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DISPERSION OF ELECTROMAGNETIC WAVES GUIDED BY AN OPEN TAPE HELIX II

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Abstract—The dispersion equation for electromagnetic waves guided by an open tape helix for the standard model of an infinitesimally thin and perfectly conducting tape is derived from an exact solution of a homogeneous boundary value problem for Maxwell's equations. A numerical analysis of the dispersion equation reveals that the tape current density component perpendicular to the winding direction does not affect the dispersion characteristic to any significant extent. In fact, there is a significant deviation from the dominant-mode sheath-helix dispersion curve only in the third allowed region and towards the end of the second allowed region. It may be concluded that the anisotropically conducting model of the tape helix that neglects the above transverse-current contribution is a good approximation to the isotropically conducting model that takes into account this contribution except at high frequencies even for wide tapes.

1. INTRODUCTION

In the first part of this contribution [1], the dispersion characteristic of electromagnetic waves guided by an open tape helix was derived neglecting the contribution of the tape current density component perpendicular to the winding direction. In this part, the method of deriving the dispersion equation for the above anisotropically conducting model of the tape helix is extended to the case where the tape is modeled to be only infinitesimally thin and perfectly conducting in all directions. Once again, there arises neither a need for any apriori assumption regarding the tape-current distribution nor is there a need for satisfying the tape-helix boundary conditions in any approximate sense.

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2. DERIVATION OF THE DISPERSION EQUATION

We make use of the tape-helix model introduced in Part I except that we no longer require the tape to the perfectly conducting only along the winding direction. We use the notation of Part I, and make use of the prefix I while referring an equation number of Part I. Accordingly, we consider a tape helix of infinite length, constant pitch, constant tape width and infinitesimal thickness surrounded by free space. We take the axis of the helix along the z-coordinate of a cylindrical coordinate system $(\rho,\ \varphi,\ z)$. The radius of the helix is a, the pitch is p and the width of the tape in the axial direction is w. The pitch angle ψ is therefore given by $\cot\psi = 2\pi a/p$.

Floquet's theorem for a periodic structure in conjunction with the property of invariance exhibited by an infinite helical structure under a translation Δz in the axial direction and a simultaneous rotation by $2\pi\Delta z/p$ around the axis permits an infinite-series expansion for any field component (phasor) $F\left(\rho,\phi,z\right)$ in the form [2]

$$F(\rho,\phi,z) = \sum_{n=-\infty}^{\infty} F_n\left(\rho\right) e^{j(n\varphi-\beta_n z)} \tag{1}$$

where

$$\beta_n = \beta_0 + 2\pi n/p \tag{2}$$

and where $\beta_0 = \beta_0(\omega)$ is the guided wave propagation constant at the radian frequency ω. Each term in the series-expansion (1) has to satisfy the Helmholtz equation in cylindrical coordinates. Hence, the Borgnis potentials [2] $U(\rho,\varphi,z)$ and $V(\rho,\varphi,z)$ for guided-wave solutions, at the radian frequency ω, may be assumed in the form

$$\begin{aligned}
[U,V]^T &= \sum_{n=-\infty}^{\infty} \left[A_n, B_n\right]^T I_n(\tau_n\rho) e^{j(n\varphi-\beta_n z)} \quad \text{for } 0 \le \rho < a, \\
&= \sum_{n=-\infty}^{\infty} \left[C_n, D_n\right]^T K_n(\tau_n\rho) e^{j(n\varphi-\beta_n z)} \quad \text{for } \rho > a,
\end{aligned} \tag{3}$$

where A_n, B_n, C_n and D_n, $n\in\mathbb{Z}$, are (complex) constants to be determined by the tape-helix boundary conditions, I_n and K_n are nth order modified Bessel functions of the first and the second kind respectively, and the superscript 'T' denotes the transpose. In (3), the transverse mode number $\tau_n,\ n\in\mathbb{Z}$, is defined by

$$\beta_n^2 - \tau_n^2 = k_0^2 \underline{\underline{\triangle}}\, \omega^2\mu\varepsilon \tag{4}$$

where μ is the permeability and ϵ is the permittivity of the ambient space. Expressing the tangential field components in terms of the

Borgnis potentials [2], we have

$$E_z = \frac{\partial^2 U}{\partial z^2} + k_0^2 U = \sum_{n=-\infty}^{\infty} -\tau_n^2 A_n I_n(\tau_n \rho) e^{j(n\varphi - \beta_n z)} \quad \text{for } 0 \leq \rho < a,$$

$$\sum_{n=-\infty}^{\infty} -\tau_n^2 C_n K_n(\tau_n \rho) e^{j(n\varphi - \beta_n z)} \quad \text{for } \rho > a \quad \text{(5a)}$$

$$\begin{aligned} E_\phi &= \frac{1}{\rho}\frac{\partial^2 U}{\partial \varphi \partial z} + j\omega\mu \frac{\partial V}{\partial \rho} \\ &= \sum_{n=-\infty}^{\infty} \left[n\beta_n A_n I_n(\tau_n \rho)/\rho + j\omega\mu\tau_n B_n I'_n(\tau_n \rho) \right] e^{j(n\varphi - \beta_n z)} \\ &\qquad \text{for } 0 \leq \rho < a, \\ &\sum_{n=-\infty}^{\infty} \left[n\beta_n C_n K_n(\tau_n \rho)/\rho + j\omega\mu\tau_n D_n K'_n(\tau_n \rho) \right] e^{j(n\varphi - \beta_n z)} \\ &\qquad \text{for } \rho > a \quad \text{(5b)} \end{aligned}$$

$$H_z = \frac{\partial^2 V}{\partial z^2} + k_0^2 V = \sum_{n=-\infty}^{\infty} -\tau_n^2 B_n I_n(\tau_n \rho) e^{j(n\varphi - \beta_n z)} \quad \text{for } 0 \leq \rho < a,$$

$$\sum_{n=-\infty}^{\infty} -\tau_n^2 D_n K_n(\tau_n \rho) e^{j(n\varphi - \beta_n z)} \quad \text{for } \rho > a \quad \text{(6a)}$$

$$\begin{aligned} H_\phi &= \frac{1}{\rho}\frac{\partial^2 V}{\partial \varphi \partial z} - j\omega\epsilon \frac{\partial U}{\partial \rho} \\ &= \sum_{n=-\infty}^{\infty} \left[-j\omega\epsilon\tau_n A_n I'_n(\tau_n \rho) + n\beta_n B_n I_n(\tau_n \rho)/\rho \right] e^{j(n\varphi - \beta_n z)} \\ &\qquad \text{for } 0 \leq \rho < a, \\ &\sum_{n=-\infty}^{\infty} \left[-j\omega\epsilon\tau_n C_n K'_n(\tau_n \rho) + n\beta_n D_n K_n(\tau_n \rho)/\rho \right] e^{j(n\varphi - \beta_n z)} \\ &\qquad \text{for } \rho > a \quad \text{(6b)} \end{aligned}$$

In the expressions (5) and (6) for the tangential field components, I'_n and K'_n denote the derivatives of I_n and K_n with respect to their arguments.

The boundary conditions at $\rho = a$ for the perfectly conducting model of the tape helix are

(i) The tangential electric field is continuous for all values of φ and z.

(ii) The discontinuity in the tangential magnetic field equals the surface current density on the tape surface.

(iii) The tangential electric field vanishes on the tape surface.

Thus

$$E_z(a-,\varphi,z) - E_z(a+,\varphi,z) = 0 \tag{7a}$$
$$E_\varphi(a-,\varphi,z) - E_\varphi(a+,\varphi,z) = 0 \tag{7b}$$
$$H_z(a-,\varphi,z) - H_z(a+,\varphi,z) = J_\varphi(\varphi,z) \tag{7c}$$
$$H_\varphi(a-,\varphi,z) - H_\varphi(a+,\varphi,z) = -J_z(\varphi,z) \tag{7d}$$
$$E_z(a,\varphi,z)g(\varphi,z) = 0 \tag{7e}$$
$$E_\varphi(a,\varphi,z)g(\varphi,z) = 0 \tag{7f}$$

where $J_z(\varphi,z)$ and $J_\varphi(\varphi,z)$ are the axial and the azimuthal components of the surface current density, which is confined only to the tape surface, and the function $g(\varphi,z)$, defined in terms of the indicator functions of the disjoint (for the same value of φ) intervals

$$I_l(\varphi) \underline{\triangle} \left[(l+\varphi/2\pi)p - w/2, (l+\varphi/2\pi)p + w/2\right], \quad l \in \mathbb{Z},$$

by

$$g(\varphi,z) \underline{\triangle} \sum_{l=-\infty}^{\infty} 1_{I_l(\varphi)}(z)$$

will be equal to 1 on the tape surface and 0 elsewhere (i.e., over the gaps) on the surface of the (infinite) cylinder $\rho = a$. In (7a)–(7d)

$$F(a\pm,\varphi,z) \underline{\triangle} \lim_{\delta \downarrow 0} F(a \pm \delta, \varphi, z)$$

for any field component $F(\rho,\varphi,z)$. The functional dependence of the surface current density components $J_m(\varphi,z)$, $m = z,\varphi$, which are confined to the two-dimensional region corresponding to the tape-surface, on φ and z is governed by the periodicity and the symmetry conditions imposed by the helix geometry. Accordingly, $J_m(\varphi,z)$, $m = z,\varphi$, admit the representations

$$J_m(\varphi,z) = \left(\sum_{n=-\infty}^{\infty} J_{mn} e^{j(n\varphi - \beta_n z)}\right) g(\varphi,z), \quad m = z,\varphi, \tag{8}$$

where the (complex) constant coefficients J_{zn} and $J_{\varphi n}$, $n \in \mathbb{Z}$, appearing in the expansion (8) of the surface current density components are to be determined (in terms of any one of the constants J_{z0} and $J_{\varphi 0}$) by the tape-helix boundary conditions. Since

$$e^{j(n\varphi - \beta_n z)} = e^{-j\beta_0 z} e^{-j\frac{2\pi n}{p}(z - \varphi p/2\pi)},$$

and

$$1_{I_l(\varphi)}(z) = 1_{[lp-w/2,\, lp+w/2]}\left(z - \varphi p/2\pi\right),$$

the surface current density expansions of (8) may recast into the form

$$J_m(\varphi, z) = e^{-j\beta_0 z} f_m(\zeta), \quad m = z, \varphi, \tag{9}$$

where

$$f_m(\zeta) = \sum_{l=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} J_{mn} e^{-j2\pi n\zeta/p}\right) 1_{[lp-w/2,\, lp+w/2]}(\zeta), \quad m = z, \varphi, \tag{10}$$

and

$$\zeta = z - \varphi p/2\pi \tag{11}$$

The functions f_m, $m = z, \varphi$, being periodic in ζ with period p, may be expanded in Fourier series

$$f_m(\zeta) = \sum_{k=-\infty}^{\infty} \Gamma_{mk} e^{-j2\pi k\zeta/p} \tag{12}$$

where the Fourier coefficients $\Gamma_{mk}, k \in \mathbb{Z}, m = z, \varphi$, are given by

$$\begin{aligned}\Gamma_{mk} &= (1/p)\int_{-p/2}^{p/2} f_m(\zeta) e^{j2\pi k\zeta/p} d\zeta \\ &= \hat{w} \sum_{k=-\infty}^{\infty} J_{mn} \sin\mathrm{c}(n-k)\hat{w}, \quad m = z, \varphi \end{aligned}\tag{13}$$

In (13), $\hat{w} = w/p$ and $\mathrm{sinc}\, X \underline{\triangle} \sin \pi X/\pi X$. Thus

$$J_m(\varphi, z) = \sum_{n=-\infty}^{\infty} \Gamma_{mn} e^{j(n\varphi - \beta_n z)}, \quad m = z, \varphi \tag{14}$$

Making use of the expansions (14) for $J_m(\varphi, z)$, $m = z, \varphi$, in (7c) and (7d), we may deduce from the first four of the tape-helix boundary conditions (7a)–(7d), the following expressions for the coefficients A_n, B_n, C_n and $D_n, n \in \mathbb{Z}$, appearing in the field expansions of (5) and (6), in terms of the coefficients Γ_{zn} and $\Gamma_{\varphi,n}, n \in \mathbb{Z}$, of the current-density expansions (14) on equating the 'coefficients' of $e^{j(n\varphi - \beta_n z)}$ on both sides:

$$C_n = \left(I_{na}/K_{na}\right) A_n \tag{15a}$$

$$D_n = \left(I_{na}{}'/K'_{na}\right) B_n \tag{15b}$$

$$A_n = aK_{na}\left[\Gamma_{zn} - \left(n\beta_{na}/\tau_{na}^2\right)\Gamma_{\varphi n}\right]/j\omega\epsilon \tag{15c}$$

$$B_n = a^2 K'_{na} \Gamma_{\varphi n}/\tau_{na} \tag{15d}$$

where we have resorted to the following abbreviations

$$I_{na} \underset{=}{\triangle} I_n(\tau_n a), \qquad K_{na} \underset{=}{\triangle} K_n(\tau_n a),$$
$$I'_{na} \underset{=}{\triangle} I'_n(\tau_n a), \qquad K'_{na} \underset{=}{\triangle} K'_n(\tau_n a) \tag{16}$$

for the modified Bessel functions and their derivatives evaluated at $\rho = a$. In (15), (16), and in the sequel

$$k_{0a} \underset{=}{\triangle} ak_0, \ \beta_{0a} \underset{=}{\triangle} a\beta_0, \ \beta_{na} \underset{=}{\triangle} a\beta_n = \beta_{0a} + n\cot\psi, \ \tau_{na} \underset{=}{\triangle} a\tau_n \tag{17}$$

Finally,the enforcement of the homogeneous boundary conditions (7e) and (7f) on the tangential electric field components leads to the two equations

$$h_z(\zeta) \underset{=}{\triangle} \sum_{l=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} (\mu_n \Gamma_{zn} - \nu_n \Gamma_{\varphi n})\, e^{-j2\pi n\zeta/p} \right) 1_{[lp-w/2,\, lp+w/2]}(\zeta)$$
$$= 0 \tag{18a}$$

$$h_\varphi(\zeta) \underset{=}{\triangle} \sum_{l=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} (\nu_n \Gamma_{zn} - \eta_n \Gamma_{\varphi n})\, e^{-j2\pi n\zeta/p} \right) 1_{[lp-w/2,\, lp+w/2]}(\zeta)$$
$$= 0 \tag{18b}$$

on cancelation of the non-zero factor $je^{-j\beta_0 z}/\omega\epsilon a$. In (18)

$$\mu_n \underset{=}{\triangle} \tau_{na}^2 I_{na} K_{na} \tag{19a}$$

$$\nu_n \underset{=}{\triangle} n\beta_{na} I_{na} K_{na} \tag{19b}$$

$$\eta_n \underset{=}{\triangle} k_{0a}^2 I'_{na} K'_{na} + (n\beta_{na}/\tau_{na})^2 I_{na} K_{na} \tag{19c}$$

Equations (18) imply that each Fourier coefficient of the two periodic functions $h_m(\zeta)$, $m = z, \ \varphi$, of ζ (with period p) must vanish, that is

$$\sum_{n=-\infty}^{\infty} (\mu_n \Gamma_{zn} - \nu_n \Gamma_{\varphi n}) \operatorname{sinc}(n-k)\hat{w} = 0, \tag{20a}$$

$$\sum_{n=-\infty}^{\infty} (\nu_n \Gamma_{zn} - \eta_n \Gamma_{\varphi n}) \operatorname{sinc}(n-k)\hat{w} = 0 \qquad \text{for } k \in \mathbb{Z} \tag{20b}$$

Substituting for Γ_{zn} and $\Gamma_{\varphi n}$ from (14), Equations (20) may be put in the form

$$\sum_{q=-\infty}^{\infty} (\mu_{kq} J_{zq} - \nu_{kq} J_{\varphi q}) = 0, \quad k \in \mathbb{Z}, \tag{21a}$$

$$\sum_{q=-\infty}^{\infty} (\nu_{kq} J_{zq} - \eta_{kq} J_{\varphi q}) = 0, \quad k \in \mathbb{Z} \tag{21b}$$

where

$$\mu_{kq} = \sum_{n=-\infty}^{\infty} \mu_n \operatorname{sinc}(k-n)\hat{w}\ \operatorname{sinc}(q-n)\hat{w}, \tag{22a}$$

$$\nu_{kq} = \sum_{n=-\infty}^{\infty} \nu_n \operatorname{sinc}(k-n)\hat{w}\ \operatorname{sinc}(q-n)\hat{w}, \tag{22b}$$

$$\eta_{kq} = \sum_{n=-\infty}^{\infty} \eta_n \operatorname{sinc}(k-n)\hat{w}\ \operatorname{sinc}(q-n)\hat{w}, \tag{22c}$$

$$k, q \in \mathbb{Z}.$$

In terms of the three infinite-order matrices $\mathbf{A}_\mu \underline{\triangle} [\mu_{kq}]_{k,q\in\mathbb{Z}}$, $\mathbf{A}_\nu \underline{\triangle} [\nu_{kq}]_{k,q\in\mathbb{Z}}$ and $\mathbf{A}_\eta \underline{\triangle} [\eta_{kq}]_{k,q\in\mathbb{Z}}$, and the two infinite-dimensional vectors $\mathbf{J}_m = [\ldots, J_{m\bar{2}}, J_{m\bar{1}}, J_{m0}, J_{m1}, J_{m2}, \ldots]^T$, $m = z, \varphi$, the two infinite sets of linear homogeneous Equations (21) for determining the two infinite sets of coefficients J_{zq} and $J_{\varphi q}$, $q \in \mathbb{Z}$, may be expressed compactly as

$$\mathbf{A}_\mu \mathbf{J}_z - \mathbf{A}_\nu \mathbf{J}_\varphi = \mathbf{0} \tag{23a}$$

$$\mathbf{A}_\nu \mathbf{J}_z - \mathbf{A}_\eta \mathbf{J}_\varphi = \mathbf{0} \tag{23b}$$

where $\mathbf{0}$ denotes the column vector of infinite number of zeros. Solving (23a) for $\mathbf{J}_\varphi$ in terms of $\mathbf{J}_z$ as

$$\mathbf{J}_\varphi = \mathbf{A}_\nu^{-1} \mathbf{A}_\mu \mathbf{J}_z \tag{24}$$

and substituting into (23b), we have the infinite set of equations

$$\left[\mathbf{A}_\nu - \mathbf{A}_\eta \mathbf{A}_\nu^{-1} \mathbf{A}_\mu\right] \mathbf{J}_z = \mathbf{0} \tag{25}$$

for determining the infinite set of coefficients J_{zq}, $q \in \mathbb{Z}$. For a nontrivial solution for $\mathbf{J}_z$, it is necessary that

$$\left|\mathbf{A}_\nu - \mathbf{A}_\eta \mathbf{A}_\nu^{-1} \mathbf{A}_\mu\right| = 0 \tag{26}$$

The determinantal condition (26) gives, in principle, the dispersion equation for the cold-wave modes supported by an open perfectly conducting infinite tape helix of infinitesimal thickness and finite width.

It may be appropriate to note at this juncture that the dispersion equation derived by Samuel Sensiper for a perfectly conducting tape helix in his doctoral thesis [3] also has a form resembling (26); however, the expressions for the matrix entries μ_{kq}, ν_{kq} and η_{kq}, k, $q \in \mathbb{Z}$, obtained by him are radically different from those given by (22). In fact, these entries (after correcting a misprint in his thesis), in our notation, are

$$\mu_{kq}^{(S)} = \mu_k M_{kq}, \quad \nu_{kq}^{(S)} = \nu_k M_{kq} \quad \text{and} \quad \eta_{kq}^{(S)} = \eta_k M_{kq}, \quad k, q \in \mathbb{Z},$$

where

$$M_{kq} = \begin{cases} \left(e^{j2\pi(k-q\hat{w})} - 1\right)/j2\pi(k-q\hat{w}) & \text{for } k - q\hat{w} \neq 0, \\ 1 & \text{for } k - q\hat{w} = 0 \end{cases}$$

In terms of the associated infinite-order matrices $\mathbf{A}_\mu^{(S)}$, $\mathbf{A}_\nu^{(S)}$ and $\mathbf{A}_\eta^{(S)}$, the dispersion equation of Sensiper becomes

$$\left| \mathbf{A}_\eta^{(S)} - \mathbf{A}_\nu^{(S)} \left(\mathbf{A}_\mu^{(S)}\right)^{-1} \mathbf{A}_\nu^{(S)} \right| = 0 \tag{27}$$

In order to see that (27) cannot be the correct dispersion equation for a tape helix, let us consider the zeroth-order approximation to the dispersion equation resulting from a truncation of the infinite-order matrices $\mathbf{A}_\mu^{(S)}$, $\mathbf{A}_\nu^{(S)}$ and $\mathbf{A}_\eta^{(S)}$ to the 1×1 matrices $[\mu_{00}^{(S)}]$, $[\nu_{00}^{(S)}]$ and $[\eta_{00}^{(S)}]$. Since $\nu_{00}^{(S)} \equiv 0$ and $M_{00} = 1$, the dispersion equation corresponding to the above zeroth order truncation becomes

$$k_{0a}^2 I_1(\tau_{0a}) K_1(\tau_{0a}) = 0 \tag{28}$$

Equation (28) has only the trivial solution $k_{0a} \equiv 0$. This circumstance may be traced to the inherent inability of the form of series expansion assumed by Sensiper for the surface current density components to correctly confine the surface current to the region of the tape only. The same drawback persists even in the analysis of the tape-helix model that neglects the transverse component of the tape-current density necessitating an ad hoc assumption regarding the tape-current distribution as a consequence of which it is not possible to satisfy the tangential electric field boundary conditions over the entire width of the tape. On the contrary, when the infinite-order matrices appearing in (26) are truncated to 1×1 matrices, the corresponding zeroth order dispersion equation following from our exact analysis is

$$\nu_{00}^2 - \mu_{00}\eta_{00} = 0 \tag{29}$$

where

$$\mu_{00} = \mu_0 + \sum_{n=1}^{\infty} (\mu_n + \mu_{-n}) \operatorname{sinc}^2 n\hat{w}, \tag{30a}$$

$$\nu_{00} = \sum_{n=1}^{\infty} (\nu_n + \nu_{-n}) \operatorname{sinc}^2 n\hat{w}, \tag{30b}$$

$$\eta_{00} = \eta_0 + \sum_{n=1}^{\infty} (\eta_n + \eta_{-n}) \operatorname{sinc}^2 n\hat{w} \tag{30c}$$

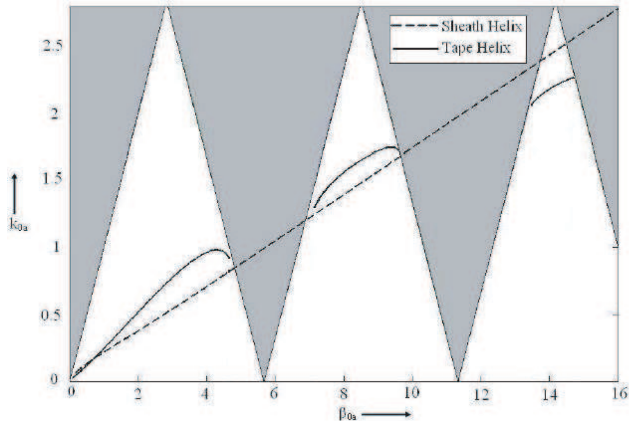

Figure 1. Zeroth order dispersion curve including the transverse-current contribution for $\hat{w} = 1/7$ and $\psi = 10°$.

The zeroth-order dispersion curve for the choice of $\hat{w} = 1/7$ and $\psi = 10°$, when the infinite series for μ_{00}, ν_{00} and η_{00} are truncated at the 6th term (that is, when the contributions from the main lobe of the 'sinc^2' functions only are retained), is plotted in Fig. 1. The plot of Fig. 1, however, should not be misconstrued to be representative of the dispersion characteristics of an open tape helix including the transverse-current distribution since (29) is too crude an approximation to the true dispersion equation for this case. As will be demonstrated in the sequel for the choice of parameter values $\hat{w} = 1/2$ and $\psi = 10°$, the infinite-order matrices $\mathbf{A}_\mu$, $\mathbf{A}_\nu$, and $\mathbf{A}_\eta$, need to be symmetrically truncated to order at least 7×7 in order to get a reasonably good approximation to the dispersion characteristic for the assumed model of the tape helix.

3. NUMERICAL SOLUTION OF THE TRUNCATED DISPERSION EQUATION

As was done in Part I, we resort to a symmetric truncation of the infinite-order matrices $\mathbf{A}_\mu$, $\mathbf{A}_\nu$, and $\mathbf{A}_\eta$, to the $(2N+1) \times (2N+1)$ matrices $\hat{\mathbf{A}}_\mu$, $\hat{\mathbf{A}}_\nu$, and $\hat{\mathbf{A}}_\eta$. Our objective, as for the the anisotropically conducting model, is to study for $\hat{w} = 1/2$, the behavior of the dispersion characteristic with respect to the truncation order N, and deduce the smallest value $\hat{N}$ of N such that there is no perceptible difference between the dispersion curves for $\hat{N}$ and $\hat{N}+1$. It is readily seen from the expressions (22a)–(22c) that only the main lobes of the sinc functions contribute significantly to the values of μ_{kq}, ν_{kq} and η_{kq}. Thus, for the choice of $\hat{w} = 1/2$, the infinite series for them get

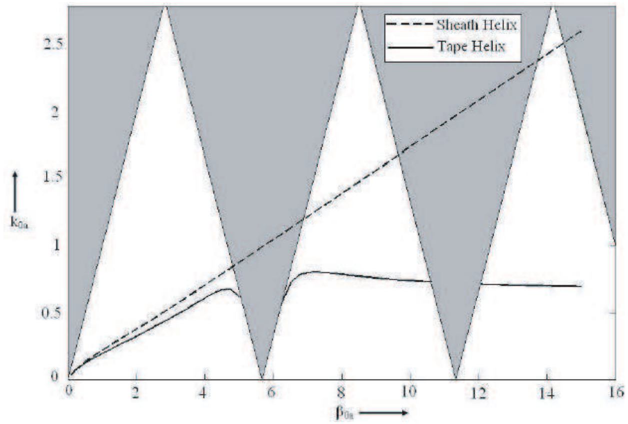

Figure 2. Dispersion characteristic of tape helix for truncation order $N = 1$.

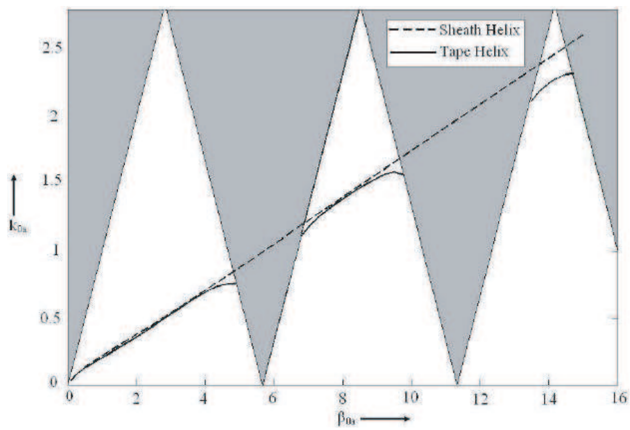

Figure 3. Dispersion characteristic of tape helix for truncation order $N = 2$.

truncated to [1]

$$b_{kq} \underline{\triangle} \hat{\mu}_{kq} = \sum_{n=\max(k,q)-1}^{\min(k,q)+1} \mu_n \text{sinc}\left(\frac{k-n}{2}\right) \text{sinc}\left(\frac{q-n}{2}\right) \quad (31a)$$

$$c_{kq} \underline{\triangle} \hat{\nu}_{kq} = \sum_{n=\max(k,q)-1}^{\min(k,q)+1} \nu_n \text{sinc}\left(\frac{k-n}{2}\right) \text{sinc}\left(\frac{q-n}{2}\right) \quad (31b)$$

$$d_{kq} \underline{\triangle} \hat{\eta}_{kq} = \sum_{n=\max(k,q)-1}^{\min(k,q)+1} \eta_n \text{sinc}\left(\frac{k-n}{2}\right) \text{sinc}\left(\frac{q-n}{2}\right) \quad (31c)$$

When the contributions from the main lobes of the sinc functions only are retained in the expressions for b_{kq}, c_{kq} and d_{kq}, $-N \leq k,\ q \leq N$, there will only be three types of non-zero entries in the $(2N+1) \times (2N+1)$ symmetric matrices $\hat{\mathbf{A}}_\mu$, $\hat{\mathbf{A}}_\nu$, and $\hat{\mathbf{A}}_\eta$ for the choice of $\hat{w} = 1/2$, viz.,

$$\begin{aligned} b_{kk} &= \mu_k + (2/\pi)^2(\mu_{k-1} + \mu_{k+1}) && -N \leq k \leq N, \\ b_{k,k+1} &= b_{k+1,k} = (2/\pi)(\mu_k + \mu_{k+1}) && -N \leq k \leq N-1, \\ b_{k,k+2} &= b_{k+2,k} = (2/\pi)^2 \mu_{k+1} && -N \leq k \leq N-2 \end{aligned}$$

with similar relations for $c_{k,k+i}$ and $d_{k,k+i}$, $i = 0, 1, 2$. Thus, the truncated coefficient matrices $\hat{\mathbf{A}}_\mu$, $\hat{\mathbf{A}}_\nu$, and $\hat{\mathbf{A}}_\eta$ will be banded symmetric matrices with nonzero entries only along the main diagonal and the four symmetrically located subdiagonals adjacent to the main diagonal. Thus, the approximate dispersion equation corresponding to a truncation order equal to N becomes

$$f^{(N)}(k_{0a}; \beta_{0a}) \underline{\triangle} \left| \hat{\mathbf{A}}_\nu - \hat{\mathbf{A}}_\eta \hat{\mathbf{A}}_\nu^{-1} \hat{\mathbf{A}}_\mu \right| = 0 \quad (32)$$

The approximate dispersion Equation (32) is solved numerically on a computer for truncation orders of $N = 0$, 1, 2, 3 and 4, by seeking a real root of (32) for $\hat{k}_{0a}\,(\beta_{0a})$ for various values of $\beta_{0a} \in (0, 16)$, and the resulting family of tape-helix dispersion curves for the choice of the pitch angle $\psi = 10°$ are plotted in Figs. 2–5 for truncation orders 1, 2, 3 and 4 respectively. The dominant-mode dispersion curve of the sheath helix (for the same value of $\psi = 10°$) is also plotted in the figures for comparison. Tape-helix dispersion curve for the truncation order of 0 is not shown because of its limited range of validity. It may be seen from the plots that a truncation order as low as $N = 3$ is adequate to deliver a fairly accurate estimate of the dispersion curve for a tape width-to-pitch ratio of 0.5. However, a fairly large number N of the 'modal amplitudes' $\hat{J}_{zn}$ and $\hat{J}_{\varphi n}$, $0 \leq |n| \leq N$ corresponding to $\hat{\beta}_{0a}(k_{0a})$

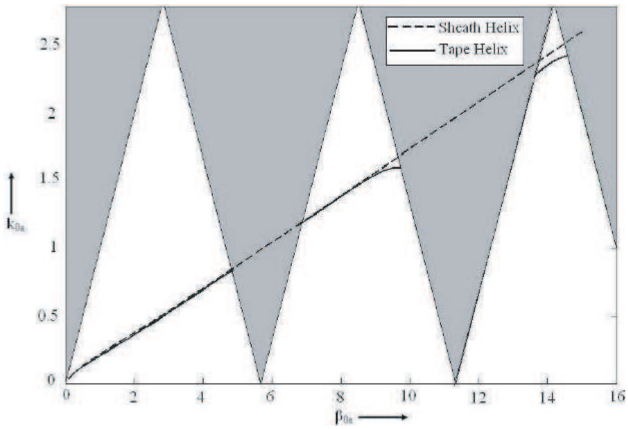

Figure 4. Dispersion characteristic of tape helix for truncation order $N = 3$.

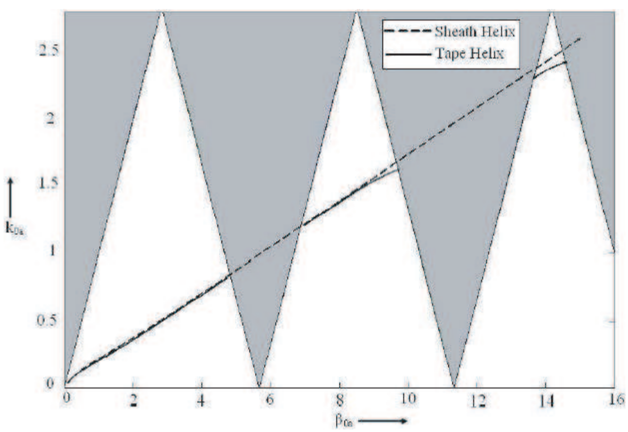

Figure 5. Dispersion characteristic of tape helix for truncation order $N = 4$.

(where $\hat{\beta}_{0a}(k_{0a})$ is the estimate of the normalized propagation constant corresponding to the normalized frequency k_{0a}, that is, the unique root of the equation $\hat{k}_{0a}(\beta_{0a}) = k_{0a}$ for β_{0a}) are needed in the series representation (8) of the surface current density components $J_z(\varphi, z)$ and $J_\varphi(\varphi, z)$ in order to be assured of a reasonably good approximation for the tape-current density, and hence for the electromagnetic field vectors.

Stacking the first $(2N + 1)$ (for N large enough) lowest order coefficients in the expansion (8) of the surface-current density components into the two $(2N+1)$-dimensional vectors $\hat{\mathbf{J}}_m = [\hat{J}_{m\overline{N}}, \hat{J}_{m\overline{(N-1)}}, \ldots, \hat{J}_{m\overline{1}}, \hat{J}_{m0}, \hat{J}_{m1}, \ldots, \hat{J}_{m(N-1)}, \hat{J}_{mN}]^T$, $m = z, \varphi$, the truncated versions of (24) and (25) may be expressed as

$$\hat{\mathbf{J}}_\varphi = \hat{\mathbf{A}}_\nu^{-1}\hat{\mathbf{A}}_\mu\hat{\mathbf{J}}_z \tag{33}$$

and

$$\left[\hat{\mathbf{A}}_\nu - \hat{\mathbf{A}}_\eta\hat{\mathbf{A}}_\nu^{-1}\hat{\mathbf{A}}_\mu\right]\hat{\mathbf{J}}_z = 0 \tag{34}$$

where $\mathbf{0}$ denotes the column vector of $(2N+1)$ zeros. Thus, the task before us is to find the null-space vector of a $(2N+1)\times(2N+1)$ (N large) rank-deficient matrix corresponding the the already located root $\hat{k}_{0a}(\beta_{0a})$ of the determinantal Equation (32). A direct method of doing this is likely to be computationally very intensive because of the large size of the coefficient matrix in (34). Fortunately, there is an alternative computationally more efficient method for iteratively computing the null-space vector.

We begin by recasting the two infinite sets of Equation (21) into a single infinite set of equations with matrix coefficients

$$\sum_{q=-\infty}^{\infty} \boldsymbol{\alpha}_{kq}\mathbf{J}_q = 0, \qquad k \in \mathbb{Z}, \tag{35}$$

for determining the two-component vectors

$$\mathbf{J}_q \triangleq [J_{zq}, J_{\varphi q}]^T, \qquad q \in \mathbb{Z},$$

where the 2×2 matrices $\boldsymbol{\alpha}_{kq}$, k, $q \in \mathbb{Z}$, are given by

$$\boldsymbol{\alpha}_{kq} = \begin{bmatrix} \mu_{kq} & -\nu_{kq} \\ \nu_{kq} & -\eta_{kq} \end{bmatrix}$$

When the infinite set of Equations (35) is symmetrically truncated to $(2N+1)$ equations, and the infinite-series representation for μ_{kq}, ν_{kq} and η_{kq}, $-N \le k$, $q \le N$, is truncated retaining only the contributions from the main lobes of the sinc functions, the truncated versions of (35) go over into

$$\sum_{q=-N}^{N} \mathbf{a}_{kq}\hat{\mathbf{J}}_q = 0, \qquad -N \le k \le N, \tag{36}$$

where

$$\mathbf{a}_{kq} = \hat{\boldsymbol{\alpha}}_{kq} = \begin{bmatrix} b_{kq} & -c_{kq} \\ c_{kq} & -d_{kq} \end{bmatrix}, \qquad -N \le k, q \le N,$$

and the b_{kq}, c_{kq} and d_{kq} are defined in (31). The $(2N+1)$-dimensional vector $\hat{\mathbf{J}}_z$ made up of the $(2N+1)$ first components of the $(2N+1)$ two-component vectors

$$\hat{\mathbf{J}}_q = \left[\hat{J}_{zq}, \quad \hat{J}_{\varphi q}\right]^T, \qquad -N \le q \le N,$$

is of course the sought-after null-space vector of the $(2N+1)\times(2N+1)$ matrix appearing in (34) corresponding to the already determined root $\hat{k}_{0a}(\beta_{0a})$ of the 'Nth order' dispersion Equation (32). The $\hat{\mathbf{J}}_q$, $-N \le q \le N$, make up the first $(2N+1)$ lowest order coefficients in the truncated expansion for the surface current density vector. We now make the key observation that the set of Equations (36) has a form identical to (I36) except for the replacement of the scalar coefficients a_{kq}, $-N \le k$, $q \le N$, by the matrix coefficients $\mathbf{a}_{kq}$, $-N \le k$, $q \le N$, and the replacement of the $(2N+1)$ scalar components $\hat{J}_q$, $-N \le q \le N$, of the null-space vector by the $(2N+1)$ two-component vectors $\hat{\mathbf{J}}_q$, $-N \le q \le N$. Moreover, the $(2N+1)\times(2N+1)$ matrix $[\mathbf{a}_{kq}]_{-N\le k,q\le N}$, whose entries are themselves 2×2 matrices, has the same banded structure as that of the $(2N+1)\times(2N+1)$ matrix $[a_{kq}]_{-N\le k,q\le N}$ of Part I with scalar entries. Thus, it becomes feasible to adapt the recursive method (of successive substitutions and eliminations) employed in Part I to arrive at the approximate dispersion equation to the present context to recursively solve for $\hat{\mathbf{J}}_q$ in terms of $\hat{\mathbf{J}}_{(|q|-1)sgnq}$ and $\hat{\mathbf{J}}_{(|q|-2)sgnq}$, $2 \le |q| \le N$. Both $\hat{\mathbf{J}}_{\bar{1}}$ and $\hat{\mathbf{J}}_1$ are solved in terms of $\hat{\mathbf{J}}_0$. Since $\hat{\mathbf{J}}_0 \underline{\triangle} [\hat{J}_{z0}, \hat{J}_{\varphi 0}]^T$ satisfies an equation of the form

$$\mathbf{a}_{00}^{(N)}\hat{\mathbf{J}}_0 = \mathbf{0}, \tag{37}$$

where the 2×2 coefficient matrix $\mathbf{a}_{00}^{(N)}$ is singular, only the ratio of $\hat{J}_{z0}$ to $\hat{J}_{\varphi 0}$ is fixed by (36). Thus, all of $\hat{\mathbf{J}}_q$, $-N \le q \le N$, are determined modulo an arbitrary (complex) mulitiplicative constant. We now give below the relations required for carrying out the recursive computation of the coefficients $\hat{\mathbf{J}}_q$, $-N \le q \le N$, appearing in the expansion for the surface current density vector. We assume N to be equal to 2 at least.

$$\hat{\mathbf{J}}_{N-i} = -\left(\mathbf{a}_{N-i,N-i}^{(i)}\right)^{-1}\left[\mathbf{a}_{N-i,N-i-1}^{(i)}\hat{\mathbf{J}}_{N-i-1} + \mathbf{a}_{N-i,N-i-2}\hat{\mathbf{J}}_{N-i-2}\right] \quad \text{for } 0 \le i \le N-2, \tag{38}$$

$$\mathbf{a}^{(i)}_{N-i,N-i} = \mathbf{a}^{(i-1)}_{N-i,N-i} - \mathbf{a}^{(i-1)}_{N-i,N-i+1}\left(\mathbf{a}^{(i-1)}_{N-i+1,N-i+1}\right)^{-1}\mathbf{a}^{(i-1)}_{N-i+1,N-i},$$

$$\mathbf{a}^{(i)}_{N-i,N-i-1} = \mathbf{a}_{N-i,N-i-1} - \mathbf{a}^{(i-1)}_{N-i,N-i+1}\left(\mathbf{a}^{(i-1)}_{N-i+1,N-i+1}\right)^{-1}$$
$$\mathbf{a}_{N-i+1,N-i-1},$$

$$\mathbf{a}^{(i)}_{N-i-1,N-i} = \mathbf{a}_{N-i-1,N-i} - \mathbf{a}_{N-i-1,N-i+1}\left(\mathbf{a}^{(i-1)}_{N-i+1,N-i+1}\right)^{-1}$$
$$\mathbf{a}^{(i-1)}_{N-i+1,N-i},$$

$$\mathbf{a}^{(i)}_{N-i-1,N-i-1} = \mathbf{a}_{N-i-1,N-i-1} - \mathbf{a}_{N-i-1,N-i+1}\left(\mathbf{a}^{(i-1)}_{N-i+1,N-i+1}\right)^{-1}$$
$$\mathbf{a}_{N-i+1,N-i-1} \qquad \text{for } 1 \le i \le N-2, \qquad (39)$$

together with a corresponding set of relations with an overbar over the suffices. In (38), (39), and in any of the subsequent formulae, $\mathbf{a}^{(0)}_{kl}$, $-N \le k,\ l \le N$, is to be interpreted simply as $\mathbf{a}_{kl}$. The recursive relation for $\hat{\mathbf{J}}_1$ is

$$\hat{\mathbf{J}}_1 = \mathbf{a}^{(N)}_{10}\hat{\mathbf{J}}_0 = \left[\left(\mathbf{a}^{(N-1)}_{\bar{1}\bar{1}}\right)^{-1}\mathbf{a}_{\bar{1}1} - \mathbf{a}^{-1}_{1\bar{1}}\mathbf{a}^{(N-1)}_{11}\right]^{-1}$$
$$\left[\mathbf{a}^{-1}_{1\bar{1}}\mathbf{a}^{(N-1)}_{10} - \left(\mathbf{a}^{(N-1)}_{\bar{1}\bar{1}}\right)^{-1}\mathbf{a}^{(N-1)}_{\bar{1}0}\right]\hat{\mathbf{J}}_0, \qquad (40a)$$

and the expression for $\mathbf{a}^{(N)}_{\bar{1}0}$ in the recursive relation

$$\hat{\mathbf{J}}_{\bar{1}} = \mathbf{a}^{(N)}_{\bar{1}0}\hat{\mathbf{J}}_0 \qquad (40b)$$

for $\hat{\mathbf{J}}_{\bar{1}}$ is obtained from that for $\mathbf{a}^{(N)}_{10}$ by complementing all the suffixes where $\bar{\bar{1}}\underline{\triangle}1$ and $\bar{0}\underline{\triangle}0$. In the expression for $\mathbf{a}^{(N)}_{10}$

$$\mathbf{a}^{(N-1)}_{10} = \mathbf{a}_{10} - \mathbf{a}^{(N-2)}_{12}\left(\mathbf{a}^{(N-2)}_{22}\right)^{-1}\mathbf{a}_{20},$$
$$\mathbf{a}^{(N-1)}_{11} = \mathbf{a}^{(N-2)}_{11} - \mathbf{a}^{(N-2)}_{12}\left(\mathbf{a}^{(N-2)}_{22}\right)^{-1}\mathbf{a}^{(N-2)}_{21} \qquad (41)$$

together with a corresponding set of relations obtained by complementing the suffixes. Finally, the 2×2 rank-one matrix $\mathbf{a}^{(N)}_{00}$ appearing in (37) is given by

$$\mathbf{a}^{(N)}_{00} = \mathbf{a}_{00} + \mathbf{a}^{(N-1)}_{0\bar{1}}\mathbf{a}^{(N)}_{\bar{1}0} + \mathbf{a}^{(N-1)}_{01}\mathbf{a}^{(N)}_{10}$$
$$- \mathbf{a}_{0\bar{2}}\left(\mathbf{a}^{(N-2)}_{\bar{2}\bar{2}}\right)^{-1}\mathbf{a}_{\bar{2}0} - \mathbf{a}_{02}\left(\mathbf{a}^{(N-2)}_{22}\right)^{-1}\mathbf{a}_{20} \qquad (42)$$

where

$$\mathbf{a}_{0\bar{1}}^{(N-1)} = \mathbf{a}_{0\bar{1}} - \mathbf{a}_{0\bar{2}} \left(\mathbf{a}_{\bar{2}\bar{2}}^{(N-2)}\right)^{-1} \mathbf{a}_{\bar{2}\bar{1}}^{(N-2)} \tag{43a}$$

and

$$\mathbf{a}_{01}^{(N-1)} = \mathbf{a}_{01} - \mathbf{a}_{02} \left(\mathbf{a}_{22}^{(N-2)}\right)^{-1} \mathbf{a}_{21}^{(N-2)} \tag{43b}$$

Denoting the entries of the rank-one matrix $\mathbf{a}_{00}^{(N)}$ by $a_{kl}^{(N)}$, k, $l = 1$, 2, we have

$$\hat{J}_{z0}/\hat{J}_{\varphi 0} = -a_{12}^{(N)}/a_{11}^{(N)} = -a_{22}^{(N)}/a_{21}^{(N)} = \lambda_N \left(\hat{\beta}_{0a}(k_{0a})\right) \text{ (say)}$$

or

$$\hat{J}_{z0} = \hat{\lambda}_N(k_{0a})\hat{J}_{\varphi 0} \tag{44}$$

where $\hat{\lambda}_N(k_{0a}) \underline{\triangle} \lambda_N(\hat{\beta}_{0a}(k_{0a}))$. Working backward from (44) through (40) to (38) with the help of (39), (40), (43) and (42), we can determine all the $2(2N+1)$ coefficients in the truncated expansion of the surface current density components in terms of one undetermined (complex) constant $\hat{J}_{\varphi 0}$.

Before concluding the paper, it seems appropriate to correct a statement made towards the end of Section 2 of Part I regarding the interval of existence for real solutions to (I30). It has been erroneously concluded, on the basis of the particular fixed point format (I32), that the truncated version of (I30) does not possess any real solution for $k_{0a}(\beta_{0a})$ beyond $\beta_{0a} = 1.543$. As a matter of fact, it is possible to extend the interval of existence beyond the stipulated value by resorting to an alternate fixed-point format for the truncated version of the dispersion equation (I30) which may be expressed as

$$\sigma_0 + \sum_{n=1}^{W} \gamma_n(\sigma_n + \sigma_{-n}) = 0 \tag{45}$$

where

$$\gamma_n \underline{\triangle} \operatorname{sinc}^2 n\hat{w}, \qquad n = 1, 2, 3, \ldots$$

Making use of the expressions for σ_0 and σ_n, $|n| \geq 1$, (45) may be written in terms of k_{0a} and β_{0a} as

$$\begin{aligned} &\left[C(\tau_{0a}) + \sum_{n=1}^{W} \gamma_n(C(\tau_{na}) + C(\tau_{-na}))\right] k_{0a}^2 \\ &+ \left[D(\tau_{0a}) + \sum_{n=1}^{W} \gamma_n(D(\tau_{na}) + D(\tau_{-na}))\right] k_{0a}^2 \cot^2 \psi \\ &= \left[C(\tau_{0a}) + \sum_{n=1}^{W} \gamma_n(C(\tau_{na}) + C(\tau_{-na}))\right] \beta_{0a}^2 \end{aligned} \tag{46}$$

where

$$\begin{aligned} C(\tau_{na}) =& I_{|n|}(\tau_{na})K_{|n|}(\tau_{na}), \\ D(\tau_{na}) =& I_{|n|+1}(\tau_{na})K_{|n|+1}(\tau_{na}) \\ &+ (|n|/\tau_{na})\left(I_{|n|}(\tau_{na})K_{|n|-1}(\tau_{na}) - I_{|n|+1}(\tau_{na})K_{|n|}(\tau_{na})\right) \\ &\text{for } |n| \geq 0 \end{aligned}$$

Defining

$$\chi_1^{(W)}(k_{0a};\beta_{0a}) \triangleq C(\tau_{0a}) + \sum_{n=1}^{W} \gamma_n (C(\tau_{na}) + C(\tau_{-na}))$$

and

$$\chi_2^{(W)}(k_{0a};\beta_{0a}) \triangleq D(\tau_{0a}) + \sum_{n=1}^{W} \gamma_n (D(\tau_{na}) + D(\tau_{-na})),$$

Equation (46) may be put in the alternate fixed-point format

$$\begin{aligned} k_{0a} &= G_W^{(0)}(k_{0a};\beta_{0a}) \\ &\triangleq \left\{\beta_{0a}^2 / \left(1 + \chi_2^{(W)}(k_{0a};\beta_{0a}) \cot^2\psi / \chi_1^{(W)}(k_{0a};\beta_{0a})\right)\right\}^{1/2} \end{aligned} \quad (47)$$

Equation (47) may be solved numerically for $k_{0a}(\beta_{0a})$, $\beta_{0a} > 0$, for the choice of $\hat{w} = 0.1/\pi$, by the method of successive substitutions to find any fixed point of the 'operator' for k_{0a} in the range $0 < k_{0a} < (1/2)\cot 10°$. The resulting dispersion curve is plotted in Fig. 6.

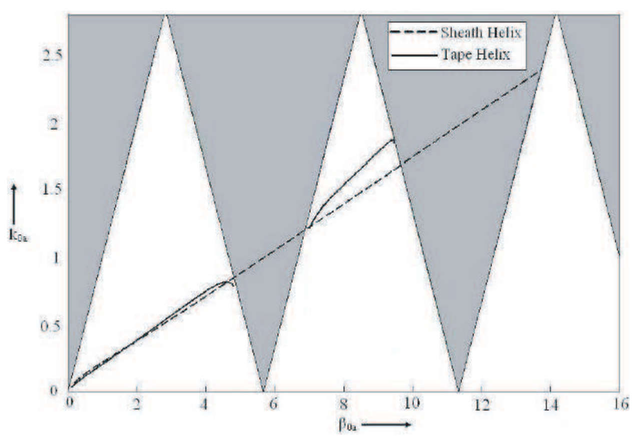

Figure 6. Zeroth order dispersion curve excluding transverse-current contribution for $\hat{w} = 0.1/\pi$ and $\psi = 10°$.

together with the dominant-mode dispersion curve of the sheath helix. It may be seen from Fig. 6. that a real solution of the dispersion equation exists for values of β_{0a} up to 9.45 in the complement of the forbidden regions. However, the tape-helix dispersion curve (for the truncation order $N = 0$) crosses the sheath helix dispersion curve from below around a value of $\beta_{0a} = 1.8$ and stays above it for the remaining values of β_{0a} in the allowed regions except for a second crossing from above at around $\beta_{0a} = 4.5$ near the first forbidden-region boundary. It may therefore be inferred that the behavior of the 'zeroth-order' tape-helix dispersion curve of Fig. 6. may not be acceptable beyond $\beta_{0a} = 1.8$ notwithstanding the fact that a real root $k_{0a}(\beta_{0a})$ of the dispersion Equation (47) is available for larger values of β_{0a}. This is again a confirmation of the already established fact that the zeroth order ($N = 0$) truncation of the tape-helix dispersion equation is too crude to reveal the true nature of the dispersion characteristics.

4. CONCLUSION

In this paper, we have demonstrated the feasibility of an exact analysis of guided electromagnetic wave propagation through an open tape helix including the effect of the transverse component of the tape-current density. The main conclusion that may be drawn from the analysis is that the tape-current density component perpendicular to the winding direction does not affect the dispersion characteristic to any significant extent except for a small decrease in the phase speed of the cold wave supported by the helix towards the high-frequency end.

Work on the extension of analysis presented in both parts of this paper to a full field analysis of the practically important case of a dielectric-loaded tape-helix enclosed in a coaxial perfectly conducting cylindrical shell is currently in progress and will be reported in due course.

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