xi \neq 0$.

The Maxwell system (1) and the invertibility assumption imply the Gauss equations

$$\nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{H} = 0 \tag{3}$$

We substitute (2) into (1) to obtain

$$\begin{cases} \mathrm{i}\omega(\varepsilon\mathbf{E} + \xi\mathbf{H}) = -\nabla\times\mathbf{H} \\ \mathrm{i}\omega(\zeta\mathbf{E} + \mu\mathbf{H}) = \nabla\times\mathbf{E} \end{cases} \tag{4}$$

We choose to work with the electric field here; then the knowledge of $\mathbf{E}$ implies the knowledge of $\mathbf{H}$ also, via the equation

$$\mathbf{H} = \frac{1}{\mathrm{i}\omega\mu}\left(-\mathrm{i}\omega\zeta\mathbf{E} + \nabla\times\mathbf{E}\right) \tag{5}$$

In other words, the time harmonic Maxwell system can be reduced to an equation involving only three components. After some straightforward calculations, we find that $\mathbf{E}$ satisfies the vectorial PDE

$$\nabla\times\nabla\times\mathbf{E} - 2\omega\chi\nabla\times\mathbf{E} - \omega^2 d\mathbf{E} = 0 \tag{6}$$

where the parameter χ is defined by

$$\chi := \mathrm{i}\frac{\zeta - \xi}{2}$$

This is in general a complex number and it serves as a measure of the chirality of the material. The Equation (6) models the propagation of the electric field inside an arbitrary domain Ω which is occupied by a bi-isotropic material. Observe that the material parameters enter only via the chirality χ and the determinant d.

At this moment, we would like to make a comment about a notational convenience. We formally consider the *curl* operator as an antisymmetric matrix with entries partial derivatives, which applies in 3-fields:

$$\nabla\times = \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix}$$

More precisely, *curl* can be seen as a block matrix as follows

$$\nabla\times = \begin{bmatrix} \begin{pmatrix} 0 & -\partial_z \\ \partial_z & 0 \end{pmatrix} & \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} \\ \begin{pmatrix} -\partial_y & \partial_x \end{pmatrix} & 0 \end{bmatrix} \tag{7}$$

Visualization (7) is particularly useful when we split the fields in a transverse and a longitudinal part, that is if we write a field

$$\mathbf{A} := \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

in the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_\perp \\ A_z \end{pmatrix}$$

where, of course,

$$\mathbf{A}_\perp = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

3. WAVEGUIDES AND BOUNDARY CONDITION

We now turn our attention to cylindrical domains which model waveguides. Especially, we consider an infinite waveguide described in the space by

$$\Omega := \Omega_\perp \times \mathbf{R}$$

where $\Omega_\perp$ is a bounded, open, connected domain in the transverse plane with a sufficiently smooth boundary $\Gamma_\perp := \partial\Omega_\perp$, see Figure 1. We refer to $\Omega_\perp$ as the cross section of the waveguide. The axis $\mathbf{R}$ of the waveguide is often referred to as the longitudinal line. The wall of the waveguide is the boundary $\Gamma := \Gamma_\perp \times \mathbf{R}$. We denote by $\hat{\boldsymbol{\nu}} = \nu_x\hat{\mathbf{x}} + \nu_y\hat{\mathbf{y}} \in \mathbf{R}^2$ the outward normal to the curve $\Gamma_\perp$ in the transverse plane. Then, as a vector in $\mathbf{R}^3$, $\hat{\mathbf{n}} = \nu_x\hat{\mathbf{x}} + \nu_y\hat{\mathbf{y}}$ is the outward normal to the wall Γ. We make the following assumption:

PEC The wall Γ is a perfect conductor.

This is applicable, for instance, if the wall is metallic. The PEC wall is modeled by the boundary condition

$$\hat{\mathbf{n}} \times \mathbf{E} = 0 \text{ on } \Gamma \tag{8}$$

A simple calculation gives

$$\hat{\mathbf{n}} \times \mathbf{E} = \nu_y E_z\hat{\mathbf{x}} - \nu_x E_z\hat{\mathbf{y}} + (\nu_x E_y - \nu_y E_x)\hat{z} \tag{9}$$

Since ν_x, ν_y cannot vanish simultaneously, we have

$$E_z = 0 \text{ on } \Gamma \tag{10}$$

Consider, in the sequel, the transverse field

$$\mathbf{E}_\perp := E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}$$

and the 2×2 matrix

$$V := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Seen as an operator in $\mathbf{R}^2$, V acts as a clockwise $\pi/2$ rotation. Furthermore, V is antisymmetric and $V^2 = -I_2$ (the 2×2 identity matrix). Actually, V represents the Levi-Civita antisymmetric tensor ε_{ijk} with $k = 3$. Now $\hat{\boldsymbol{\tau}} = -V\hat{\boldsymbol{\nu}}$ is a tangential unit vector on $\Gamma_\perp$ and from (9) we deduce that

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{E}_\perp = 0 \text{ on } \Gamma \tag{11}$$

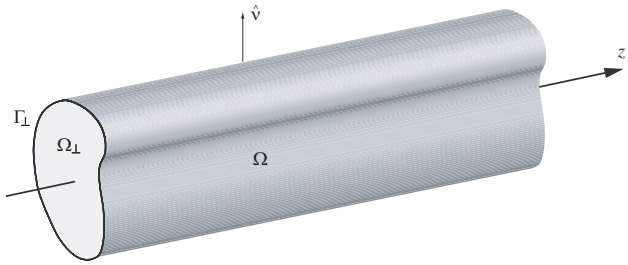

Figure 1. The waveguide geometry of the problem.

4. THE SPECTRAL PROBLEM

We now search for special solutions of (6) of the form

$$\mathbf{E}(\mathbf{r}) = \mathrm{e}^{\gamma z}\mathbf{u}(\boldsymbol{\rho}), \quad z \in \mathbf{R}, \quad \boldsymbol{\rho} \in \Omega_\perp \tag{12}$$

Here, γ denotes a complex constant and $\mathbf{u}$ a complex vector field dependent on the transverse variables $\boldsymbol{\rho} := x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \in \Omega_\perp$. We are now in a position to pose the propagation problem in a metallic bi-isotropic waveguide. It is often convenient to discriminate $\mathbf{u}$ in a transverse and a longitudinal part as follows:

$$\mathbf{u} := \begin{pmatrix} \mathbf{u}_\perp \\ \mathbf{u}_z \end{pmatrix}$$

The *curl* operator, for example, restricted to the subspace of functions of the form (12), is formally given by the matrix

$$C_\gamma := \begin{bmatrix} -\gamma V & V\nabla_\perp \\ \nabla_\perp \cdot V & 0 \end{bmatrix}$$

See (7) and note that partial derivation with respect to z becomes a multiplication with γ. The dot denotes the usual inner product. The transverse gradient is formally defined by

$$\nabla_\perp := \partial_x\hat{\mathbf{x}} + \partial_y\hat{\mathbf{y}}$$

The Gauss equation for $\mathbf{E}$ reads

$$\nabla_\perp \cdot \mathbf{u}_\perp + \gamma u_z = 0 \tag{13}$$

and the boundary conditions are now written as

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{u}_\perp = 0 \ \text{ and } \ u_z = 0 \ \text{ on } \Gamma_\perp \tag{14}$$

Equation (6) becomes

$$C_\gamma^2\mathbf{u} - 2\omega\chi C_\gamma \mathbf{u} - \omega^2 d\mathbf{u} = 0 \tag{15}$$

We agree to call a pair $\gamma \in \mathbf{C}$, $\mathbf{u} : \Omega_\perp \to \mathbf{C}^3$ (non–zero function) a *solution* of (15) if it turns the equation into an identity. The complex number γ is called an *eigenvalue* or a *propagation constant* and the function $\mathbf{u}$ a corresponding *mode* of the waveguide. According to this terminology, the bi-isotropic waveguide propagation problem is called *well-posed* if the propagation constants of the waveguide form a sequence $(\gamma_n) \subset \mathbf{C}$ with the following properties:

- it has no accumulation points other than ∞, and
- it can be ordered in such a way that

$$|\gamma_1| \leqslant \ldots \leqslant |\gamma_n| \leqslant \ldots \to \infty$$

We will now reformulate (15) as an eigenvalue problem. Let

$$P(\gamma) = C_\gamma^2 - 2\omega\chi C_\gamma - \omega^2 dI$$

This is an operator pencil with parameter γ. We say that γ is an *eigenvalue* of $P(\cdot)$ if there is a non-zero field $\mathbf{u}$ (corresponding *eigenvector*) such that $P(\gamma)\mathbf{u} = 0$. Using this terminology, γ, $\mathbf{u}$, is a solution of (15) i.e., γ is a propagation constant and $\mathbf{u}$ a corresponding mode if and only if γ is an eigenvalue of $P(\cdot)$ with corresponding eigenvector $\mathbf{u}$.

5. FORMULATION

It is straightforward to "factorize" the pencil $P(\cdot)$ as follows

$$P(\gamma) = (C_\gamma - \kappa_+ I)(C_\gamma - \kappa_- I) = (C_\gamma - \kappa_- I)(C_\gamma - \kappa_+ I) \tag{16}$$

where

$$\kappa_\pm = \kappa_\pm(\omega) := \omega\left(\chi \pm \sqrt{\chi^2 + d}\right)$$

are the roots of the quadratic equation

$$x^2 - 2\omega\chi x - \omega^2 d = 0$$

Another useful observation is that $\kappa_\pm$ are the eigenvalues of the matrix

$$\mathrm{i}\omega \begin{bmatrix} -\zeta & -\mu \\ \varepsilon & \xi \end{bmatrix} = -\mathrm{i}\omega V M \tag{17}$$

(16) shows that Equation (15), equivalently $P(\gamma)\mathbf{u} = 0$, holds for $\mathbf{u} \neq 0$ if and only if one of the following holds:

(C1) $\mathbf{u}$ is an eigenvector of C_γ corresponding to κ_+.

(C2) $\mathbf{u}$ is an eigenvector of C_γ corresponding to κ_-.

(C3) $\mathbf{w}_\pm := C_\gamma \mathbf{u} - \kappa_\mp \mathbf{u}$ is an eigenvector of C_γ corresponding to $\kappa_\pm$.

We have thus reduced the problem to the eigenvalue–eigenvector problem for operator C_γ.

5.1. Eigenvalues and Eigenvectors of C_γ

The equation $C_\gamma \mathbf{w} = \lambda \mathbf{w}$ is equivalent to the system

$$\begin{cases} -\gamma V \mathbf{w}_\perp + V \nabla_\perp w_z = \lambda \mathbf{w}_\perp \\ \nabla_\perp V \mathbf{w}_\perp = \lambda w_z \end{cases} \tag{18}$$

Solving with respect to the longitudinal field w_z we find

$$\nabla_\perp^2 w_z + k^2 w_z = 0 \tag{19}$$

where

$$k^2 := \lambda^2 + \gamma^2 \tag{20}$$

Thereby, w_z is a solution of the Helmholtz Equation (19). The transverse field is then calculated by the first equation in (18) and it is given analytically as

$$\mathbf{w}_\perp = \frac{\gamma}{k^2}\nabla_\perp w_z + \frac{\lambda}{k^2} V \nabla_\perp w_z \tag{21}$$

As a consequence, $\mathbf{w}$ is completely determined by its longitudinal part.

5.2. The Trivial Cases (C1), (C2)

Suppose that $\mathbf{u}$ is an eigenvalue of C_γ corresponding to κ_+ (resp. to κ_-). Then, due to (14) and (19), u_z is a solution to the Helmholtz equation with the parameter $k_+^2 := \kappa_+^2 + \gamma^2$ (resp. $k_-^2 := \kappa_-^2 + \gamma^2$), satisfying the Dirichlet boundary condition. (21) also implies

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{u}_\perp = \frac{\gamma}{k_+^2}\frac{\partial u_z}{\partial \hat{\boldsymbol{\tau}}} - \frac{\kappa_+}{k_+^2}\frac{\partial u_z}{\partial \hat{\boldsymbol{\nu}}} \text{ , resp. } \hat{\boldsymbol{\tau}} \cdot \mathbf{u}_\perp = \frac{\gamma}{k_-^2}\frac{\partial u_z}{\partial \hat{\boldsymbol{\tau}}} - \frac{\kappa_-}{k_-^2}\frac{\partial u_z}{\partial \hat{\boldsymbol{\nu}}} \tag{22}$$

Consequently, by (14), u_z satisfies also the Neumann boundary condition

$$\frac{\partial u_z}{\partial \hat{\boldsymbol{\nu}}} = 0$$

Thus $u_z = 0$ and $\mathbf{u}$ is the zero field.

5.3. The Non-trivial Case (C3)

The initial field $\mathbf{u}$ can be recovered by

$$\mathbf{u} = \frac{\mathbf{w}_+ - \mathbf{w}_-}{\kappa_+ - \kappa_-} \tag{23}$$

Equation (19) shows that

$$\nabla_\perp w_{\pm z} + k_\pm^2 w_{\pm z} = 0 \tag{24}$$

Equivalently, we formulate the formal eigenvalue problem for a matrix operator

$$\begin{bmatrix} -\nabla_\perp^2 - \kappa_+^2 I & 0 \\ 0 & -\nabla_\perp^2 - \kappa_-^2 I \end{bmatrix} \begin{pmatrix} w_{+z} \\ w_{-z} \end{pmatrix} = \gamma^2 \begin{pmatrix} w_{+z} \\ w_{-z} \end{pmatrix} \tag{25}$$

The boundary conditions are coupled and can be found by using (23); actually,

$$w_{+z} - w_{-z} = (\kappa_+ - \kappa_-)u_z, \quad \hat{\boldsymbol{\tau}} \cdot \mathbf{w}_{+\perp} - \hat{\boldsymbol{\tau}} \cdot \mathbf{w}_{-\perp} = (\kappa_+ - \kappa_-)\hat{\boldsymbol{\tau}} \cdot \mathbf{u}_\perp$$

The first equation gives

$$w_{+z} = w_{-z} \text{ on } \Gamma_\perp \tag{26}$$

whereas, by using (21), we calculate

$$\mathbf{w}_{\pm\perp} = \frac{\gamma}{k_\pm^2}\nabla_\perp w_{\pm z} + \frac{\kappa_\pm}{k_\pm^2} V \nabla_\perp w_{\pm z}$$

and by taking the scalar product with $\hat{\boldsymbol{\tau}}$ we find

$$\frac{\gamma}{k_+^2}\frac{\partial w_{+z}}{\partial \hat{\boldsymbol{\tau}}} - \frac{\kappa_+}{k_+^2}\frac{\partial w_{+z}}{\partial \hat{\boldsymbol{\nu}}} = \frac{\gamma}{k_-^2}\frac{\partial w_{-z}}{\partial \hat{\boldsymbol{\tau}}} - \frac{\kappa_-}{k_-^2}\frac{\partial w_{-z}}{\partial \hat{\boldsymbol{\nu}}} \text{ on } \Gamma_\perp \tag{27}$$

The problem (25)–(27) resembles the interior transmission problem, which arises in the inverse scattering theory, and has initiated large research efforts during the last years, see [25]. The big difficulty here is that the eigenvalue is involved in the boundary condition (27). Actually, as a general remark, we can underline that (27) couples the three elements of the waveguide (geometry, walls, material) and the propagation properties in a highly complicated manner; this is a characteristic feature of chirality. We note also that our results agree with [2, Ch. 4].

6. SOLVABILITY

We now show that the solvability of the problem (25)–(27) is found in the Dirichlet problem for the Helmholtz equation

$$\nabla_\perp^2 w + k^2 w = 0 \quad \text{in } \Omega_\perp \tag{28}$$

$$w = \varphi \quad \text{on } \Gamma_\perp \tag{29}$$

This classical problem is documented, among others, in [26]. To solve (28), (29) means to perform one of the following procedures:

(i) *To construct the Dirichlet Green function.* A general method for that is to begin with a fundamental solution, that is a function $\Phi_k(\boldsymbol{\rho};\boldsymbol{\rho}')$ which solves the problem

$$\nabla_{\perp\boldsymbol{\rho}}^2 \Phi_k(\boldsymbol{\rho};\boldsymbol{\rho}') + k^2 \Phi_k(\boldsymbol{\rho};\boldsymbol{\rho}') = \delta_{\boldsymbol{\rho}'} \tag{30}$$

Here δ denotes the delta function of Dirac. A usual choice is

$$\Phi_k(\boldsymbol{\rho};\boldsymbol{\rho}') = \frac{\mathrm{i}}{4} H_0^{(1)}\left(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|\right) \tag{31}$$

where $H_0^{(1)}$ is the Hankel function of first kind. Choose now a $\boldsymbol{\rho} \in \Omega_\perp$ and specify a function $h^{\boldsymbol{\rho}}(\boldsymbol{\rho}')$, that satisfies the Dirichlet problem, see [27]

$$\nabla_{\perp\boldsymbol{\rho}'}^2 h^{\boldsymbol{\rho}}(\boldsymbol{\rho}') + k^2 h^{\boldsymbol{\rho}}(\boldsymbol{\rho}') = 0, \quad \boldsymbol{\rho}' \in \Omega_\perp$$

$$h^{\boldsymbol{\rho}}(\boldsymbol{\rho}') = -\Phi(\boldsymbol{\rho};\boldsymbol{\rho}'), \quad \boldsymbol{\rho}' \in \Gamma_\perp$$

The Dirichlet Green function is defined as

$$G_k(\boldsymbol{\rho};\boldsymbol{\rho}') := -\frac{\partial}{\partial\hat{\boldsymbol{\nu}}(\boldsymbol{\rho}')}\left[\Phi_k(\boldsymbol{\rho};\boldsymbol{\rho}') + h^{\boldsymbol{\rho}}(\boldsymbol{\rho}')\right] \tag{32}$$

and then solution of (28), (29) is represented as

$$w(\boldsymbol{\rho}) = \oint_{\Gamma_\perp} G_k(\boldsymbol{\rho};\boldsymbol{\rho}')\varphi(\boldsymbol{\rho}')\, d\ell(\boldsymbol{\rho}'), \quad \boldsymbol{\rho}\in\Omega_\perp \tag{33}$$

(ii) *To construct the Dirichlet-to-Neumann operator.* That is, given the solution w of (28), (29), to specify the mapping

$$\Lambda_k \,:\, \varphi \mapsto \frac{\partial w}{\partial\hat{\boldsymbol{\nu}}} \tag{34}$$

With this at hand, we can represent the solution by the formula

$$w(\boldsymbol{\rho}) = \oint_{\Gamma_\perp}\left(\Phi_k(\boldsymbol{\rho};\boldsymbol{\rho}')(\Lambda_k\varphi)(\boldsymbol{\rho}') - \frac{\partial\Phi_k(\boldsymbol{\rho};\boldsymbol{\rho}')}{\partial\hat{\boldsymbol{\nu}}(\boldsymbol{\rho}')}\varphi(\boldsymbol{\rho}')\right) d\ell(\boldsymbol{\rho}') \tag{35}$$

(iii) *To apply a separation of variables technique.* This is recommended in certain and relatively simple geometries $\Gamma_\perp$. To fix ideas, consider a change of coordinates

$$\begin{cases}\psi = \psi(x,y)\\ \eta = \eta(x,y)\end{cases}$$

such that $\Gamma_\perp$ can be described by the simple equations $\psi = c$, $a \leqslant \eta \leqslant b$. Equation (28) is accordingly transformed and we search for solutions of the form

$$W(\psi,\eta) := w(x(\psi,\eta), y(\psi,\eta)) = \Psi(\psi)H(\eta) \tag{36}$$

Suppose that we can find a sequence $(\Psi_n(\psi)H_n(\eta))$ of such solutions and that the initial data can be expanded with respect to $(H_n(\eta))$ (which is assumed, consequently, to be a basis for the initial data)

$$\phi(\eta) := \varphi(x(c,\eta), y(c,\eta)) = \sum_n \hat{\phi}_n H_n(\eta) \tag{37}$$

$\{\hat{\phi}_n\}_n$ being the expansion coefficients. The solution of (28), (29) is represented as a series

$$W(\psi,\eta) = \sum_n \frac{\hat{\phi}_n}{\Psi_n(c)}\Psi_n(\psi)H_n(\eta) \tag{38}$$

We return now to the problem (25)–(27). According to (26), we insert an auxiliary function φ to express the common boundary values of $w_{\pm z}$, i.e.,

$$w_{+z} = \varphi = w_{-z} \text{ on } \Gamma_\perp \tag{39}$$

If we can solve the problem (28), (29) for $k = k_\pm$, then we have analytic expressions for the tangential and normal derivatives of $w_{\pm z}$, which will have a functional dependence on φ. Consequently, substituting to (27), we obtain a functional equation with respect to φ. This equation is homogeneous; in order to admit non-trivial solutions, the parameters involved (γ, ω, $\kappa_\pm$) must satisfy a certain equation, namely the dispersion equation. By this we can calculate the propagation constants. With the corresponding φ at hand, we can retrieve $w_{\pm z}$ and with these we can reconstruct the modes.

7. THE NULL-FIELD EQUATIONS

In this section, we employ a general method to solve the problem (25)–(27). Our approach resembles the null-field method [18, 19]. First of all, it is easy to see that if f, g are functions defined in a domain containing $\Gamma_\perp$, then

$$\oint_{\Gamma_\perp} \frac{\partial f}{\partial \hat{\boldsymbol{\tau}}} g \, d\ell = - \oint_{\Gamma_\perp} f \frac{\partial g}{\partial \hat{\boldsymbol{\tau}}} \, d\ell \tag{40}$$

If, in addition, f, g are solutions of a Helmholtz equation in $\Omega_\perp$ then the second Green identity implies

$$\oint_{\Gamma_\perp} \frac{\partial f}{\partial \hat{\boldsymbol{\nu}}} g \, d\ell = \oint_{\Gamma_\perp} f \frac{\partial g}{\partial \hat{\boldsymbol{\nu}}} \, d\ell \tag{41}$$

The central idea is to expand the unknown fields in appropriate bases, not necessarily orthonormal. We work in the space $L^2(\Gamma_\perp)$ with the usual inner product

$$(f, g) = \oint_{\Gamma_\perp} f \bar{g} \, d\ell$$

Bar denotes the complex conjugation. We now consider an infinite sequence (υ_n^k) of linearly independent solutions of the Helmholtz equation with parameter k in a domain larger than $\Omega_\perp$. Put $\upsilon_n^\pm := \upsilon_n^{k_\pm}$ for convenience; actually they represent solutions of (24). The reader can have always in mind the standing wave system of the null-field method, see [18]. Equation (41) implies

$$\oint_{\Gamma_\perp} \frac{\partial w_{+z}}{\partial \hat{\boldsymbol{\nu}}} \upsilon_n^+ \, d\ell = \oint_{\Gamma_\perp} \varphi \frac{\partial \upsilon_n^+}{\partial \hat{\boldsymbol{\nu}}} \, d\ell \tag{42}$$

We make now the following hypothesis.

Basis property Both the restrictions

$$f_n^* := \overline{\upsilon_n^+}\,\Big|\,\Gamma_\perp, \quad g_n^* := \frac{\partial\overline{\upsilon_n^+}}{\partial\hat{\boldsymbol{\nu}}}\,\Big|\,\Gamma_\perp$$

define bases of $L^2(\Gamma_\perp)$.

By a "basis" here we mean that each square integrable function can be expanded as a series converging in norm. What we really need, actually, is that there exist sequences (f_n), (g_n) such that each function $h \in L^2(\Gamma_\perp)$ is expanded as follows [28]

$$h = \sum_n (h, f_n^*) f_n = \sum_n (h, g_n^*) g_n \tag{43}$$

It should be noted that the sequences (f_n) and (f_n^*), (g_n) and (g_n^*), are indeed bi-orthogonal, i.e.,

$$(f_m^*, f_n) = (g_m^*, g_n) = \delta_{mn} \tag{44}$$

where now δ stand for the Kronecker delta. Consequently, we have the expansions

$$\varphi = \sum_n c_n g_n, \qquad \frac{\partial w_{+z}}{\partial\hat{\boldsymbol{\nu}}} = \sum_n c_n f_n, \tag{45}$$

c_n being the common value of the integrals in (42). The assumption of basis is related to the Rayleigh hypothesis, and we refer the reader to Ref. [18] for more details on this intriguing subject.

In the sequel, we multiply both sides of (27) with υ_m^- and we integrate on $\Gamma_\perp$. By using again (40), (41) we obtain

$$\begin{aligned} &\frac{\gamma}{k_+^2}\oint_{\Gamma_\perp}\frac{\partial\upsilon_m^-}{\partial\hat{\boldsymbol{\tau}}}\varphi\,d\ell + \frac{\kappa_+}{k_+^2}\oint_{\Gamma_\perp}\upsilon_m^-\frac{\partial w_{+z}}{\partial\hat{\boldsymbol{\nu}}}\,d\ell \\ &= \frac{\gamma}{k_-^2}\oint_{\Gamma_\perp}\frac{\partial\upsilon_m^-}{\partial\hat{\boldsymbol{\tau}}}\varphi\,d\ell + \frac{\kappa_-}{k_-^2}\oint_{\Gamma_\perp}\frac{\partial\upsilon_m^-}{\partial\hat{\boldsymbol{\nu}}}\varphi\,d\ell \end{aligned} \tag{46}$$

Inserting (45) into (46), we obtain an infinite homogeneous linear system having the sequence (c_n) as the unknown:

$$\sum_n Q_{mn} c_n = 0\,, \tag{47}$$

where

$$\begin{aligned} Q_{mn} :\ = \ &\frac{\gamma}{k_+^2}\oint_{\Gamma_\perp}\frac{\partial\upsilon_m^-}{\partial\hat{\boldsymbol{\tau}}}g_n\,d\ell + \frac{\kappa_+}{k_+^2}\oint_{\Gamma_\perp}\upsilon_m^- f_n\,d\ell \\ &-\frac{\gamma}{k_-^2}\oint_{\Gamma_\perp}\frac{\partial\upsilon_m^-}{\partial\hat{\boldsymbol{\tau}}}g_n\,d\ell - \frac{\kappa_-}{k_-^2}\oint_{\Gamma_\perp}\frac{\partial\upsilon_m^-}{\partial\hat{\boldsymbol{\nu}}}g_n\,d\ell \end{aligned} \tag{48}$$

We now make the following statement, for a proof concerning the standing wave system see [29, 30].

Completeness Both the restrictions

$$\overline{\upsilon_n^-} \,\Big|\, \Gamma_\perp,\ \frac{\partial \overline{\upsilon_n^-}}{\partial \hat{\boldsymbol{\nu}}} \,\Big|\, \Gamma_\perp$$

define complete systems in $L^2(\Gamma_\perp)$, that is

$$\text{either } \left(f, \overline{\upsilon_n^-}\right) = 0 \text{ or } \left(f, \frac{\partial \overline{\upsilon_n^-}}{\partial \hat{\boldsymbol{\nu}}}\right) = 0 \text{ for every } n$$

imply $f = 0$.

After this, we expect the system (47) to admit only the trivial solution unless a certain condition among the parameters is fulfilled. This condition is the determinant of the infinite matrix $Q := [Q_{mn}]$ to vanish and provides the implicit dispersion relation from which the propagation constants can be calculated. Note that the results presented in this section depend on two nontrivial hypotheses. In the following section, they are validated for the case of circular geometry.

8. THE CIRCULAR GEOMETRY

We focus now on the case where $\Omega_\perp$ is a circular disc of radius a. The choice here is quite straightforward and this is the sequence proposed by the null-field method.

$$\upsilon_n^k(\rho, \theta) := J_n(k\rho)\mathrm{e}^{\mathrm{i}n\theta} \tag{49}$$

where (ρ, θ) denote the polar coordinates. Then, the conjugate restrictions of υ_n^+ on the circle $\rho = a$

$$f_n^*(\theta) = \overline{J_n(k_+a)}\mathrm{e}^{-\mathrm{i}n\theta}, \quad g_n^*(\theta) = \overline{k_+ J_n'(k_+a)}\mathrm{e}^{-\mathrm{i}n\theta} \tag{50}$$

do define (orthogonal) bases for $L^2(\Gamma_\perp) = L^2[0, 2\pi]$ (n takes values in the integers). The corresponding sequences of functions are defined by

$$f_n(\theta) = \frac{1}{J_n(k_+a)}\mathrm{e}^{-\mathrm{i}n\theta}, \quad g_n(\theta) = \frac{1}{k_+ J_n'(k_+a)}\mathrm{e}^{-\mathrm{i}n\theta} \tag{51}$$

Substituting to (48) we obtain

$$\begin{aligned} Q_{mn} \;=\; 2\pi\delta_{mn} \Bigg(& \frac{\mathrm{i}\gamma m J_m(k_-a)}{a k_+^3 J_n'(k_+a)} + \frac{\kappa_+ J_m(k_-a)}{k_+^2 J_n(k_+a)} \\ & - \frac{\mathrm{i}\gamma m J_m(k_-a)}{a k_-^2 k_+ J_n'(k_+a)} - \frac{\kappa_- k_- J_m'(k_-a)}{k_-^2 k_+ J_n'(k_+a)} \Bigg) \end{aligned} \tag{52}$$

In this case, namely, the matrix Q is diagonal and its determinant vanishes if one of the diagonal elements does so. Thereby, for each integer m, we take the equation

$$\begin{aligned} & k_+^2 J_m(k_+a)\left[\mathrm{i}m\gamma J_m(k_-a) + a\kappa_- k_- J'_m(k_-a)\right] \\ = \; & k_-^2 J_m(k_-a)\left[\mathrm{i}m\gamma J_m(k_+a) + a\kappa_+ k_+ J'_m(k_+a)\right] \end{aligned} \tag{53}$$

The well known relation for Bessel functions

$$J_{-m}(\cdot) = (-1)^m J_m(\cdot), \quad J'_{-m}(\cdot) = (-1)^m J'_m(\cdot)$$

show that (53) can also be written, if we substitute m with $-m$,

$$\begin{aligned} & k_+^2 J_m(k_+a)\left[\mathrm{i}m\gamma J_m(k_-a) - a\kappa_- k_- J'_m(k_-a)\right] \\ = \; & k_-^2 J_m(k_-a)\left[\mathrm{i}m\gamma J_m(k_+a) - a\kappa_+ k_+ J'_m(k_+a)\right] \end{aligned} \tag{54}$$

Compare with the corresponding equations in [12], [2, Ch. 4]. The only problem here is to match the different notations.

9. NUMERICAL ILLUSTRATION

In this section, we illustrate the analysis presented in this paper by a numerical example, and, for simplicity, we adopt a waveguide with a circular cross section. The dispersion relation for this geometry is given as the roots to a transcendental equation explicitly given in Section 8, see (54). Dispersion relations for this geometry and constant material parameters have previously been published in the literature, see e.g., [11, 12]. These materials are all lossless, and therefore the propagating modes show no damping. However, dispersion effects are always present at least to some extent, and here we illustrate the theory with a more realistic material model that includes dispersion and losses.

All realistic material shows dispersion effects — at least in some frequency interval, and, moreover, the chirality parameter χ cannot assume a non-zero value at zero frequency. To our knowledge, the only known dispersion model for bi-isotropic effects is Condon's model [2, 31]. In order to model a passive material, the proper models for the permittivity and the permeability have to be adopted. A dispersive model, that is passive, is the following combined Lorentz-Debye model for the permittivity and permeability and Condon's model for the chirality parameter, i.e.,

$$\begin{cases} \epsilon(\omega) = 1 - \dfrac{\omega_{\mathrm{pe}}^2}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu_{\mathrm{e}}} + \dfrac{\alpha\tau}{1 - \mathrm{i}\omega\tau} \\ \mu(\omega) = 1 - \dfrac{\omega_{\mathrm{pm}}^2}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu_{\mathrm{m}}} \\ \xi(\omega) = -\zeta(\omega) = \dfrac{\mathrm{i}\omega\omega_{\mathrm{c}}}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu_{\mathrm{c}}} \end{cases} \tag{55}$$

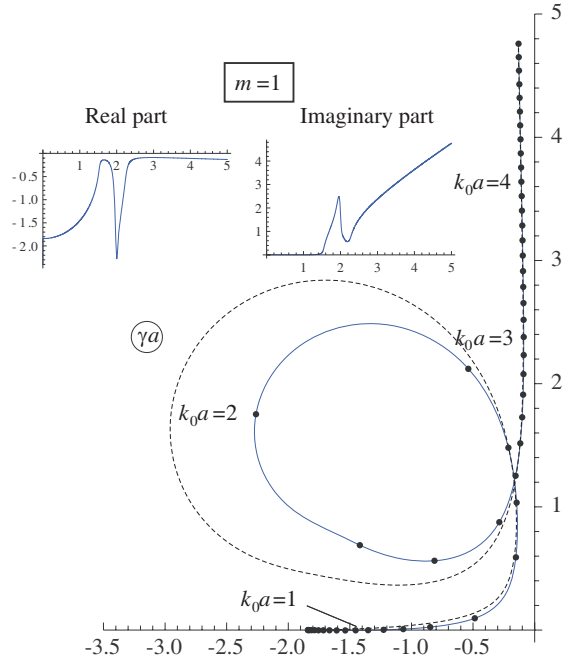

Figure 2. The dispersion relation of the lowest order mode for material data in Table 1 in the complex γa-plane. The curve shows the propagation constant γa as a function of the normalized frequency $k_0 a$ for $m = 1$. The dots along the curve show the normalized frequency $k_0 a = 0, 0.1, 0.2, \ldots, 5$. The curve starts at $\gamma a = -1.841$ at zero frequency. The inserts show the real and imaginary parts of γa as a function of $k_0 a$. The dashed line shows the dispersion curve for the non-chiral case ($\omega_c = 0$).

The dispersion curve, i.e., the curve of γa in the complex γa-plane as a function of the normalized frequency $k_0 a = \omega a / c_0$, for the lowest order mode is displayed in Figures 2 and 3 for three different values of the azimuthal index $m = \pm 1$. The data of the dispersive material is explicitly displayed in Table 1.

The curves start at $\gamma a = -1.841$ (TE) at zero frequency, and they immediately assume complex values. For comparison, the dashed line shows the dispersion curve for the non-chiral case ($\omega_c = 0$). The loop is inherent with the resonance characteristic of the material, and notice also that the propagation constant is no longer purely real or imaginary, but in general a complex number due to losses. The loop is traversed very quickly — a fact that is due to the small losses of the material.

Table 1. The data of the dispersive bi-isotropic material, see (55), in Figures 2 and 3.

$\omega_{\rm pe}a/c_0 = \omega_{\rm pm}a/c_0 = 0.5$
$\omega_0 a/c_0 = 2$
$\nu_{\rm e}a/c_0 = \nu_{\rm m}a/c_0 = \nu_{\rm c}a/c_0 = 0.1$
$\omega_{\rm c}a/c_0 = 0.04$
$\alpha\tau = 0.1$
$\tau c_0/a = 0.2$

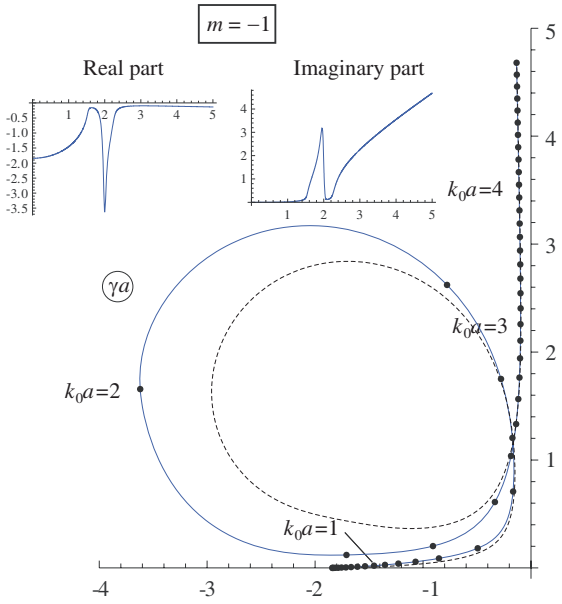

Figure 3. The dispersion relation of the lowest order mode for material data in Table 1 in the complex γa-plane. The curve shows the propagation constant γa as a function of the normalized frequency $k_0 a$ for $m = -1$. The dots along the curve show the normalized frequency $k_0 a = 0, 0.1, 0.2, \ldots, 5$. The curve starts at $\gamma a = -1.841$ at zero frequency. The inserts show the real and imaginary parts of γa as a function of $k_0 a$. The dashed line shows the dispersion curve for the non-chiral case ($\omega_{\rm c} = 0$).

The Figures 2 and 3 show the distinctive difference between the different modes corresponding to $m = +1$, and $m = -1$ — the loop is more narrow in $m = +1$ than in $m = -1$, and non-chiral case

lies somewhere in between. This is due to the sign of our chirality parameter.

The asymptotic behavior of the curve γa as $k_0 a \to \infty$ approaches $\Re\gamma a \approx -0.1$, which corresponds well to the order of magnitude of the losses in the material, i.e., $\alpha\tau = 0.1$ and $\tau c_0/a = 0.2$. In fact, under the assumption that the frequency is high enough so that only the Debye term in $\epsilon(\omega)$ contributes to its imaginary part, and $\Re\epsilon \approx 1$, $\mu \approx 1$, and $\xi \approx 0$, we have

$$\gamma a \approx -\frac{k_0 a \Im\epsilon}{2} + \mathrm{i}k_0 a \quad \Rightarrow \quad \gamma a \approx -\frac{1}{2}\frac{k_0^2 a\alpha\tau^2 c_0}{1 + k_0^2\tau^2 c_0^2} + \mathrm{i}k_0 a$$

$$\Rightarrow \quad \gamma a \approx -0.125 + 5\mathrm{i}$$

10. CONCLUSION

In this paper, we have analyzed the propagation problem inside a metallic waveguide of arbitrary cross-section, filled with a homogeneous bi-isotropic medium. We formulate the relevant spectral problem and we develop the theory to treat it. We propose also a general method (based on the null-field method and two hypotheses) which can solve the problem. The validity of this method is tested in the well studied case of the circular waveguide, where we obtain the known results.

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