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WEAKLY CONDITIONALLY STABLE AND UNCONDITIONALLY STABLE FDTD SCHEMES FOR 3D MAXWELL'S EQUATIONS

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Abstract—To overcome the Courant limit on the time step size of the conventional finite-difference time-domain (FDTD) method, some weakly conditionally stable and unconditionally stable FDTD methods have been developed recently. To analyze the relations between these methods theoretically, they are all viewed as approximations of the conventional FDTD scheme in present discussion. The errors between these methods and the conventional FDTD method are presented analytically, and the numerical performances, including computation accuracy, efficiency, and memory requirements, are discussed, by comparing with those of the conventional FDTD method.

1. INTRODUCTION

The finite-difference time-domain (FDTD) method [1–4] has been proven to be an effective means that provides accurate predictions of field behaviors for varieties of electromagnetic interaction problems. However, as it is based on an explicit finite-difference algorithm, the Courant-Friedrich-Levy (CFL) condition [5] must be satisfied when this method is used. Therefore, a maximum time-step size is limited by minimum cell size in a computational domain, which makes this method inefficient for the problems where fine scale dimensions are used.

To remove the CFL constraint on the time step size of the FDTD method, some unconditionally stable methods such as the alternating-direction implicit (ADI) FDTD [6–10] scheme and Crank-Nicolson

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(CN) FDTD [11–16] method have been studied extensively. Although the time step size in the ADI-FDTD simulation is no longer bounded by the Courant-Friedrich-Levy criterion, the method exhibits a splitting error associated with the square of the time step size [17, 18], which limits the accuracy of the ADI-FDTD method. The CN-FDTD scheme is believed to have higher numerical accuracy than the ADI-FDTD method, but with a huge spare irreducible matrix. Directly solving this matrix by Gaussian elimination or an iterative method is so CPU intensive that the CN scheme is hardly usable for practical problems.

To overcome the above drawbacks, the hybrid implicit-explicit (HIE) FDTD method [19–27] has been developed recently. In this method, the CFL constraint is not removed totally, but being weaker than that of conventional FDTD method. The time step size of the HIE-FDTD method is only determined by two space discretizations, which is useful for problems with very fine structures in one direction [24]. The HIE-FDTD method has better accuracy and higher computation efficiency than the ADI-FDTD method, especially for larger field variation [21]. While maintain the same time step size, the CPU time for the HIE–FDTD method can be reduced to about 1/2 of that for the ADI-FDTD method [21].

Although the HIE-FDTD method has higher accuracy and efficiency than the ADI-FDTD method, it is with confined usage, because the time step size in this method is limited by two space discretizations, which makes it only useful for the problems with very fine structures in one direction. To solve the electromagnetic simulation with fine scale dimensions in two directions, the weakly conditionally stable (WCS) FDTD method is developed [28–32]. The time step size in this method is only determined by one space discretization. The stability condition of this method becomes weaker further compared with that of the HIE-FDTD method, and the computation accuracy and efficiency of this method are better than those of the ADI-FDTD method yet [31].

The HIE-FDTD method, WCS-FDTD method and ADI-FDTD method are all referred as fast FDTD scheme which is an integrated system with the conventional FDTD method. But what are the relations between these methods, and how could the time step size affect the accuracy of these methods? To the knowledge of the authors, there are no related works.

The present discussion starts with the Maxwell's equations, and then presents the basic formulation of the conventional FDTD method. Based on different transformations of the equations of the conventional FDTD method, the formulations of HIE-FDTD, WCS-FDTD, and ADI-FDTD method are explored, respectively. In such a case, the relations between these methods are inferred theoretically, and the approximation errors between these methods and the conventional FDTD method are presented analytically. The numerical performances of these methods, including computation accuracy, efficiency, and memory requirements, are discussed by comparing with those of the conventional FDTD method, and the conclusions are demonstrated by numerical examples.

2. FORMULATIONS

For a medium with permittivity ε and permeability μ, assuming no free charges or currents, the 3D Maxwell's equations can be written,

$$\left.\begin{array}{l}\frac{\partial E_x}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\\ \frac{\partial E_y}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\\ \frac{\partial E_z}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\end{array}\right\}\quad \left.\begin{array}{l}\frac{\partial H_x}{\partial t}=-\frac{1}{\mu}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\\ \frac{\partial H_y}{\partial t}=-\frac{1}{\mu}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\\ \frac{\partial H_z}{\partial t}=-\frac{1}{\mu}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\end{array}\right\} \tag{1}$$

where E_x, E_y, E_z, H_x, H_y, and H_z denote the components of the electric field E and magnetic field H.

2.1. Formulations of Conventional FDTD Method

In Eq. (1), approximating each derivative in time by finite difference, the formulation of the conventional FDTD method is obtained,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & -\frac{\Delta t\partial}{\mu\partial z} & \frac{\Delta t\partial}{\mu\partial y} & 1 & 0 & 0\\ \frac{\Delta t\partial}{\mu\partial z} & 0 & -\frac{\Delta t\partial}{\mu\partial x} & 0 & 1 & 0\\ -\frac{\Delta t\partial}{\mu\partial y} & \frac{\Delta t\partial}{\mu\partial x} & 0 & 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} E_x^{n+1}\\ E_y^{n+1}\\ E_z^{n+1}\\ H_x^{n+1}\\ H_y^{n+1}\\ H_z^{n+1}\end{bmatrix}$$
$$=\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{\Delta t\partial}{\varepsilon\partial z} & \frac{\Delta t\partial}{\varepsilon\partial y}\\ 0 & 1 & 0 & \frac{\Delta t\partial}{\varepsilon\partial z} & 0 & -\frac{\Delta t\partial}{\varepsilon\partial x}\\ 0 & 0 & 1 & -\frac{\Delta t\partial}{\varepsilon\partial y} & \frac{\Delta t\partial}{\varepsilon\partial x} & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} E_x^{n}\\ E_y^{n}\\ E_z^{n}\\ H_x^{n}\\ H_y^{n}\\ H_z^{n}\end{bmatrix} \tag{2}$$

where n and Δt are the index and size of time-step.

For simplicity, we define matrices $A_{p1,2}$, $B_{p1,2}$, $p = x, y, z$, as follows

$$[A_{x1}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\mu\partial x} & 0 & 0 & 0 & 0 \end{bmatrix} [A_{y1}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\mu\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A_{z1}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\mu\partial z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} [B_{x1}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\mu\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B_{y1}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\mu\partial y} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad [B_{z1}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\mu\partial z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A_{x2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\partial}{\varepsilon\partial x} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} [A_{y2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\varepsilon\partial y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A_{z2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\varepsilon\partial z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} [B_{x2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\varepsilon\partial x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B_{y2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\varepsilon\partial y} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} [B_{z2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\varepsilon\partial z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, Eq. (2) is equivalent to the following:

$$\begin{aligned}&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x1}]+[A_{y1}]+[A_{y1}]+[A_{z1}]+[A_{z1}]\right)\right.\\&\left.-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x1}]+[B_{y1}]+[B_{y1}]+[B_{z1}]+[B_{z1}]\right)\right)U^{n+1}\\=&\left(I+\frac{\Delta t}{2}\left([A_{x2}]+[A_{x2}]+[A_{y2}]+[A_{y2}]+[A_{z2}]+[A_{z2}]\right)\right.\\&\left.+\frac{\Delta t}{2}\left([B_{x2}]+[B_{x2}]+[B_{y2}]+[B_{y2}]+[B_{z2}]+[B_{z2}]\right)\right)U^{n}\end{aligned}\tag{3}$$

with,

$$\begin{aligned}U^{n+1}&=\begin{bmatrix}E_x^{n+1} & E_y^{n+1} & E_z^{n+1} & H_x^{n+1} & H_y^{n+1} & H_z^{n+1}\end{bmatrix}',\\U^{n}&=\begin{bmatrix}E_x^{n} & E_y^{n} & E_z^{n} & H_x^{n} & H_y^{n} & H_z^{n}\end{bmatrix}',\end{aligned}$$

I is 6×6 identity matrix. It is noted that Eq. (3) is very important, because all the methods discussed later are based on the transformations of this equation.

In Eq. (2), approximating each derivative in space by centered second-order finite differences, we obtain,

$$\begin{aligned}&E_x^{n+1}(i+1/2,j,k)=E_x^{n}(i+1/2,j,k)\\&+\frac{\Delta t}{\varepsilon}\left[\frac{H_z^n(i+1/2,j+1/2,k)-H_z^n(i+1/2,j-1/2,k)}{\Delta y}\right.\\&\left.-\frac{H_y^n(i+1/2,j,k+1/2)-H_y^n(i+1/2,j,k-1/2)}{\Delta z}\right]\end{aligned}\tag{4a}$$

$$\begin{aligned}&E_y^{n+1}(i,j+1/2,k)=E_y^{n}(i,j+1/2,k)\\&+\frac{\Delta t}{\varepsilon}\left[\frac{H_x^n(i,j+1/2,k+1/2)-H_x^n(i,j+1/2,k-1/2)}{\Delta z}\right.\\&\left.-\frac{H_z^n(i+1/2,j+1/2,k)-H_z^n(i-1/2,j+1/2,k)}{\Delta x}\right]\end{aligned}\tag{4b}$$

$$\begin{aligned}&E_z^{n+1}(i,j,k+1/2)=E_z^{n}(i,j,k+1/2)\\&+\frac{\Delta t}{\varepsilon}\left[\frac{H_y^n(i+1/2,j,k+1/2)-H_y^n(i-1/2,j,k+1/2)}{\Delta x}\right.\\&\left.-\frac{H_x^n(i,j+1/2,k+1/2)-H_x^n(i,j-1/2,k+1/2)}{\Delta y}\right]\end{aligned}\tag{4c}$$

$$H_x^{n+1}(i,j+1/2,k+1/2) = H_x^{n}(i,j+1/2,k+1/2) - \frac{\Delta t}{\mu}\left[\frac{E_z^{n+1}(i,j+1,k+1/2)-E_z^{n+1}(i,j,k+1/2)}{\Delta y} - \frac{E_y^{n+1}(i,j+1/2,k+1)-E_y^{n+1}(i,j+1/2,k)}{\Delta z}\right] \tag{4d}$$

$$H_y^{n+1}(i+1/2,j,k+1/2) = H_y^{n}(i+1/2,j,k+1/2) - \frac{\Delta t}{\mu}\left[\frac{E_x^{n+1}(i+1/2,j,k+1)-E_x^{n+1}(i+1/2,j,k)}{\Delta z} - \frac{E_z^{n+1}(i+1,j,k+1/2)-E_z^{n+1}(i,j,k+1/2)}{\Delta x}\right] \tag{4e}$$

$$H_z^{n+1}(i+1/2,j+1/2,k) = H_z^{n}(i+1/2,j+1/2,k) - \frac{\Delta t}{\mu}\left[\frac{E_y^{n+1}(i+1,j+1/2,k)-E_y^{n+1}(i,j+1/2,k)}{\Delta x} - \frac{E_x^{n+1}(i+1/2,j+1,k)-E_x^{n+1}(i+1/2,j,k)}{\Delta y}\right] \tag{4f}$$

here, i, j, and k denote the indices of spatial increments; Δx, Δy and Δz are the spatial increments respectively in x, y and z directions.

Equation (4) is the conventional Yee's FDTD method. It is noted that the forward differencing in time is employed here instead of central differencing. This is purely mathematical license and the actually implemented codes maintain central time differencing. This discrepancy doesn't affect the time-marching process. It is also a recursive time-marching algorithm where the field solution at the current time step is deduced from the field values calculated previously. So the CFL condition [5]:

$$\Delta t \le 1\Big/\left(c\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2 + (1/\Delta z)^2}\right) \tag{5}$$

must be satisfied when this method is used, here, $c = 1/\sqrt{\varepsilon\mu}$ is the speed of light in the medium. Therefore, a maximum time-step size is limited by minimum cell size in the computational domain, which means that if an object of analysis has fine scale dimensions compared with wavelength, calculation time will be increased due to the small time-step size, making this method inefficient for the problems where fine scale dimensions are used.

To overcome the above drawbacks fast FDTD schemes, including HIE-FDTD, WCS-FDTD, and ADI-FDTD methods in which the restraint of the CFL condition is eliminated or relaxed are developed.

2.2. Formulation of the HIE-FDTD Method

Write Eq. (3) in a new form:

$$\begin{aligned}&\left(\begin{array}{l} I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x1}]+[A_{y1}]+[A_{y2}]+[A_{z1}]+[A_{z1}]\right) \\ -\frac{\Delta t}{2}\left([B_{x1}]+[B_{x1}]+[B_{y1}]+[B_{y2}]+[B_{z1}]+[B_{z1}]\right)\end{array}\right)U^{n+1}\\ =&\left(\begin{array}{l} I+\frac{\Delta t}{2}\left([A_{x2}]+[A_{x2}]+[A_{y1}]+[A_{y2}]+[A_{z2}]+[A_{z2}]\right) \\ +\frac{\Delta t}{2}\left([B_{x2}]+[B_{x2}]+[B_{y1}]+[B_{y2}]+[B_{z2}]+[B_{z2}]\right)\end{array}\right)U^{n}\\ &+\frac{\Delta t}{2}\left(E_1\right)\left(U^{n+1}-U^{n}\right)\end{aligned} \tag{6}$$

with,

$$[E_1]=\left(\begin{array}{l}[A_{y1}]-[A_{y2}] \\ +[B_{y1}]-[B_{y2}]\end{array}\right)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{\partial}{\varepsilon\partial y} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\varepsilon\partial y} & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\mu\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\mu\partial y} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Ignore the last term in Eq. (6), we get

$$\begin{aligned}&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x1}]+[A_{y1}]+[A_{y2}]+[A_{z1}]+[A_{z1}]\right)\right.\\ &\left.-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x1}]+[B_{y1}]+[B_{y2}]+[B_{z1}]+[B_{z1}]\right)\right)U^{n+1}\\ =&\left(I+\frac{\Delta t}{2}\left([A_{x2}]+[A_{x2}]+[A_{y1}]+[A_{y2}]+[A_{z2}]+[A_{z2}]\right)\right.\\ &\left.+\frac{\Delta t}{2}\left([B_{x2}]+[B_{x2}]+[B_{y1}]+[B_{y2}]+[B_{z2}]+[B_{z2}]\right)\right)U^{n}\end{aligned} \tag{7}$$

Eq. (7) is a transformation of (3). It introduces an error, of the form

$$\frac{\Delta t}{2}\left(E_1\right)\left(U^{n+1}-U^{n}\right) \tag{8}$$

to the solution. The effect of this error is proportional to the time step size and the spatial variation rate of field.

Considering the expression of matrices A and B, Eq. (7) is equivalent to the following:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -\frac{\Delta t\partial}{2\varepsilon\partial y} \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{\Delta t\partial}{2\varepsilon\partial y} & 0 & 0 \\
0 & -\frac{\Delta t\partial}{\mu\partial z} & \frac{\Delta t\partial}{2\mu\partial y} & 1 & 0 & 0 \\
\frac{\Delta t\partial}{\mu\partial z} & 0 & -\frac{\Delta t\partial}{\mu\partial x} & 0 & 1 & 0 \\
-\frac{\Delta t\partial}{2\mu\partial y} & \frac{\Delta t\partial}{\mu\partial x} & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{bmatrix} E_x^{n+1} \\ E_y^{n+1} \\ E_z^{n+1} \\ H_x^{n+1} \\ H_y^{n+1} \\ H_z^{n+1} \end{bmatrix}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 & -\frac{\Delta t\partial}{\varepsilon\partial z} & \frac{\Delta t\partial}{2\varepsilon\partial y} \\
0 & 1 & 0 & \frac{\Delta t\partial}{\varepsilon\partial z} & 0 & -\frac{\Delta t\partial}{\varepsilon\partial x} \\
0 & 0 & 1 & -\frac{\Delta t\partial}{2\varepsilon\partial y} & \frac{\Delta t\partial}{\varepsilon\partial x} & 0 \\
0 & 0 & -\frac{\Delta t\partial}{2\mu\partial y} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\frac{\Delta t\partial}{2\mu\partial y} & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{bmatrix} E_x^{n} \\ E_y^{n} \\ E_z^{n} \\ H_x^{n} \\ H_y^{n} \\ H_z^{n} \end{bmatrix} \tag{9}
$$

Approximating each derivative in space by centered second-order finite differences in Eq. (9), it is obtained:

$$
\begin{aligned}
&E_y^{n+1}\left(i, j+\frac{1}{2}, k\right) = E_y^{n}\left(i, j+\frac{1}{2}, k\right) \\
&+\frac{\Delta t}{\varepsilon\Delta z}\left[H_x^n\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) - H_x^n\left(i, j+\frac{1}{2}, k-\frac{1}{2}\right)\right] \\
&-\frac{\Delta t}{\varepsilon\Delta x}\left[H_z^n\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right) - H_z^n\left(i-\frac{1}{2}, j+\frac{1}{2}, k\right)\right]
\end{aligned} \tag{10a}
$$

$$
\begin{aligned}
&E_x^{n+1}\left(i+\frac{1}{2}, j, k\right) = E_x^{n}\left(i+\frac{1}{2}, j, k\right) \\
&-\frac{\Delta t}{\varepsilon\Delta z}\left[H_y^n\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right) - H_y^n\left(i+\frac{1}{2}, j, k-\frac{1}{2}\right)\right] \\
&+\frac{\Delta t}{2\varepsilon\Delta y}\left[H_z^{n+1}\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right) - H_z^{n+1}\left(i+\frac{1}{2}, j-\frac{1}{2}, k\right)\right. \\
&\left.+H_z^n\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right) - H_z^n\left(i+\frac{1}{2}, j-\frac{1}{2}, k\right)\right]
\end{aligned} \tag{10b}
$$

$$H_z^{n+1}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)=H_z^n\left(i+\frac{1}{2},j+\frac{1}{2},k\right)$$
$$-\frac{\Delta t}{\mu\Delta x}\left[E_y^{n+1}\left(i+1,j+\frac{1}{2},k\right)-E_y^{n+1}\left(i,j+\frac{1}{2},k\right)\right]$$
$$+\frac{\Delta t}{2\mu\Delta y}\left[E_x^{n+1}\left(i+\frac{1}{2},j+1,k\right)-E_x^{n+1}\left(i+\frac{1}{2},j,k\right)\right.$$
$$\left.+E_x^n\left(i+\frac{1}{2},j+1,k\right)-E_x^n\left(i+\frac{1}{2},j,k\right)\right] \tag{10c}$$

$$E_z^{n+1}\left(i,j,k+\frac{1}{2}\right)=E_z^n\left(i,j,k+\frac{1}{2}\right)$$
$$+\frac{\Delta t}{\varepsilon\Delta x}\left[H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^n\left(i-\frac{1}{2},j,k+\frac{1}{2}\right)\right]$$
$$-\frac{\Delta t}{2\varepsilon\Delta y}\left[H_x^{n+1}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^{n+1}\left(i,j-\frac{1}{2},k+\frac{1}{2}\right)\right.$$
$$\left.+H_x^n\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^n\left(i,j-\frac{1}{2},k+\frac{1}{2}\right)\right] \tag{10d}$$

$$H_x^{n+1}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)=H_x^n\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)$$
$$+\frac{\Delta t}{\mu\Delta z}\left[E_y^{n+1}\left(i,j+\frac{1}{2},k+1\right)-E_y^{n+1}\left(i,j+\frac{1}{2},k\right)\right]$$
$$-\frac{\Delta t}{2\mu\Delta y}\left[E_z^{n+1}\left(i,j+1,k+\frac{1}{2}\right)-E_z^{n+1}\left(i,j,k+\frac{1}{2}\right)\right.$$
$$\left.+E_z^n\left(i,j+1,k+\frac{1}{2}\right)-E_z^n\left(i,j,k+\frac{1}{2}\right)\right] \tag{10e}$$

$$H_y^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)=H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)$$
$$-\frac{\Delta t}{\mu\Delta z}\left[E_x^{n+1}\left(i+\frac{1}{2},j,k+1\right)-E_x^{n+1}\left(i+\frac{1}{2},j,k\right)\right]$$
$$+\frac{\Delta t}{\mu\Delta x}\left[E_z^{n+1}\left(i+1,j,k+\frac{1}{2}\right)-E_z^{n+1}\left(i,j,k+\frac{1}{2}\right)\right] \tag{10f}$$

Equation (10) is the basic formulations of the HIE-FDTD method. Obviously, the HIE-FDTD method is an approximation scheme of the conventional FDTD method, and the approximation error is expressed by Eq. (8).

Updating of the E_x component, as shown in Eq. (10b), needs the unknown H_z component at the same time, thus the E_x component has to be updated implicitly. By substituting (10c) into (10b), the equation for E_x field is given as

$$\begin{aligned}
&[1+2\lambda_y]E_x^{n+1}(i+\frac{1}{2},j,k)-\lambda_y\left[E_x^{n+1}\left(i+\frac{1}{2},j+1,k\right)+E_x^{n+1}\left(i+\frac{1}{2},j-1,k\right)\right]\\
&=[1-2\lambda_y]E_x^{n}\left(i+\frac{1}{2},j,k\right)+\lambda_y\left[E_x^{n}\left(i+\frac{1}{2},j+1,k\right)+E_x^{n}\left(i+\frac{1}{2},j-1,k\right)\right]\\
&+\frac{\Delta t}{\varepsilon\Delta y}\left[H_z^n\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^n\left(i+\frac{1}{2},j-\frac{1}{2},k\right)\right]\\
&-\frac{\Delta t}{\varepsilon\Delta z}\left[H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^n\left(i+\frac{1}{2},j,k-\frac{1}{2}\right)\right]\\
&-\frac{\Delta t^2}{2\mu\varepsilon\Delta x\Delta y}\left[E_y^{n+1}\left(i+1,j+\frac{1}{2},k\right)-E_y^{n+1}\left(i,j+\frac{1}{2},k\right)\right.\\
&\left.-E_y^{n+1}\left(i+1,j-\frac{1}{2},k\right)+E_y^{n+1}\left(i,j-\frac{1}{2},k\right)\right] \qquad (11)
\end{aligned}$$

where, $\lambda_y = \Delta t^2 / 4\varepsilon\mu\Delta y^2$.

In the same way, updating of the E_z component needs the unknown H_x component at the same time step. By substituting (10e) into (10d), the equation for E_z field is given as:

$$\begin{aligned}
&[1+2\lambda_y]E_z^{n+1}(i,j,k+\frac{1}{2})-\lambda_y\left[E_z^{n+1}\left(i,j+1,k+\frac{1}{2}\right)+E_z^{n+1}\left(i,j-1,k+\frac{1}{2}\right)\right]\\
&=[1-2\lambda_y]E_z^{n}\left(i,j,k+\frac{1}{2}\right)+\lambda_y\left[E_z^{n}\left(i,j+1,k+\frac{1}{2}\right)+E_z^{n}\left(i+\frac{1}{2},j-1,k\right)\right]\\
&-\frac{\Delta t}{\varepsilon\Delta y}\left[H_x^n\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^n\left(i,j-\frac{1}{2},k+\frac{1}{2}\right)\right]\\
&+\frac{\Delta t}{\varepsilon\Delta x}\left[H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^n\left(i-\frac{1}{2},j,k+\frac{1}{2}\right)\right]\\
&-\frac{\Delta t^2}{2\mu\varepsilon\Delta z\Delta y}\left[E_y^{n+1}\left(i,j+\frac{1}{2},k+1\right)-E_y^{n+1}\left(i,j+\frac{1}{2},k\right)\right.\\
&\left.-E_y^{n+1}\left(i,j-\frac{1}{2},k+1\right)+E_y^{n+1}\left(i,j-\frac{1}{2},k\right)\right] \qquad (12)
\end{aligned}$$

Therefore, updating of field components will be achieved by using Eqs. (10a), (11), (10c), (10), (10e) and (10f). The field component E_y is explicitly updated first. Components E_x and E_z are updated implicitly by solving the tridiagonal matrix equations through Eqs. (11) and (12). After the E_x and E_z are obtained at each time step, the components H_z, H_x and H_y can be explicitly updated straightforward by using

Eqs. (10c), (10e) and (10f). Thus at each time step, two tridiagonal matrices and four explicit update are needed for the field development.

We now demonstrated the weakly conditional stability of this HIE-FDTD method. Without loss of generality, the field components can be written as follows

$$\phi_w^n(x,y,z) = \varphi_{\phi w}\zeta^n f(x,y,z) \tag{13a}$$

$$f(x,y,z) = \exp(\overleftarrow{j} k_x x + \overleftarrow{j} k_y y + \overleftarrow{j} k_z z) \tag{13b}$$

where ϕ denotes E or H, $w = x,y,z$, $\overleftarrow{j} = \sqrt{-1}$. k_x, k_y, and k_z are wave numbers. ζ indicates growth factor. $\varphi_{\phi w}$ is the amplitude of the field components. Substituting these expressions into Eq. (9), approximating each derivative in space by centered second-order finite differences, the matrix becomes:

$$\begin{pmatrix} \frac{(\zeta-1)}{a} & 0 & 0 & 0 & \sigma_z & -\frac{\sigma_y(\zeta+1)}{2} \\ 0 & \frac{(\zeta-1)}{a} & 0 & -\sigma_z & 0 & \sigma_x \\ 0 & 0 & \frac{(\zeta-1)}{a} & \frac{\sigma_y(\zeta+1)}{2} & -\sigma_x & 0 \\ 0 & -\sigma_z\zeta & \frac{\sigma_y(\zeta+1)}{2} & \frac{(\zeta-1)}{b} & 0 & 0 \\ \sigma_z\zeta & 0 & -\sigma_x\zeta & 0 & \frac{(\zeta-1)}{b} & 0 \\ -\frac{\sigma_y(\zeta+1)}{2} & \sigma_x\zeta & 0 & 0 & 0 & \frac{(\zeta-1)}{b} \end{pmatrix} [U^n] = 0 \tag{14}$$

where, $a = \Delta t/\varepsilon$, $b = \Delta t/\mu$, $\sigma_w = \overleftarrow{j}\sin\left(\frac{k_w \Delta w}{2}\right)\Big/\left(\frac{\Delta w}{2}\right)$, $w = x,y,z$.

For a nontrivial solution of (14), the determinant of the coefficient matrix in (14) should be zero. With some manipulations, the amplification factor of this scheme can be obtained as

$$\zeta_{1,2} = 1 \tag{15}$$

$$\zeta_{3,4} = \zeta_{5,6} = \frac{\left(1+ab\sigma_y^2/4+2M\right) \pm \sqrt{\left(1+ab\sigma_y^2/4+2M\right)^2 - \left(1-ab\sigma_y^2/4\right)^2}}{\left(1-ab\sigma_y^2/4\right)} \tag{16}$$

where, $M = ab\sigma_x^2/4 + ab\sigma_z^2/4$.

To satisfy the stability condition during field advancement, the module of growth factor ζ can't be larger than 1. It is evident that the module of ζ_{12} is unity. For the values of ζ_{34} and ζ_{56}, when the condition $1 + M \geq 0$ is satisfied, $|\zeta_{34}| = 1$ and $|\zeta_{56}| = 1$ can be obtained. Thus, the limitation for time-step size can be calculated as follows

$$(\sin(k_x\Delta x/2)/\Delta x)^2 + (\sin(k_z\Delta z/2)/\Delta z)^2 \leq (1/\Delta x)^2 + (1/\Delta z)^2 \leq 1/ab$$
$$\Rightarrow \Delta t \leq 1\Big/c\sqrt{(1/\Delta x)^2 + (1/\Delta z)^2} \tag{17}$$

This scheme is conditionally stable. The stability condition is weaker than that of the conventional FDTD method. The time step in this method is only determined by two space discretizations Δx and Δz. This method is useful for problems with very fine structures in one direction, such as the simulations of planar structures of patch antennas [24]

If we write Eq. (3) in new forms:

$$\begin{aligned}&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y1}]+[A_{y1}]+[A_{z1}]+[A_{z1}]\right)\right.\\&\left.-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y1}]+[B_{y1}]+[B_{z1}]+[B_{z1}]\right)\right)U^{n+1}\\=&\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y2}]+[A_{y2}]+[A_{z2}]+[A_{z2}]\right)\right.\\&\left.+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y2}]+[B_{y2}]+[B_{z2}]+[B_{z2}]\right)\right)U^{n} \quad (18)\end{aligned}$$

or,

$$\begin{aligned}&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x1}]+[A_{y1}]+[A_{y1}]+[A_{z1}]+[A_{z2}]\right)\right.\\&\left.-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x1}]+[B_{y1}]+[B_{y1}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n+1}\\=&\left(I+\frac{\Delta t}{2}\left([A_{x2}]+[A_{x2}]+[A_{y2}]+[A_{y2}]+[A_{z1}]+[A_{z2}]\right)\right.\\&\left.+\frac{\Delta t}{2}\left([B_{x2}]+[B_{x2}]+[B_{y2}]+[B_{y2}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n} \quad (19)\end{aligned}$$

the weakly conditional stability not related with the space discretization Δx (or Δz) will be obtained, which can be demonstrated by following same analysis of Eq. (17).

It is noted that there exists some discrepancy between the formulations of the HIE-FDTD method presented here and those in Ref. [24]. The components E_y and H_y are all defined at time steps $n+1/2$ in Ref. [24], and are all updated explicitly before the solving of the tridiagonal matrix equations. However, this discrepancy doesn't affect the time-marching process and the weakly conditional stability of this method. It is only the trivial difference on the expression.

2.3. Formulations of WCS-FDTD Methods

Equation (3) can also be written as:

$$\begin{aligned}
&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y1}]+[A_{y1}]+[A_{z1}]+[A_{z2}]\right)\right.\\
&\left.-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y1}]+[B_{y1}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n+1}\\
=&\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y2}]+[A_{y2}]+[A_{z1}]+[A_{z2}]\right)\right.\\
&\left.+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y2}]+[B_{y2}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n}\\
&+\frac{\Delta t}{2}\left([A_{x1}]-[A_{x2}]+[B_{x1}]-[B_{x2}]+[A_{z1}]-[A_{z2}]+[B_{z1}]-[B_{z2}]\right)\\
&\left(U^{n+1}-U^{n}\right)
\end{aligned}\tag{20}$$

Ignore the last term in (20), we obtain:

$$\begin{aligned}
&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y1}]+[A_{y1}]+[A_{z1}]+[A_{z2}]\right)\right.\\
&\left.-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y1}]+[B_{y1}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n+1}\\
=&\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y2}]+[A_{y2}]+[A_{z1}]+[A_{z2}]\right)\right.\\
&\left.+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y2}]+[B_{y2}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n}
\end{aligned}\tag{21}$$

Eq. (21) also introduces an error, of the form

$$\left(\Delta t\,[E_2]\left(U^{n+1}-U^{n}\right)\right)/2 \tag{22}$$

to the solution, where:

$$\begin{aligned}
[E_2]&=\begin{pmatrix}[A_{x1}]-[A_{x2}]+[B_{x1}]-[B_{x2}]\\ +[A_{z1}]-[A_{z2}]+[B_{z1}]-[B_{z2}]\end{pmatrix}\\
&=\begin{pmatrix}
0 & 0 & 0 & 0 & \frac{\partial}{\varepsilon\partial z} & 0\\
0 & 0 & 0 & -\frac{\partial}{\varepsilon\partial z} & 0 & \frac{\partial}{\varepsilon\partial x}\\
0 & 0 & 0 & 0 & -\frac{\partial}{\varepsilon\partial x} & 0\\
0 & \frac{\partial}{\mu\partial z} & 0 & 0 & 0 & 0\\
-\frac{\partial}{\mu\partial z} & 0 & \frac{\partial}{\mu\partial x} & 0 & 0 & 0\\
0 & -\frac{\partial}{\mu\partial x} & 0 & 0 & 0 & 0
\end{pmatrix}
\end{aligned}$$

Same as in the HIE-FDTD method, the effect of this error is proportional to the time step size and the spatial variation rate of field.

To solve the Eq. (21), there are three methods with different updating equations.

2.3.1. The WCS-FDTD-1 Method

Considering the expression of matrices A and B in Eq. (21), we obtain:

$$E_x^{n+1} = E_x^n + \frac{\Delta t}{\varepsilon} D_y H_z^n - \frac{\Delta t}{2\varepsilon} D_z \left[H_y^{n+1} + H_y^n\right] \quad (23a)$$

$$E_z^{n+1} = E_z^n - \frac{\Delta t}{\varepsilon} D_y H_x^n + \frac{\Delta t}{2\varepsilon} D_x \left[H_y^{n+1} + H_y^n\right] \quad (23b)$$

$$H_y^{n+1} = H_y^n + \frac{\Delta t}{2\mu} D_x \left[E_z^{n+1} + E_z^n\right] - \frac{\Delta t}{2\mu} D_z \left[E_x^{n+1} + E_x^n\right] \quad (23c)$$

$$H_x^{n+1} = H_x^n - \frac{\Delta t}{\mu} D_y E_z^{n+1} + \frac{\Delta t}{2\mu} D_z \left[E_y^{n+1} + E_y^n\right] \quad (23d)$$

$$H_z^{n+1} = H_z^n + \frac{\Delta t}{\mu} D_y E_x^{n+1} - \frac{\Delta t}{2\mu} D_x \left[E_y^{n+1} + E_y^n\right] \quad (23e)$$

$$E_y^{n+1} = E_y^n + \frac{\Delta t}{2\varepsilon} D_z \left[H_x^{n+1} + H_x^n\right] - \frac{\Delta t}{2\varepsilon} D_x \left[H_z^{n+1} + H_z^n\right] \quad (23f)$$

here, $D_w = \partial/\partial w$ $(w = x, y, z)$ represents the first derivative with respect to w.

Obviously, updating of H_y component, as shown in Eq. (23c), needs the unknown E_x and E_z components at the same time, thus the H_y component has to be updated implicitly. Substituting Eqs. (23a) and (23b) into Eq. (23c), the equation for H_y field is given as:

$$\begin{aligned}\left(1 - b'^2 D_{2x} - b'^2 D_{2z}\right) H_y^{n+1} &= \left(1 + b'^2 D_{2x} + b'^2 D_{2z}\right) H_y^n \\ &-2b'^2 D_x D_y H_x^n - 2b'^2 D_z D_y H_z^n + b D_x E_z^n - b D_z E_x^n \end{aligned} \quad (24)$$

where $b'^2 = \Delta t^2/4\varepsilon\mu$; D_{2x} and D_{2z} are the second derivative.

Similarly, updating of E_y component needs the unknown H_x and H_z components at the same time-step. Substituting Eqs. (23d) and (23e) into Eq. (23f), we obtain the discrete equation for E_y field:

$$\begin{aligned}\left(1 - b'^2 D_{2x} - b'^2 D_{2z}\right) E_y^{n+1} &= \left(1 + b'^2 D_{2x} + b'^2 D_{2z}\right) E_y^n \\ &-2b'^2 D_x D_y E_x^{n+1} - 2b'^2 D_z D_y E_z^{n+1} - a D_x H_z^n + a D_z H_x^n \end{aligned} \quad (25)$$

Equations (24) and (25) are broadly-banded matrix equation, sometimes called a 'tri-diagonal matrix with fringes' or a 'block tri-diagonal matrix'. There are many methods for solving such a matrix.

For example, Gaussian elimination and the banded matrix method are direct methods, and successive over-relaxation (SOR) and the alternate-direction implicit (ADI) method are iterative procedures. However, these are all expensive compared to solving a simple tri-diagonal matrix.

Equation (24) is equivalent to:

$$\left(1 - b'^2 D_{2x}\right)\left(1 - b'^2 D_{2z}\right) H_y^{n+1} = \left(1 + b'^2 D_{2x}\right)\left(1 + b'^2 D_{2z}\right) H_y^n + f_0' + b'^4 D_{2x} D_{2z}\left(H_y^{n+1} - H_y^n\right) \tag{26}$$

where f_0' is the last four terms in Eq. (24).

Ignoring the last term in (26), we obtain:

$$\left(1-b'^2 D_{2x}\right)\left(1-b'^2 D_{2z}\right) H_y^{n+1} = \left(1+b'^2 D_{2x}\right)\left(1+b'^2 D_{2z}\right) H_y^n + f_0' \tag{27}$$

Subdivide it into two sub-steps:

$$\left(1 - b'^2 D_{2x}\right) H_y^* = \left(1 + b'^2 D_{2x}\right)\left(1 + b'^2 D_{2z}\right) H_y^n + f_0' \tag{28a}$$

$$\left(1 - b'^2 D_{2z}\right) H_y^{n+1} = H_y^* \tag{28b}$$

where H_y^* denotes the intermediate value of the magnetic field. Component H_y is updated by solving the tridiagonal matrix equations by using Eqs. (28a) and (28b). Similarly, component E_y is updated by solving the tridiagonal matrix equations as follows:

$$\left(1 - b'^2 D_{2x}\right) E_y^* = \left(1 + b'^2 D_{2x}\right)\left(1 + b'^2 D_{2z}\right) E_y^n + f_1' \tag{29a}$$

$$\left(1 - b'^2 D_{2z}\right) E_y^{n+1} = E_y^* \tag{29b}$$

where f_1' is the last four terms in Eq. (25). E_y^* denotes the intermediate value of the electric field.

This method is referred as WCS-FDTD-1 method and is updated by using Eqs. (23a), (23b), (23d), (23e), (28a), (28b), (29a) and (29b). Thus at each time step, the algorithm requires the solution of four tridiagonal matrices and four explicit updates.

Applying the same stability analysis as in the HIE-FDTD method, the stability condition of this method is

$$\Delta t \leq \Delta y / c \tag{30}$$

This scheme is also weakly conditionally stable. The time step size in this method is only determined by one space discretization Δy.

2.3.2. The WCS-FDTD-2 Method

In the WCS-FDTD-1 method, the H_y component is updated implicitly by solving tridiagonal matrix, which results in a large inaccuracy in the implementation of the perfect-electric-conductor (PEC) condition for the E_x and E_z components. To circumvent this problem, a new weakly conditionally stable FDTD (referred as WCS-FDTD-2 method) has been presented [29].

We write Eq. (21) in a new form:

$$\begin{aligned}&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{z1}]+[A_{z2}]+2\left[B_{y1}\right]\right)\right.\\&\left.-\frac{\Delta t}{2}\left([B_{z1}]+[B_{z2}]+[B_{x1}]+[B_{x2}]+2\left[A_{y1}\right]\right)\right)U^{n+1}\\=&\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{z1}]+[A_{z2}]+2\left[B_{y2}\right]\right)\right.\\&\left.+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{z1}]+[B_{z2}]+2\left[A_{y2}\right]\right)\right)U^{n}\end{aligned}\tag{31}$$

It is equivalent to:

$$\begin{aligned}&\left(I-\frac{\Delta t}{2}\left([B_{z1}]+[B_{z2}]+[B_{x1}]+[B_{x2}]+2\left[A_{y1}\right]\right)\right)\\&\times\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{z1}]+[A_{z2}]+2\left[B_{y1}\right]\right)\right)U^{n+1}\\=&\left(I+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{z1}]+[B_{z2}]+2\left[A_{y2}\right]\right)\right)\\&\times\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{z1}]+[A_{z2}]+2\left[B_{y2}\right]\right)\right)U^{n}\\&+\frac{\Delta t^2}{4}\left[E_2'\right]\left(U^{n+1}-U^{n}\right)\end{aligned}\tag{32}$$

with,

$$\begin{aligned}\left[E_2'\right]&=([B_{z1}]+[B_{z2}]+[B_{x1}]+[B_{x2}])\times([A_{x1}]+[A_{x2}]+[A_{z1}]+[A_{z2}])\\&=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\-\frac{\partial^2}{\varepsilon\mu\partial z\partial x}&0&0&0&0&0\\0&0&0&0&0&-\frac{\partial^2}{\varepsilon\mu\partial z\partial x}\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\end{aligned}$$

Ignoring the last term in (32), we obtain:

$$U^{n+1}=\frac{\left(I+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{z1}]+[B_{z2}]+2\left[A_{y2}\right]\right)\right)\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{z1}]+[A_{z2}]+2\left[B_{y2}\right]\right)\right)}{\left(I-\frac{\Delta t}{2}\left([B_{z1}]+[B_{z2}]+[B_{x1}]+[B_{x2}]+2\left[A_{y1}\right]\right)\right)\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{z1}]+[A_{z2}]+2\left[B_{y1}\right]\right)\right)}U^{n}\tag{33}$$

Introducing a intermediate term U^* in Eq. (33),

$$U^{n+1} = \frac{\left(I + \frac{\Delta t}{2}\left([B_{x1}] + [B_{x2}] + [B_{z1}] + [B_{z2}] + 2\,[A_{y2}]\right)\right)}{\left(I - \frac{\Delta t}{2}\left([B_{z1}] + [B_{z2}] + [B_{x1}] + [B_{x2}] + 2\,[A_{y1}]\right)\right)} U^* \quad (34a)$$

$$U^* = \frac{\left(I + \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2\,[B_{y2}]\right)\right)}{\left(I - \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2\,[B_{y1}]\right)\right)} U^n \quad (34b)$$

then, we have,

$$\left(I - \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2\,[B_{y1}]\right)\right) U^*$$
$$= \left(I + \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2\,[B_{y2}]\right)\right) U^n \quad (35a)$$

$$\left(I - \frac{\Delta t}{2}\left([B_{z1}] + [B_{z2}] + [B_{x1}] + [B_{x2}] + 2\,[A_{y1}]\right)\right) U^{n+1}$$
$$= \left(I + \frac{\Delta t}{2}\left([B_{x1}] + [B_{x2}] + [B_{z1}] + [B_{z2}] + 2\,[A_{y2}]\right)\right) U^* \quad (35b)$$

Equation (35) is the basic formulation of the WCS-FDTD-2 method. It is also an approximation of the conventional FDTD scheme, and is with the approximation error

$$\left(\Delta t\,[E_2]\left(U^{n+1} - U^n\right)\right)/2 + \left(\Delta t^2\left[E_2'\right]\left(U^{n+1} - U^n\right)/4\right) \quad (36)$$

Considering the expression of matrices A and B, and approximating each derivative in space by centered second-order finite differences in Eq. (35), we obtain the equations of the WCS-FDTD-2 method detailedly,

$$E_x^*\left(i+\frac{1}{2},j,k\right) = E_x^n\left(i+\frac{1}{2},j,k\right)$$
$$+\frac{\Delta t}{\varepsilon\Delta y}\left[H_z^n\left(i+\frac{1}{2},j+\frac{1}{2},k\right) - H_z^n\left(i+\frac{1}{2},j-\frac{1}{2},k\right)\right]$$
$$-\frac{\Delta t}{2\varepsilon\Delta z}\left[H_y^*\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) - H_y^*\left(i+\frac{1}{2},j,k-\frac{1}{2}\right)\right.$$
$$\left.+H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) - H_y^n\left(i+\frac{1}{2},j,k-\frac{1}{2}\right)\right] \quad (37a)$$

$$E_y^*\left(i,j+\frac{1}{2},k\right) = E_y^n\left(i,j+\frac{1}{2},k\right)$$
$$-\frac{\Delta t}{2\varepsilon\Delta x}\left[H_z^*\left(i+\frac{1}{2},j+\frac{1}{2},k\right) - H_z^*\left(i-\frac{1}{2},j+\frac{1}{2},k\right)\right.$$
$$\left.+H_z^n\left(i+\frac{1}{2},j+\frac{1}{2},k\right) - H_z^n\left(i-\frac{1}{2},j+\frac{1}{2},k\right)\right] \quad (37b)$$

$$E_z^*\left(i,j,k+1/2\right) = E_z^n\left(i,j,k+1/2\right) \tag{37c}$$

$$H_x^*\left(i,j+1/2,k+1/2\right) = H_x^n\left(i,j+1/2,k+1/2\right) \tag{37d}$$

$$\begin{aligned} H_y^*\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) &= H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) \\ &- \frac{\Delta t}{2\mu\Delta z}\left[E_x^*\left(i+\frac{1}{2},j,k+1\right) - E_x^*\left(i+\frac{1}{2},j,k\right)\right. \\ &\left.+E_x^n\left(i+\frac{1}{2},j,k+1\right) - E_x^n\left(i+\frac{1}{2},j,k\right)\right] \end{aligned} \tag{37e}$$

$$\begin{aligned} H_z^*\left(i+\frac{1}{2},j+\frac{1}{2},k\right) &= H_z^n\left(i+\frac{1}{2},j+\frac{1}{2},k\right) \\ &+ \frac{\Delta t}{\mu\Delta y}\left[E_x^*\left(i+\frac{1}{2},j+1,k\right) - E_x^*\left(i+\frac{1}{2},j,k\right)\right] \\ &- \frac{\Delta t}{2\mu\Delta x}\left[E_y^*\left(i+1,j+\frac{1}{2},k\right) - E_y^*\left(i,j+\frac{1}{2},k\right)\right. \\ &\left.+E_y^n\left(i+1,j+\frac{1}{2},k\right) - E_y^n\left(i,j+\frac{1}{2},k\right)\right] \end{aligned} \tag{37f}$$

$$E_x^{n+1}\left(i+1/2,j,k\right) = E_x^*\left(i+1/2,j,k\right)(\tag{38a}$$

$$\begin{aligned} &E_y^{n+1}\left(i,j+\frac{1}{2},k\right) = E_y^*\left(i,j+\frac{1}{2},k\right) \\ &+\frac{\Delta t}{2\varepsilon\Delta z}\left[H_x^{n+1}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right) - H_x^{n+1}\left(i,j+\frac{1}{2},k-\frac{1}{2}\right)\right. \\ &\left.+H_x^*\left(i,j+\frac{1}{2},k+\frac{1}{2}\right) - H_x^*\left(i,j+\frac{1}{2},k-\frac{1}{2}\right)\right] \end{aligned} \tag{38b}$$

$$\begin{aligned} &E_z^{n+1}\left(i,j,k+\frac{1}{2}\right) = E_z^*\left(i,j,k+\frac{1}{2}\right) \\ &-\frac{\Delta t}{\varepsilon\Delta y}\left[H_x^*\left(i,j+\frac{1}{2},k+\frac{1}{2}\right) - H_x^*\left(i,j-\frac{1}{2},k+\frac{1}{2}\right)\right] \\ &+\frac{\Delta t}{2\varepsilon\Delta x}\left[H_y^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) - H_y^{n+1}\left(i-\frac{1}{2},j,k+\frac{1}{2}\right)\right. \\ &\left.+H_y^*\left(i+\frac{1}{2},j,k+\frac{1}{2}\right) - H_y^*\left(i-\frac{1}{2},j,k+\frac{1}{2}\right)\right] \end{aligned} \tag{38c}$$

$$H_x^{n+1}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)=H_x^{*}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)$$
$$-\frac{\Delta t}{\mu\Delta y}\left[E_z^{n+1}\left(i,j+1,k+\frac{1}{2}\right)-E_z^{n+1}\left(i,j,k+\frac{1}{2}\right)\right]$$
$$+\frac{\Delta t}{2\mu\Delta z}\left[E_y^{n+1}\left(i,j+\frac{1}{2},k+1\right)-E_y^{n+1}\left(i,j+\frac{1}{2},k\right)\right.$$
$$\left.+E_y^{*}\left(i,j+\frac{1}{2},k+1\right)-E_y^{*}\left(i,j+\frac{1}{2},k\right)\right] \quad (38d)$$
$$H_y^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)=H_y^{*}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)$$
$$\frac{\Delta t}{2\mu\Delta x}\left[E_z^{n+1}\left(i+1,j,k+\frac{1}{2}\right)-E_z^{n+1}\left(i,j,k+\frac{1}{2}\right)\right.$$
$$\left.+E_z^{*}\left(i+1,j,k+\frac{1}{2}\right)-E_z^{*}\left(i,j,k+\frac{1}{2}\right)\right] \quad (38e)$$
$$H_z^{n+1}\left(i+1/2,j+1/2,k\right)=H_z^{*}\left(i+1/2,j+1/2,k\right) \quad (38f)$$

In the WCS-FDTD-2 method, the calculation for one discrete time step is performed using two procedures. The first procedure is based on (37) and the second procedure is based on (38).

In the first procedure, updating of E_x^* component, as shown in Eq. (37a), needs the unknown H_y^* components at the same time; thus the E_x^* component has to be updated implicitly. Substituting Eq. (37e) into Eq. (37a), the equation for E_x^* field is given as

$$\left(1+\frac{2\Delta t^2}{4\varepsilon\mu\Delta z^2}\right)E_x^{*}\left(i+\frac{1}{2},j,k\right)$$
$$-\frac{\Delta t^2}{4\varepsilon\mu\Delta z^2}\left(E_x^{*}\left(i+\frac{1}{2},j,k+1\right)+E_x^{*}\left(i+\frac{1}{2},j,k-1\right)\right)$$
$$=\left(1-\frac{2\Delta t^2}{4\varepsilon\mu\Delta z^2}\right)E_x^{n}\left(i+\frac{1}{2},j,k\right)$$
$$+\frac{\Delta t^2}{4\varepsilon\mu\Delta z^2}\left(E_x^{n}\left(i+\frac{1}{2},j,k+1\right)+E_x^{n}\left(i+\frac{1}{2},j,k-1\right)\right)$$
$$+\frac{\Delta t}{\varepsilon\Delta y}\left[H_z^{n}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^{n}\left(i+\frac{1}{2},j-\frac{1}{2},k\right)\right]$$
$$-\frac{\Delta t}{\varepsilon\Delta z}\left[H_y^{n}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^{n}\left(i+\frac{1}{2},j,k-\frac{1}{2}\right)\right] \quad (39)$$

With same manipulation for the E_y^*, E_y^{n+1}, E_z^{n+1} components, the WCS-FDTD-2 method requires the solution of four tridiagonal matrices and four explicit updates at each time step, which is same as that in the WCS-FDTD-1 method.

The components E_x^*, E_y^*, E_z^*, H_x^*, H_y^*, H_z^*, which are written as $E_x^{n+1/2}$, $E_y^{n+1/2}$, $E_z^{n+1/2}$, $H_x^{n+1/2}$, $H_y^{n+1/2}$, $H_z^{n+1/2}$ in Ref. [29], denote the intermediate values of the field and are without any physical meaning.

Applying the same stability analysis as in the HIE-FDTD method, the constrain on the time step size of the WCS-FDTD-2 method is [28]

$$\Delta t \leq \Delta y / c \tag{40}$$

This scheme is also weakly conditionally stable. The time step in this method is only determined by one space discretization Δy, as in the WCS-FDTD-1 method.

2.3.3. The WCS-FDTD-3 Method

Writing Eq. (19) in another form,

$$\begin{aligned} & \left(I - \frac{\Delta t}{2}\left([B_{z1}] + [B_{z2}] + [B_{x1}] + [B_{x2}] + 2\,[A_{y1}]\right)\right) \\ & \times \left(I - \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2\,[B_{y1}]\right)\right) U^{n+1} \\ = & \left(I + \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2\,[B_{y2}]\right)\right) \\ & \times \left(I + \frac{\Delta t}{2}\left([B_{x1}] + [B_{x2}] + [B_{z1}] + [B_{z2}] + 2\,[A_{y2}]\right)\right) U^{n} \\ & + \Delta t^2 / 4\,[\Gamma_1]\,U^{n+1} - \Delta t^2/4\,[\Gamma_2]\,U^{n} \end{aligned} \tag{41}$$

with,

$$\begin{aligned} \Gamma_1 = & \left([B_{z1}] + [B_{z2}] + [B_{x1}] + [B_{x2}]\right) \times \left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}]\right) \\ = & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\partial^2}{\varepsilon\mu\partial z\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\partial^2}{\varepsilon\mu\partial z\partial x} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\Gamma_2 = ([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}]) \times ([B_{z1}] + [B_{z2}] + [B_{x1}] + [B_{x2}])$$
$$= \begin{bmatrix} 0 & 0 & -\frac{\partial^2}{\varepsilon\mu\partial z\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial^2}{\varepsilon\mu\partial z\partial x} & 0 & 0 \end{bmatrix}$$

Ignoring the last two terms in (41), and splitting the equation into two time steps, we obtain:

$$\left(I - \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2[B_{y1}]\right)\right) U^{n+\frac{1}{2}}$$
$$= \left(I + \frac{\Delta t}{2}\left([B_{x1}] + [B_{x2}] + [B_{z1}] + [B_{z2}] + 2[A_{y2}]\right)\right) U^{n} \quad (42a)$$
$$\left(I - \frac{\Delta t}{2}\left([B_{z1}] + [B_{z2}] + [B_{x1}] + [B_{x2}] + 2[A_{y1}]\right)\right) U^{n+1}$$
$$= \left(I + \frac{\Delta t}{2}\left([A_{x1}] + [A_{x2}] + [A_{z1}] + [A_{z2}] + 2[B_{y2}]\right)\right) U^{n+\frac{1}{2}} \quad (42b)$$

Considering the expression of matrices A and B, and approximating each derivative in space by centered second-order finite differences in Eq. (42), the formulations of the WCS-FDTD-3 method are obtained [31]. In this method, four tridiagonal matrices and six explicit update are needed for the field development at one time step.

The WCS-FDTD-3 method is also an approximation of the conventional FDTD method, and the approximation error is

$$\left(\Delta t\,[E_2]\left(U^{n+1} - U^{n}\right)\right)/2 + \Delta t^2/4\,[\Gamma_1]\,U^{n+1} - \Delta t^2/4\,[\Gamma_2]\,U^{n} \quad (43)$$

The stability condition of the WCS-FDTD-3 method is [31]

$$\Delta t \leq \Delta y/c \quad (44)$$

which can be demonstrated by following same analysis of the HIE-FDTD method.

Among all the three WCS-FDTD methods above, the WCS-FDTD-2 method is used most extensively, due to its simple updating equation and high computation efficiency. The WCS-FDTD-1 method can't be applied to the implementation of the PEC condition, which confines its usage. The WCS-FDTD-3 method has almost same numerical accuracy and dispersion error as that of the WCS-FDTD-2 method, but with larger computation burden.

2.4. Formulations of Unconditionally Stable FDTD Methods

If we write Eq. (3) in the form as:

$$\begin{aligned}&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y1}]+[A_{y2}]+[A_{z1}]+[A_{z2}]\right)\right.\\&\left.-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y1}]+[B_{y2}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n+1}\\=&\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y1}]+[A_{y2}]+[A_{z1}]+[A_{z2}]\right)\right.\\&\left.+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y1}]+[B_{y2}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n}\\&+\frac{\Delta t}{2}[E_3]\left(U^{n+1}-U^{n}\right)\end{aligned}\tag{45}$$

with,

$$\begin{aligned}[E_3]&=\begin{pmatrix}[A_{x1}]-[A_{x2}]+[A_{y1}]-[A_{y2}]+[A_{z1}]-[A_{z2}]\\+[B_{x1}]-[B_{x2}]+[B_{y1}]-[B_{y2}]+[B_{z1}]-[B_{z2}]\end{pmatrix}\\&=\begin{pmatrix}0&0&0&0&\frac{\partial}{\varepsilon\partial z}&-\frac{\partial}{\varepsilon\partial y}\\0&0&0&-\frac{\partial}{\varepsilon\partial z}&0&\frac{\partial}{\varepsilon\partial x}\\0&0&0&\frac{\partial}{\varepsilon\partial y}&-\frac{\partial}{\varepsilon\partial x}&0\\0&\frac{\partial}{\mu\partial z}&-\frac{\partial}{\mu\partial y}&0&0&0\\-\frac{\partial}{\mu\partial z}&0&\frac{\partial}{\mu\partial x}&0&0&0\\\frac{\partial}{\mu\partial y}&-\frac{\partial}{\mu\partial x}&0&0&0&0\end{pmatrix}\end{aligned}$$

ignoring the last term in (45), and introducing the values of the matrices A and B, it obtains:

$$E_x^{n+1}=E_x^n+\frac{\Delta t}{2\varepsilon}D_y\left[H_z^{n+1}+H_z^n\right]-\frac{\Delta t}{2\varepsilon}D_z\left[H_y^{n+1}+H_y^n\right]\tag{46a}$$

$$E_y^{n+1}=E_y^n+\frac{\Delta t}{2\varepsilon}D_z\left[H_x^{n+1}+H_x^n\right]-\frac{\Delta t}{2\varepsilon}D_x\left[H_z^{n+1}+H_z^n\right]\tag{46b}$$

$$E_z^{n+1}=E_z^n-\frac{\Delta t}{2\varepsilon}D_y\left[H_x^{n+1}+H_x^n\right]+\frac{\Delta t}{2\varepsilon}D_x\left[H_y^{n+1}+H_y^n\right]\tag{46c}$$

$$H_x^{n+1}=H_x^n-\frac{\Delta t}{2\mu}D_y\left[E_z^{n+1}+E_z^n\right]+\frac{\Delta t}{2\mu}D_z\left[E_y^{n+1}+E_y^n\right]\tag{46d}$$

$$H_y^{n+1}=H_y^n+\frac{\Delta t}{2\mu}D_x\left[E_z^{n+1}+E_z^n\right]-\frac{\Delta t}{2\mu}D_z\left[E_x^{n+1}+E_x^n\right]\tag{46e}$$

$$H_z^{n+1}=H_z^n+\frac{\Delta t}{2\mu}D_y\left[E_x^{n+1}+E_x^n\right]-\frac{\Delta t}{2\mu}D_x\left[E_y^{n+1}+E_y^n\right]\tag{46f}$$

Equation (46) is the formulation of the Crank-Nicolson (CN) FDTD method [12, 15]. It is an approximation of the conventional FDTD method with the approximation error,

$$\frac{\Delta t}{2}\left[E_3\right]\left(U^{n+1}-U^{n}\right) \tag{47}$$

which is also proportional to the time step size and the spatial variation rate of field.

The CN-FDTD method is unconditionally stable, which have been well demonstrated in Ref. [8]. The time step size in this method is not bounded by the space discretizations.

It can be seen from Eq. (46) that none of these equations can be solved explicitly, and all the right sides of these equations include the unknown term defined at the same time step. The updating of CN-FDTD method is with a huge spare irreducible matrix. Directly solving this matrix by Gaussian elimination or an iterative method is so CPU intensive that the CN scheme is hardly usable for practical problems.

To overcome this problem, the ADI-FDTD method is developed. It is an approximation of the CN-FDTD method with the form:

$$\begin{aligned}
&\left(I-\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y1}]+[A_{y2}]+[A_{z1}]+[A_{z2}]\right)\right)\\
&\times\left(I-\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y1}]+[B_{y2}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n+1}\\
=\;&\left(I+\frac{\Delta t}{2}\left([A_{x1}]+[A_{x2}]+[A_{y1}]+[A_{y2}]+[A_{z1}]+[A_{z2}]\right)\right)\\
&\times\left(I+\frac{\Delta t}{2}\left([B_{x1}]+[B_{x2}]+[B_{y1}]+[B_{y2}]+[B_{z1}]+[B_{z2}]\right)\right)U^{n}
\end{aligned} \tag{48}$$

Obviously, the ADI-FDTD method is also an approximation of the conventional FDTD method, and the error between these two methods is,

$$\frac{\Delta t}{2}\left[E_3\right]\left(U^{n+1}-U^{n}\right)+\frac{\Delta t^2}{4}\left[E_3'\right]\left(U^{n+1}-U^{n}\right) \tag{49}$$

with

$$[E_3'] = \begin{pmatrix} ([A_{x1}] + [A_{x2}] + [A_{y1}] + [A_{y2}] + [A_{z1}] + [A_{z2}]) \times \\ ([B_{x1}] + [B_{x2}] + [B_{y1}] + [B_{y2}] + [B_{z1}] + [B_{z2}]) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\frac{\partial^2}{\varepsilon\mu\partial z\partial x} & 0 & 0 & 0 \\ -\frac{\partial^2}{\varepsilon\mu\partial y\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial^2}{\varepsilon\mu\partial y\partial z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\partial^2}{\varepsilon\mu\partial y\partial x} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\partial^2}{\varepsilon\mu\partial y\partial z} \\ 0 & 0 & 0 & -\frac{\partial^2}{\varepsilon\mu\partial x\partial z} & 0 & 0 \end{pmatrix}$$

The term $\frac{\Delta t^2}{4}[E_3'](U^{n+1} - U^n)$, which depends on the square of the time step size and the spatial variation rate of field, is referred as split error [17, 18].

Splitting Eq. (45) into two sub-steps, we obtain the formulation of the ADI-FDTD method,

$$\left[I - \frac{\Delta t}{2}([A_{x1}] + [A_{x2}] + [A_{y1}] + [A_{y2}] + [A_{z1}] + [A_{z2}])\right] U^{n+\frac{1}{2}} \quad (50a)$$

$$= \left[I + \frac{\Delta t}{2}([B_{x1}] + [B_{x2}] + [B_{y1}] + [B_{y2}] + [B_{z1}] + [B_{z2}])\right] U^n \quad (50b)$$

$$\left[I - \frac{\Delta t}{2}([B_{x1}] + [B_{x2}] + [B_{y1}] + [B_{y2}] + [B_{z1}] + [B_{z2}])\right] U^{n+1} \quad (50c)$$

$$= \left[I + \frac{\Delta t}{2}([A_{x1}] + [A_{x2}] + [A_{y1}] + [A_{y2}] + [A_{z1}] + [A_{z2}])\right] U^{n+\frac{1}{2}} \quad (50d)$$

Considering the expression of matrices A and B, and approximating each derivative in space by centered second-order finite differences in Eq. (50), the final updated equations of the ADI-FDTD method can be obtained [6, 7]

$$E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j, k\right) = E_x^n\left(i+\frac{1}{2}, j, k\right)$$
$$+ \frac{\Delta t}{2\varepsilon\Delta y}\left[H_z^n\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right) - H_z^n\left(i+\frac{1}{2}, j-\frac{1}{2}, k\right)\right]$$
$$- \frac{\Delta t}{2\varepsilon\Delta z}\left[H_y^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right) - H_y^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j, k-\frac{1}{2}\right)\right] \quad (51a)$$

$$E_y^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k\right)=E_y^n\left(i,j+\frac{1}{2},k\right)$$
$$+\frac{\Delta t}{2\varepsilon\Delta z}\left[H_x^n\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^n\left(i,j+\frac{1}{2},k-\frac{1}{2}\right)\right]$$
$$-\frac{\Delta t}{2\varepsilon\Delta x}\left[H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2},j+\frac{1}{2},k\right)\right] \quad (51b)$$

$$E_z^{n+\frac{1}{2}}\left(i,j,k+\frac{1}{2}\right)=E_z^n\left(i,j,k+\frac{1}{2}\right)$$
$$-\frac{\Delta t}{2\varepsilon\Delta y}\left[H_x^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^{n+\frac{1}{2}}\left(i,j-\frac{1}{2},k+\frac{1}{2}\right)\right]$$
$$+\frac{\Delta t}{2\varepsilon\Delta x}\left[H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^n\left(i-\frac{1}{2},j,k+\frac{1}{2}\right)\right] \quad (51c)$$

$$H_x^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)=H_x^n\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)$$
$$-\frac{\Delta t}{2\mu\Delta y}\left[E_z^{n+\frac{1}{2}}\left(i,j+1,k+\frac{1}{2}\right)-E_z^{n+\frac{1}{2}}\left(i,j,k+\frac{1}{2}\right)\right]$$
$$+\frac{\Delta t}{2\mu\Delta z}\left[E_y^n\left(i,j+\frac{1}{2},k+1\right)-E_y^n\left(i,j+\frac{1}{2},k\right)\right] \quad (51d)$$

$$H_y^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)=H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)$$
$$+\frac{\Delta t}{2\mu\Delta x}\left[E_z^n\left(i+1,j,k+\frac{1}{2}\right)-E_z^n\left(i,j,k+\frac{1}{2}\right)\right]$$
$$-\frac{\Delta t}{2\mu\Delta z}\left[E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k+1\right)-E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k\right)\right] \quad (51e)$$

$$H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)=H_z^n\left(i+\frac{1}{2},j+\frac{1}{2},k\right)$$
$$+\frac{\Delta t}{2\mu\Delta y}\left[E_x^n\left(i+\frac{1}{2},j+1,k\right)-E_x^n\left(i+\frac{1}{2},j,k\right)\right]$$
$$-\frac{\Delta t}{2\mu\Delta x}\left[E_y^{n+\frac{1}{2}}\left(i+1,j+\frac{1}{2},k\right)-E_y^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k\right)\right] \quad (51f)$$

$$E_x^{n+1}\left(i+\frac{1}{2},j,k\right)=E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k\right)$$
$$+\frac{\Delta t}{2\varepsilon\Delta y}\left[H_z^{n+1}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^{n+1}\left(i+\frac{1}{2},j-\frac{1}{2},k\right)\right]$$
$$-\frac{\Delta t}{2\varepsilon\Delta z}\left[H_y^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k-\frac{1}{2}\right)\right] \quad (52a)$$

$$E_y^{n+1}\left(i,j+\frac{1}{2},k\right)=E_y^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k\right)$$
$$+\frac{\Delta t}{2\varepsilon\Delta z}\left[H_x^{n+1}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^{n+1}\left(i,j+\frac{1}{2},k-\frac{1}{2}\right)\right]$$
$$-\frac{\Delta t}{2\varepsilon\Delta x}\left[H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^{n+\frac{1}{2}}\left(i-\frac{1}{2},j+\frac{1}{2},k\right)\right] \quad (52b)$$

$$E_z^{n+1}\left(i,j,k+\frac{1}{2}\right)=E_z^{n+\frac{1}{2}}\left(i,j,k+\frac{1}{2}\right)$$
$$-\frac{\Delta t}{2\varepsilon\Delta y}\left[H_x^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)-H_x^{n+\frac{1}{2}}\left(i,j-\frac{1}{2},k+\frac{1}{2}\right)\right]$$
$$+\frac{\Delta t}{2\varepsilon\Delta x}\left[H_y^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^{n+1}\left(i-\frac{1}{2},j,k+\frac{1}{2}\right)\right] \quad (52c)$$

$$H_x^{n+1}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)=H_x^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k+\frac{1}{2}\right)$$
$$-\frac{\Delta t}{2\mu\Delta y}\left[E_z^{n+\frac{1}{2}}\left(i,j+1,k+\frac{1}{2}\right)-E_z^{n+\frac{1}{2}}\left(i,j,k+\frac{1}{2}\right)\right]$$
$$+\frac{\Delta t}{2\mu\Delta z}\left[E_y^{n+1}\left(i,j+\frac{1}{2},k+1\right)-E_y^{n+1}\left(i,j+\frac{1}{2},k\right)\right] \quad (52d)$$

$$H_y^{n+1}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)=H_y^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)$$
$$+\frac{\Delta t}{2\mu\Delta x}\left[E_z^{n+1}\left(i+1,j,k+\frac{1}{2}\right)-E_z^{n+1}\left(i,j,k+\frac{1}{2}\right)\right]$$
$$-\frac{\Delta t}{2\mu\Delta z}\left[E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k+1\right)-E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k\right)\right] \quad (52e)$$

$$H_z^{n+1}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)=H_z^{n+\frac{1}{2}}\left(i+\frac{1}{2},j+\frac{1}{2},k\right)$$
$$+\frac{\Delta t}{2\mu\Delta y}\left[E_x^{n+1}\left(i+\frac{1}{2},j+1,k\right)-E_x^{n+1}\left(i+\frac{1}{2},j,k\right)\right]$$
$$-\frac{\Delta t}{2\mu\Delta x}\left[E_y^{n+\frac{1}{2}}\left(i+1,j+\frac{1}{2},k\right)-E_y^{n+\frac{1}{2}}\left(i,j+\frac{1}{2},k\right)\right] \quad (52f)$$

In the ADI-FDTD method, the calculation for one discrete time step is performed using two procedures. The first procedure is based on (51) and the second procedure is based on (52).

In the first procedure, updating of $E_x^{n+1/2}$ component, as shown in Eq. (51a), needs the unknown $H_y^{n+1/2}$ components at the same time;

thus the $E_x^{n+1/2}$component has to be updated implicitly. Substituting Eq. (51e) into Eq. (51a), the equation for $E_x^{n+1/2}$ field is given as

$$\begin{aligned}&\left(1+\frac{2\Delta t^2}{4\varepsilon\mu\Delta z^2}\right)E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k\right)\\&-\frac{\Delta t^2}{4\varepsilon\mu\Delta z^2}\left(E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k+1\right)+E_x^{n+\frac{1}{2}}\left(i+\frac{1}{2},j,k-1\right)\right)\\&=-\frac{\Delta t^2}{4\varepsilon\mu\Delta z\Delta x}\left[E_z^n\left(i+1,j,k+\frac{1}{2}\right)-E_z^n\left(i,j,k+\frac{1}{2}\right)\right.\\&\left.-E_z^n\left(i+1,j,k-\frac{1}{2}\right)+E_z^n\left(i,j,k-\frac{1}{2}\right)\right]+E_x^n\left(i+\frac{1}{2},j,k\right)\\&+\frac{\Delta t}{2\varepsilon\Delta y}\left[H_z^n\left(i+\frac{1}{2},j+\frac{1}{2},k\right)-H_z^n\left(i+\frac{1}{2},j-\frac{1}{2},k\right)\right]\\&-\frac{\Delta t}{2\varepsilon\Delta z}\left[H_y^n\left(i+\frac{1}{2},j,k+\frac{1}{2}\right)-H_y^n\left(i+\frac{1}{2},j,k-\frac{1}{2}\right)\right]\end{aligned}\tag{53}$$

With same manipulation for the other electric components, the ADI-FDTD method requires the solution of six tridiagonal matrices and six explicit updates at each time step.

The ADI-FDTD method is also unconditionally stable [6, 7]. The time step size in this method is not related with the space discretizations.

3. COMPARISON

Among all the methods described above, the FDTD method is with simplest updating equations. Whereas, the time step size in this method must satisfy with the CFL condition, which makes this method inefficient for the problems where fine scale dimensions are used. To overcome the CFL constraint, the fast FDTD method, including HIE-FDTD, WCS-FDTD, and unconditionally stable FDTD methods are presented.

The HIE-FDTD method is a weakly conditionally stable method which is extremely useful for problems where a very fine mesh is needed in one direction. The time step size in this method is determined by two space discretizations. The WCS-FDTD methods are also weakly conditionally stable and the time step sizes in these methods are only determined by one space discretization. They are useful for the problems where a fine mesh is needed in two directions. In the CN-FDTD method and ADI-FDTD method, the CFL constraint is removed totally. The time step sizes in these two methods have no relations with the space discretizations.

Due to the important impact of these methods on electromagnetic computation, the numerical performance of these methods, including computation accuracy, efficiency, and memory requirements are needed to be compared detailedly.

In this section, theoretical analysis of the accuracy and efficiency of the HIE-FDTD, WCS-FDTD-2, and ADI-FDTD methods are given. Following that, numerical illustrations are presented, by comparing with that of the conventional 3D FDTD method.

It is noted that, although there are three WCS-FDTD methods and two unconditionally stable FDTD methods, only the WCS-FDTD-2 method and the ADI-FDTD method are considered here, due to their extensive usage.

3.1. Theoretical Analysis

3.1.1. Comparison of Accuracy

From the analysis above, we can see that the HIE-FDTD method, WCS-FDTD-2 method, and ADI-FDTD method can be thought as approximations of the conventional FDTD method. They all include approximation error compared with the conventional FDTD method. To clarify this point further, we recall the approximation errors of these methods.

$$\frac{\Delta t}{2}\left[E_1\right]\left(U^{n+1}-U^n\right) \tag{54}$$

$$\frac{\Delta t}{2}\left[E_2\right]\left(U^{n+1}-U^n\right)+\frac{\Delta t^2}{4}\left[E_2'\right]\left(U^{n+1}-U^n\right) \tag{55}$$

$$\frac{\Delta t}{2}\left[E_3\right]\left(U^{n+1}-U^n\right)+\frac{\Delta t^2}{4}\left[E_3'\right]\left(U^{n+1}-U^n\right) \tag{56}$$

Equations (54), (55) and (56) are the approximation errors of the HIE-FDTD, WCS-FDTD-2 and ADI-FDTD methods, respectively. It can be seen from these equations that all the approximation errors of these methods are related with the time step size and the spatial variation rate. The larger the time step size and/or the spatial variation rate, the worse the computation accuracy of these methods.

Compared with the HIE-FDTD method, due to the application of more approximation (along x and z directions), the accuracy of the WCS-FDTD-2 is degraded. The approximation error of the WCS-FDTD-2 method includes the factor related with the square of the time step size, which makes the effect of the time step size on the accuracy more considerable.

In the ADI-FDTD method, approximation along three directions are applied, as seen in Eq. (45), which makes the accuracy of the ADI-FDTD worst among all these methods, especially when the time step size is large and the computation region is with larger spatial field variation rate.

3.1.2. Comparison of Efficiency

In ADI-FDTD scheme, three time steps are used for defining the field components and two sub-iterations are required for field advancement. It must solve six tridiagonal matrices and six explicit updates for one full update cycle, which make the ADI-FDTD method computationally inefficient. For the HIE- FDTD method, only a single iteration with two tridiagonal matrices and four explicit updates are needed for the field development. For the WCS FDTD-2 method, although two sub-iterations are required, it only solves four tridiagonal matrices and four explicit updates for one full update cycle. Thus, compared with the ADI-FDTD method, the CPU time for the HIE-FDTD method and the WCS-FDTD-2 method is reduced.

To provide detailed assessment and comparison with regard to the computation efficiency of these methods, the floating point operations (flops) counts taking into account the number of multiplications/divisions (M/D) and additions/subtractions (A/S) required for one complete time step for all these algorithms are listed in Table 1, based on the right-hand sides of their respective updating equations such as (11), (2.30-1'), and (2.43-1'), etc. For simplicity, the number of electric and magnetic field components in all directions has been taken to be the same and assume that all multiplicative factors have been precomputed and stored. From the table, it is clear that, among all these methods, the HIE-FDTD method is with highest computation efficiency, followed by the WCS-FDTD-2 method. The

Table 1. Flops count for different methods.

Scheme	Algorithm	Implicit	Explicit	Total
HIE-FDTD	M/D	10	8	58
	A/S	20	20	
WCS-FDTD-2	M/D	16	6	74
	A/S	32	20	
ADI-FDTD	M/D	18	12	102
	A/S	48	24	

computation efficiency of the ADI-FDTD is worst. For the right-hand sides of the updating equations, the total flops count (M/D + A/S) of the ADI-FDTD method is almost 1.8 times as that of the HIE-FDTD method, and 1.4 times as that of the WCS-FDTD-2 method.

3.2. Numerical Validation

In order to compare the accuracy and efficiency of these methods numerically, a simulation of electromagnetic fields distributed inside an enclosed box is studied. The result calculated by the conventional FDTD method is assumed to be standard and the results from the other techniques are compared against it.

The dimension of the box is $15\,\mathrm{cm} \times 15\,\mathrm{cm} \times 3\,\mathrm{cm}$, and is discretized with a uniform grid of $30 \times 30 \times 30$ along x, y, and z directions, as shown in Figure 1. The cell size is set as $\Delta y = \Delta x = 5\Delta z = 0.5\,\mathrm{cm}$. A small current source applied along z direction is placed at the central grid point (15, 15, 15). The time dependence of the excitation function is:

$$I_z(t) = \exp\left[-\alpha(t - t_0)^2\right] \tag{57}$$

where $t_0 = 0.6\times10^{-9}\,\mathrm{s}$, $\alpha = 3.49\times10^{19}\,\mathrm{s}^{-2}$. To eliminate any unwanted truncation error, the PEC boundary condition is set at the faces of the box.

Considering the numerical model with fine mesh along the z direction, the HIE-FDTD method represented by Eq. (19) is used, in such a case, the time step size in the HIE-FDTD method is only determined by the space discretizations Δx and Δy.

Because the numerical errors of these methods are all related with the time step size and the spatial variation rate of field, different time

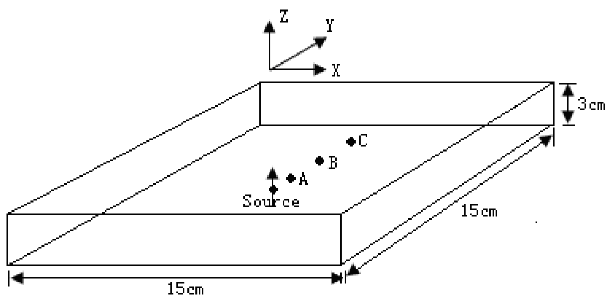

Figure 1. Geometric configuration of the enclosed box.

step sizes and observation points are set in this example. To satisfy the limitation of the stable condition in the conventional FDTD, HIE-FDTD, and WCS-FDTD-2 methods, the maximum time-step sizes are 3.60 ps, 11.78 ps, and 16.67 ps, respectively. Three observation points A, B and C, being 0.5 cm, 1.5 cm, and 3 cm far from the source, are set at the axes of the box.

In this section, the effect of the time step size and the spatial variation rate of field on the accuracy of these methods are discussed, and the computation efficiency and memory requirements of these methods are compared.

3.2.1. Effect of Time Step Size

To analyze the effect of the time step size on the accuracy of these methods, the observation point is set at C, and the simulation results for the E_z component calculated by using different methods under different time step sizes are shown in Figures 2–4. For clarity, the figures only depict the results from the time history 0.6 ns to 1.5 ns.

It can be seen from Figure 2 that the results calculated by the HIE-FDTD, WCS-FDTD-2, and ADI-FDTD methods under time step size 3.20 ps agree well with the result calculated by the conventional FDTD method, which shows that under small time step size, all these methods have high accuracy.

Increasing the time step size to 11.78 ps, the results of the HIE-FDTD method and the WCS-FDTD-2 method remain to match well with that of the FDTD method, but a deviation of the ADI-FDTD

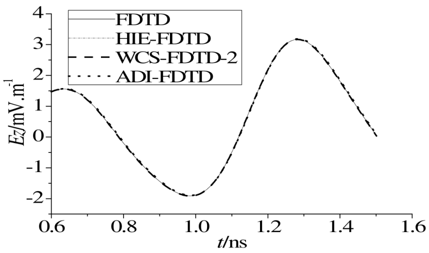

Figure 2. Numerical results calculated by using conventional FDTD, HIE-FDTD, WCS-FDTD-2 and ADI-FDTD methods under time step size 3.20 ps.

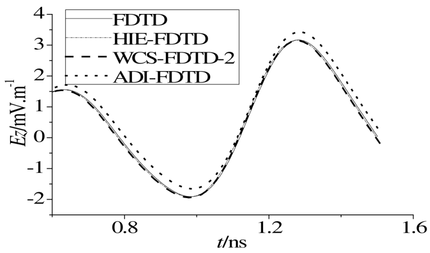

Figure 3. Numerical results calculated by using conventional FDTD (Δt = 3.20 ps), HIE-FDTD (Δt = 11.78 ps), WCS-FDTD-2 (Δt = 11.78 ps) and ADI-FDTD (Δt = 11.78 ps) methods.

method from the conventional FDTD method is observed. It is apparent that the HIE-FDTD method and WCS-FDTD-2 method have higher accuracy than the ADI-FDTD method with large time step size.

When the time step size increases to 16.66 ps, the result of the WCS-FDTD-2 method begins to deviate from the result of the conventional FDTD method, as shown in Figure 4. However, the deviation of WCS-FDTD algorithm from conventional FDTD method is much smaller than that of the ADI-FDTD scheme.

From the analysis above, we can see that both the computation accuracy of the WCS-FDTD-2 method and the ADI-FDTD method are decreased as the increase of the time step size, and the effect of the time step size on the accuracy of the ADI-FDTD method is more considerable than that on the WCS-FDTD-2 method. For the HIE-FDTD method, the results are almost unchanged when the time step increases, as shown in Figures 2 and 3, which shows that the accuracy of the HIE-FDTD method is almost unaffected by the time step size.

3.2.2. Effect of the Spatial Variation Rate

In order to observe how the spatial field variation rate affects the accuracy, three observation points A, B and C are set, and the time step size is selected as 3.20 ps. Among these points, observation point A is with largest spatial field variation rate, followed by point B. The spatial field variation rate at point C is lowest.

The simulation results for the E_z component calculated by using different methods at different observation points are shown in Figures 5–7. It can be seen from these figures that, at point A, only

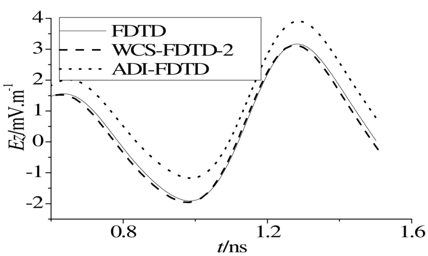

Figure 4. Numerical results calculated by using conventional FDTD (Δt = 3.20 ps), HIE-FDTD (Δt = 16.66 ps), WCS-FDTD-2 (Δt = 16.66 ps) and ADI-FDTD (Δt = 16.66 ps) methods.

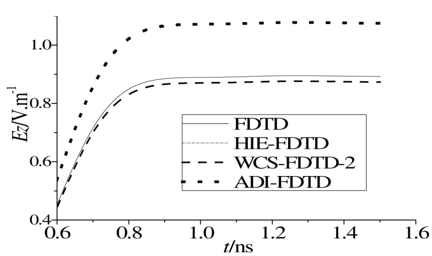

Figure 5. Numerical results calculated by using conventional FDTD HIE-FDTD, WCS-FDTD-2 and ADI-FDTD methods at point A.

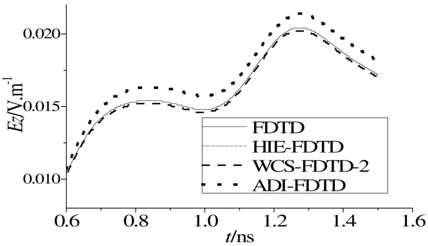

Figure 6. Numerical results calculated by using conventional FDTD HIE-FDTD, WCS-FDTD-2 and ADI-FDTD methods at point B.

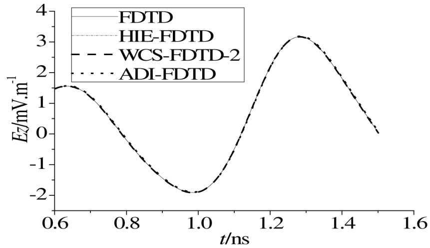

Figure 7. Numerical results calculated by using conventional FDTD HIE-FDTD, WCS-FDTD-2 and ADI-FDTD methods at point C.

the numerical results obtained by the conventional FDTD scheme and the HIE-FDTD scheme are in good agreement. Both the results of the WCS-FDTD-2 method and ADI-FDTD method deviate from that of the conventional FDTD method apparently, especially the ADI-FDTD method. The result of the ADI-FDTD is so incorrect that it can't be used in practical computation.

At observation point B, due to the smaller spatial field variation rate, both the accuracy of the WCS-FDTD-2 method and the ADI-FDTD method are improved, as shown in Figure 6. At point C, all the results calculated by the HIE-FDTD, WCS-FDTD-2, and ADI-FDTD methods agree well with the result calculated by the conventional FDTD method.

From the analysis above, we conclude that, as the spatial field variation rate increase, both the computation accuracy of the WCS-FDTD-2 method and the ADI-FDTD method are degraded, and the effect of the spatial field variation rate on the accuracy of ADI-FDTD method is more considerable than that on the WCS-FDTD-2 method, whereas, the effect on the accuracy of the HIE-FDTD method is not apparent. At all these points, the numerical results obtained by the conventional FDTD scheme and the HIE-FDTD scheme are in good agreement.

3.2.3. The Computation Efficiency of These Methods

The simulation time for different numerical methods with different time steps are summarized in Table 2. It shows that when $\Delta t = 3.20\,\text{ps}$, the simulations takes 17.46 s for the conventional FDTD method 26.08 s for the HIE-FDTD method, 33.32 s for the WCS-FDTD-2 method,

Table 2. CPU times for conventional FDTD, HIE-FDTD, WCS-FDTD-2 , and ADI-FDTD methods with different time-step sizes.

	$\Delta t = 3.20\,\text{ps}$	$\Delta t = 11.78\,\text{ps}$	$\Delta t = 16.66\,\text{ps}$
FDTD	17.46 s		
HIE-FDTD	26.08 s	7.85 s	
WCS-FDTD-2	33.32 s	9.14 s	6.55 s
ADI-FDTD	44.64 s	12.88 s	8.95 s

Table 3. The memory requirements for the conventional FDTD, HIE-FDTD, WCS-FDTD and ADI-FDTD methods.

	FDTD	HIE-FDTD	WCS-FDTD-2	ADI-FDTD
Memory(Mb)	2.73	2.74	2.75	2.80

and 44.64 s for the ADI-FDTD method, that is to say, under the same time-step size, the conventional FDTD method take the shortest computation time. All the other methods have lower efficiency than the conventional FDTD method. The ADI-FDTD method takes the most computation time, followed by WCS-FDTD-2 method. The HIE-FDTD method has the highest computation efficiency among all these weakly conditionally stable and unconditionally stable FDTD methods. The CPU time for the ADI-FDTD method is almost 1.4 times as that of the WCS-FDTD-2 method and 1.8 times as that of the HIE-FDTD method, which is consistent with the theoretical analysis. Increasing the time step size all the computation times for the HIE-FDTD, WCS-FDTD-2, and ADI-FDTD methods are reduced compared with that of the conventional FDTD method.

3.2.4. The Memory Requirements of These Methods

The memory requirements of these methods in this simulation is shown in Table 3. It is apparent that the memory requirements are almost the same for these methods. The insignificant memory discrepancy between them is due to the storage of the tridiagonal matrix.

Through the analysis above, the numerical performance of the HIE-FDTD method, WCS-FDTD-2 method, and the ADI-FDTD method can be concluded. It includes three aspects:

1) Accuracy: Among all these method, the HIE-FDTD method has the best accuracy, followed by the WCS-FDTD-2 method; the accuracy of the ADI-FDTD method is worst.

2) Computation efficiency: When maintain the same time step size, the HIE-FDTD method has the highest computation efficiency, followed by the WCS-FDTD method; the ADI-FDTD method has the lowest computation efficiency. The computation time of the ADI-FDTD method is almost 1.4 times as that of the WCS-FDTD-2 method and 1.8 times as that of the HIE-FDTD method.
3) Memory requirement: The memory requirements for the HIE-FDTD method, WCS-FDTD-2 method, and the ADI-FDTD method are almost the same, except the insignificant discrepancy at the storage of the tridiagonal matrix.

4. CONCLUSION

Take different approximations to the conventional 3-D FDTD method, we can obtain different numerical methods. Among these methods, the ADI-FDTD and CN-FDTD methods are unconditionally stable; the HIE-FDTD and WCS FDTD methods are weakly and conditionally stable. The errors between these methods and the conventional FDTD method are presented analytically, which shows that the HIE-FDTD method is most accurate, followed by WCS FDTD-2 method. The accuracy of the ADI-FDTD is worse than those of the HIE-FDTD and WCS-FDTD-2, especially when the time step is large, and the computation region is with larger spatial field variation rate. The computation times of all these methods are compared, and it shows that the HIE-FDTD and WCS-FDTD-2 methods have higher computation efficiency than the ADI-FDTD method, which is demonstrated through numerical examples.

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