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DETERMINATION THE MATERIAL PARAMETERS FOR ARBITRARY CLOAK BASED ON POISSON'S EQUATION

J.-J. Ma, X.-Y. Cao, K.-M. Yu, and T. Liu

Telecommunication Engineering Institute
Air Force University of Engineering
Xi'an, China

Abstract—We propose a general method to determine the material parameters for arbitrary shapes of cloak based on the Poisson's equation to map the coordinate transformation. As a result, we can obtain the diverse deformation material properties and then the field distribution. This method, compared with the previous technique presented in literature, can determine the countless transformation forms, so it may provide the opportunity to choose the optimization transformation and the material parameter map which is easily to be fabricated using the metamaterial technology.

1. INTRODUCTION

Based on form-invariant coordinate transformation and optical conformal map methods, the invisibility cloak [1–3] with different forms such as circular [1-3], square [4] and elliptical [5, 6] shapes were proposed. These concepts were even extended to the design of the general optical transformation media [7–9]. In order to design such transformation optical cloak or other devices, a specific coordinate transformation should be deployed first, which maps the original space with given electromagnetic material parameters into an envisioned one. As a direct consequence, the permittivity and permeability, as now described in terms of the new coordinates, have to be transformed accordingly. So, the pivotal step for designing an irregular shapes of cloak is to determine this specific coordinate transformation. Recently, some analytical methods in designing arbitrarily cloaks, such as relying on the semi-analytic or semi-numerical approaches to evaluate the

Corresponding author: J.-J. Ma (aifors@126.com).

material parameters of the cloak, have been presented in literature [10–12]. Practically, the arbitrary shapes of cloak are mostly irregular and complicated. These methods are difficult to express the corresponding coordinate transformations analytically, which can only be used in the simple case. So a pure numerical method to design cloak with arbitrary shapes is still necessary.

For a given shape of cloak, using the solution of Laplace's equation to determine the coordinate transformation is an excellent pure numerical method proposed in [13], which postulates the original coordinate; the transformed coordinate satisfies the Lalapce's equation, then solves this equation by theoretical or numerical method to determine material parameters in the new transformed coordinate system. But in fact, there are countless ways that can perform this spatial transformation. The solution of Laplace's equation can only determine one of these transformations. In this paper, we will use the Poisson's equation to determine the spatial transformation. By properly choosing different factor functions, we can get diverse spatial compression maps. So we will have the opportunity to choose the optimization transformation and material parameter which is easily to be fabricated using the metamaterial technology.

2. DESIGN METHOD

According to the coordinate transformation method, under a space transformation from a flat space x to a distorted one $x'(x)$, the tensors of permittivity ε' and permeability μ' for a linear, anisotropic, non-dispersive, non-bianisotropic medium in the transformed space can be written in a more accessible form as [4]

$$\begin{aligned} \varepsilon' &= A\varepsilon A^T / \det(A) \\ \mu' &= A\mu A^T / \det(A) \end{aligned} \tag{1}$$

where ε and μ are the permittivity and permeability of the original space. For the three dimensional, Euclidian space expressed in a Cartesian coordinate system, the ε and μ can be expressed in the form of

$$\begin{aligned} \varepsilon &= \varepsilon_0 \delta_{ij} \\ \mu &= \mu_0 \delta_{ij} \end{aligned} \tag{2}$$

where $\delta_{ij} = 1,\ i, j = 1, 2, 3$ for $i = j$ and $\delta_{ij} = 0$ elsewhere.

The matrix A is the Jacobian transformation tensor with the elements

$$A_{ij} = \partial x'_i / \partial x_j \tag{3}$$

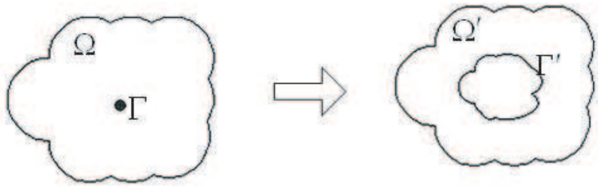

Figure 1. Illustration of the coordinate transformation for an arbitrary cloak.

Consider an arbitrary cloak as shown in Fig. 1, the inner boundary Γ in the original space expandes to Γ', and the space Ω accordingly compresses to the distorted one Ω'. Under this boundary condition, we can build the Poisson's equation which indicates the relationship between the original coordinate x and distorted coordinate x' as

$$\begin{cases} \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)x_i'=f_i(x_1,x_2,x_3),\ i=1,2,3 \\ x_i'|_{x\in\Gamma}\in\Gamma' \end{cases} \tag{4}$$

By solving Poisson's equation, we can get the coordinate relationship as $x_i'(x_1,x_2,x_3)$. In order to keep from the singular solution of the Poisson's equation, we can use the inverse form of the Poisson's equation as

$$\begin{cases} \left(\frac{\partial^2}{\partial x_1'^2}+\frac{\partial^2}{\partial x_2'^2}+\frac{\partial^2}{\partial x_3'^2}\right)x_i=f_i\left(x_1',x_2',x'_3\right),\ i=1,2,3 \\ x_i|_{x'\in\Gamma'}\in\Gamma \end{cases} \tag{5}$$

From the solution of Eq. (5), we can calculate the inverse Jacobian matrix elements $A_{ij}'=\partial x_i/\partial x_j'$, then determine the Jacobian transformation tensor via equation $A=(A')^{-1}$. The transformed material parameters now can be calculated using Eqs. (1) and (2).

3. APPLICATION TO REGULAR CLOAKS

Firstly, let's consider the standard cylindrical cloak, which is compressed from the cylindrical region $r\leq b$ into a concentric cylindrical shell $a\leq r'\leq b$. The Poisson's equation in the cylindrical coordinate system is expressed as

$$\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial u_i}{\partial r'}\right)+\frac{1}{r'^2}\frac{\partial^2 u_i}{\partial\theta'^2}+\frac{\partial^2 u_i}{\partial z'^2}=f_i\left(r',\theta',z'\right) \tag{6}$$

where u_i $(i=1,2,3)$ denotes the original coordinate component r, θ, z respectively.

For a two-dimensional cylindrical cloak, if we set $\theta = \theta'$ and $z = z'$, then the coordinate component r satisfies the following equation

$$\begin{cases} \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial r}{\partial r'}\right) = f(r', \theta', z') \\ r(r' = a) = 0, \quad r(r' = b) = b \end{cases} \tag{7}$$

For example, if the factor function is chosen with $f = b/r'(b - a)$, the solution of Eq. (7) has the form of

$$r = \frac{(r' - a)\,b}{b - a} \tag{8}$$

or equivalently

$$r' = (1 - a/b)r + a \tag{9}$$

where a and b are respectively the radii of the inner and outer boundaries of the cloak. Eq. (9) indicates the commonly linear transformation. Similarly, if we set the factor function with $f = 0$, which is just the Laplace equation, the solution of Eq. (7) has another form as

$$r = b \log_{b/a} \frac{r'}{a} \tag{10}$$

or

$$r' = a(b/a)^{r/b} \tag{11}$$

Obviously, the different factor functions can determine different coordinate transformation maps, so this method is more flexible compared with the solution of Laplace's equation method.

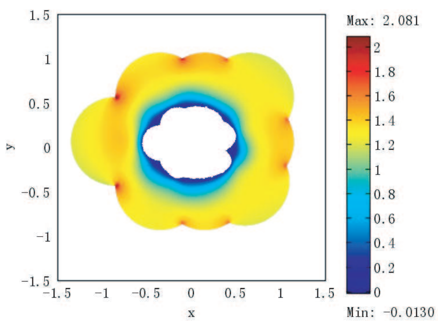

Figure 2. The contour plots of relative permittivity ε_{zz} determined by Laplace's equations.

4. APPLICATION TO IRREGULAR CLOAKS

An arbitrary cloak's boundary is difficult to be expressed in analytical form, so the Poisson's equation must be solved numerically. In this paper, we will use the commercial software COMSOL Multiphysics to solve the Poisson's equation with arbitrary boundary. As an example, we define a two-dimensional cloak as shown in Fig. 1. In the Cartesian coordinate system, the coordinate has the components x and y, so there are two Poisson's equations should be solved to get the coordinate transformations. This can be achieved by adding two PDE (partial differential equations) solver modes provided by COMSOL. In the PDE modes of Poisson's equations, we set the boundary conditions $x = \Gamma_x$ and $y = \Gamma_y$ for the Γ' boundary and $x = x'$, $y = y'$ for all other boundaries, where Γ_x and Γ_y denote the coordinate components of the Γ point. After solving these two Poisson's equations, the relationship $x(x')$ and $y(y')$ can be determined, then the material parameters of the cloak can be easily retrieved according to Eqs. (1)–(3). As an example, the relative permittivity ε_{zz} for an arbitrary cloak presented in Fig. 1 was calculated and shown in Fig. 2 by setting the factor function $f_x = f_y = 0$, which is simply the Laplace's equation.

To compare the different distributions of the material parameters, we consider four kinds of Poisson's equation with different factor functions. The relative permittivities ε_{zz} of these cloaks are shown in Fig. 3. It can be found that different parameter distributions for the same shape of cloak can be obtained through solving different Poisson's equations. Obviously, the function factor can easyly modulate the distribution maps and parameter ranges. With the help of optimization method, we even can determine the optimized material parameter which can be easily realized by the metamaterial technology.

For the verification, the full wave simulation of the cloak is solved with the help of the TE wave mode harmonic propagation solver provided by COMSOL. The cloak is embedded in the air of a square area. The upper and bottom boundaries are set with the scattering boundary conditions; left and right boundaries are set with wave port. The contour plots of the electric filed E_z for the plane waves incident on the cloak horizontally are shown in Fig. 4. Obviously, inside the cloak, the electric fields have different distributions for different coordinate transformations. The basic function of these cloaks can also be achieved, that is, the existence of the cloak does not disturb the incident waves and can shield an irregular obstacle from detection.

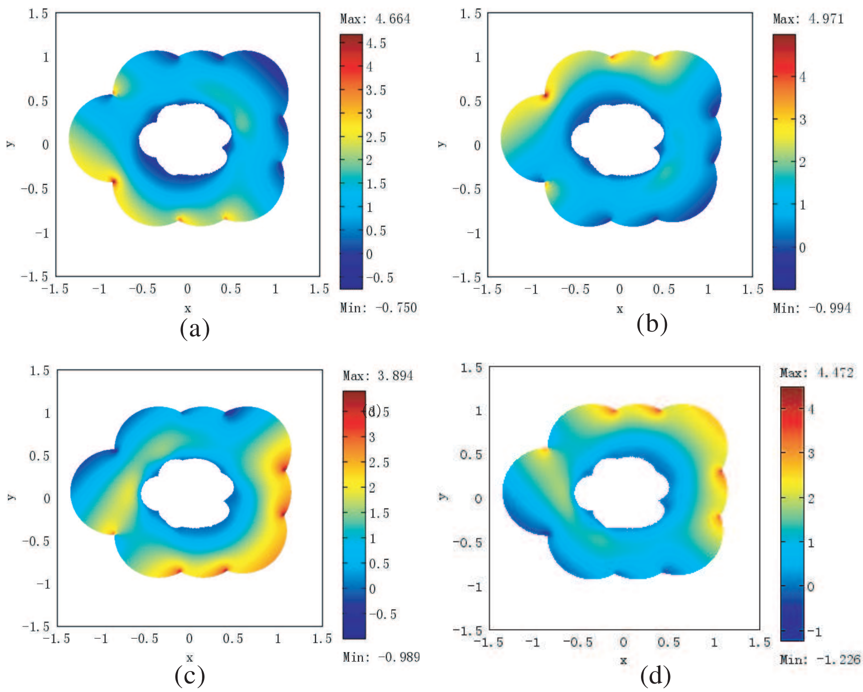

Figure 3. The contour plots of relative permittivity ε_{zz} determined by different Poisson's equations with factor function (a) $f_x = f_y = 4$, (b) $f_x = 4,\ f_y = -4$, (c) $f_x = -4,\ f_y = 4$ and (d) $f_x = f_y = -4$.

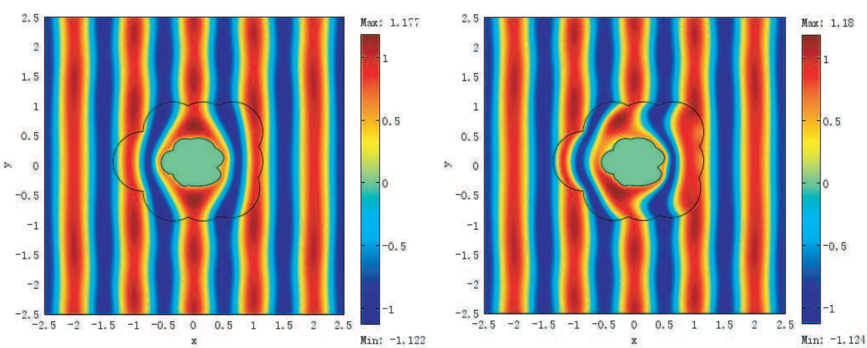

Figure 4. The contour plots of the electric fields determined by different kinds of Poisson's equations with (a) $f_x = f_y = 0$ and (b) $f_x = 4,\ f_y = -4$.

5. CONCLUSION

To conclude, we proposed a flexible method for designing arbitrary cloak based on the deformation theory. The Poisson's equations are proposed to evaluate numerically the deformation field, which are further used to compute the transformed material parameters inside the cloak. Compared with the Laplace's equation presented in [13], this method can determine diverse deformation material maps. The full wave simulation results with the help of the software COMSOL Multiphysics validate this method.

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