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# PULSE REFLECTION FROM A DIELECTRIC DISCONTINUITY IN A RECTANGULAR WAVEGUIDE

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**Abstract**—A simple, closed-form expression for the time-domain reflection coefficient for a pulsed $TE_{10}$-mode wave incident on a dielectric material discontinuity in a rectangular waveguide is presented. This formula may be used to represent the transient field reflected or transmitted by a dielectric-filled waveguide section, which is useful in material characterization routines. An exponential function approximation to the reflection coefficient is presented, and the formula is validated both numerically and experimentally.

## 1. INTRODUCTION

The propapation of transient electromagnetic fields in conducting waveguides has received a great deal of attention, with emphasis being on understanding the temporal evolution of the field under dispersive conditions [1–8]. Pulse propagation in inhomogeneously filled guides has received far less attention [9, 10], even though partially-filled guides are often used for determining the constitutive parameters of material samples [11–15]. Recently, Moradi and Abtipour [16] proposed using time-domain methods to extract the material parameters of waveguide samples. Such approaches require an understanding of the interaction of propagating transient fields with the interfaces between differing materials.

Butrym et al. [17] found a closed form expression for the impulsive field reflected by a perfect dielectric step discontinuity for both TE and TM modes using using a convolution approach, but their solution is cumbersome and is given in terms of hypergeometric functions. The result is extended in [18] to lossy dielectrics, but the resulting impulse response is very complicated, and an intuitive connection between

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the waveshape and the physical parameters is difficult to establish. Emig [19] gives a time-domain solution for a pulse incident on a material-filled waveguide section in terms of infinite series, but these provide little insight into the physical processes involved.

In this paper, the authors provide a simple expression for the impulse reflection and transmission responses for a transient $TE_{10}$ field incident upon a perfect dielectric step discontinuity in a rectangular waveguide. The expression is easy to implement, and may be used to find the transient response of a material-filled waveguide section used in time-domain material extraction techniques. Validation of the formulas is provided by comparison with a numerical transform of the frequency-domain expressions, and by comparison to swept-frequency measurements of the reflection coefficient. An exponential approximation for the reflection impulse response is also presented. Although the numerical values produced by the new expressions are not different from those produced by the expressions developed by previous authors, the simplicity of implementation should greatly enhance their usefulness in material parameter extraction schemes.

The results presented here are for lossless dielectrics. While this limits the applicability of the expressions, the formulas are very accurate for low loss materials such as plastics, which make up a large class of materials of interest to researchers. A comparison to the measured reflection from a typical plastic (acrylic) demonstrates the usefulness of the expression for describing the reflection from such materials. Extension of the present technique to lossy materials is only a matter of complexity, and is left for future research.

## 2. THEORY

Consider a rectangular waveguide loaded by a perfect dielectric of permittivity $\epsilon_2 = \epsilon_r \epsilon_0$ in the region $0 \leq z \leq d$, as shown in Figure 1. A

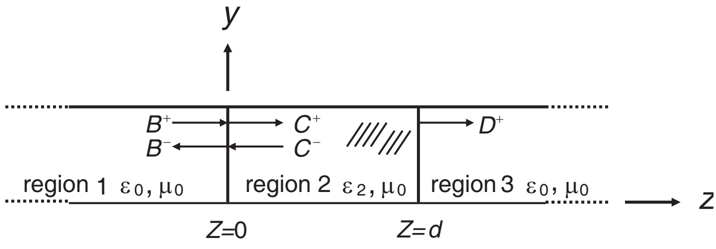


**Figure 1.** Side view of a rectangular waveguide loaded with a perfect dielectric material.

$\text{TE}_{10}$ wave of frequency $\omega = 2\pi f$ is incident from $z < 0$ on the dielectric insert, exciting an identical mode in the material region (region 2), and in the empty region $z > d$ (region 3). Thus, the transverse fields in these regions may be written as [20]

$$H_{x1} = \left(B^+ \mathrm{e}^{-jk_{z1}z} - B^- \mathrm{e}^{jk_{z1}z}\right) jk_{z1}k_c \sin(k_c x) \tag{1}$$

$$E_{y1} = -\left(B^+ \mathrm{e}^{-jk_{z1}z} + B^- \mathrm{e}^{jk_{z1}z}\right) Z_1 jk_{z1}k_c \sin(k_c x) \tag{2}$$

$$H_{x2} = \left(C^+ \mathrm{e}^{-jk_{z2}z} - C^- \mathrm{e}^{jk_{z2}z}\right) jk_{z2}k_c \sin(k_c x) \tag{3}$$

$$E_{y2} = -\left(C^+ \mathrm{e}^{-jk_{z2}z} + C^- \mathrm{e}^{jk_{z2}z}\right) Z_2 jk_{z2}k_c \sin(k_c x) \tag{4}$$

$$H_{x3} = D^+ \mathrm{e}^{-jk_{z1}(z-d)} jk_{z1}k_c \sin(k_c x) \tag{5}$$

$$E_{y3} = -D^+ \mathrm{e}^{-jk_{z1}(z-d)} Z_1 jk_{z1}k_c \sin(k_c x) \tag{6}$$

In these expressions $k_c = \pi/a$ is the cutoff wavenumber, $k_{z1} = \sqrt{k_0^2 - k_c^2}$, $k_{z2} = \sqrt{k_0^2 \epsilon_r \mu_r - k_c^2}$, $Z_1 = \eta_0 k_0 / k_{z1}$, $Z_2 = \mu_r \eta_0 k_0 / k_{z2}$, $k_0 = \omega\sqrt{\mu_0 \epsilon_0}$, and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$. For a dielectric, $\mu_r = 1$ is assumed. Also, note that the time convention $e^{j\omega t}$ is assumed.

Applying the boundary conditions of tangential field continuity at $z = 0$ and $z = d$ leads directly to formulas for the reflection coefficient $R_{oc}$ and the transmission coefficient $T$:

$$R_{oc}(\omega) = \frac{B^-}{B^+} = \Gamma \frac{1 - P^2}{1 - P^2 \Gamma^2} \tag{7}$$

$$T(\omega) = \frac{D^+}{B^+} = P \frac{1 - \Gamma^2}{1 - P^2 \Gamma^2}. \tag{8}$$

Here

$$P = \mathrm{e}^{-jk_{z2}d} \tag{9}$$

is the phase shift of a propagating wave passing through the dielectric region, and $\Gamma$ is the interfacial reflection coefficient for a wave incident on an interface between two semi-infinite waveguide regions:

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}. \tag{10}$$

A closely related problem occurs when region 3 is replaced with a perfect conductor (providing a short circuit at the end of region 2). Then the boundary condition on continuity of tangential fields at $z = d$ is replaced by the condition of zero tangential electric field, leading to the reflection coefficient

$$R_{sc}(\omega) = \frac{B^-}{B^+} = \frac{\Gamma - P^2}{1 - P^2 \Gamma}. \tag{11}$$

If the incident electric field has the frequency spectrum $E(\omega)$, then the reflected and transmitted field spectra are given simply by $E_r(\omega) = E(\omega)R(\omega)$ and $E_t(\omega) = E(\omega)T(\omega)$, respectively. Thus, the transient reflected and transmitted fields produced by pulse illumination of the dielectric insert can be found using the inverse Fourier transform and the convolution theorem:

$$E_r(t) = E(t) * R(t) \tag{12}$$

$$E_t(t) = E(t) * T(t). \tag{13}$$

Here $E_r(t) = \mathcal{F}^{-1}\{E_r(\omega)\}$, etc, and '$*$' is the convolution operator.

Although expressions for $R(t)$ and $T(t)$ may be formulated in terms of infinite series [19], expansions in terms of convolution sequences allows a more physical description of the temporal field as an infinite sequence of multiple reflections established within the dielectric region. These formulas are easily found by expanding the denominators of the frequency-domain counterparts (7), (8), and (11) in power series. For example, (11) becomes

$$\begin{aligned} R_{sc}(\omega) &= \left(\Gamma - P^2\right) \sum_{n=0}^{\infty} \left(\Gamma P^2\right)^n = \Gamma + \left(\Gamma^2 - 1\right) P^2 + \Gamma\left(\Gamma^2 - 1\right) P^4 + \ldots \\ &= \Gamma + T^+ \cdot P \cdot (-1) \cdot P \cdot T^- \\ &\quad + T^+ \cdot P \cdot (-1) \cdot P \cdot (-\Gamma) \cdot P \cdot (-1) \cdot P \cdot T^- + \ldots \end{aligned} \tag{14}$$

Here $T^+ = 1 + \Gamma$ is the interfacial transmission coefficient for a wave passing from air to dielectric, and $T^- = 1 - \Gamma$ is the interfacial transmission coefficient for a wave passing from dielectric to air. The multiple reflections may be identified by taking the inverse Fourier transform of (14), giving

$$\begin{aligned} R_{sc}(t) = {} & \Gamma(t) + T^+(t) * P(t) \cdot (-1) * P(t) * T^-(t) \\ & + T^+(t) * P(t) \cdot (-1) * P(t) * (-\Gamma(t)) \\ & * P(t) \cdot (-1) * P(t) * T^-(t) + \ldots \end{aligned} \tag{15}$$

where $P(t) = \mathcal{F}^{-1}\{P(\omega)\}$ describes the effect (time delay and dispersion) of propagation through the dielectric medium. Defining the Laplace transform variable as $s = j\omega$, $P(s)$ can be written as

$$P(s) = \mathrm{e}^{-\frac{d}{v}\sqrt{s^2 + k_c^2}}, \tag{16}$$

where $v = c/\sqrt{\epsilon_r}$. Using standard transform tables [21] immediately gives

$$P(t) = \delta\left(t - \frac{d}{v}\right) - \frac{k_c}{v} \frac{\mathrm{J}_1\left(k_c\sqrt{t^2 - (d/v)^2}\right)}{\sqrt{t^2 - (d/v)^2}} \mathrm{u}\left(t - \frac{d}{v}\right) \tag{17}$$

with $\mathrm{u}(t)$ the unit step function, $\delta(t)$ the impulse function, and $\mathrm{J}_1(x)$ the first-kind ordinary Bessel function of unity order. Here $t-d/v$ is the time delay for propagation through a distance $d$, and so the quantity $v$ is identified as the propagation velocity of the wavefront. The first term replicates the field waveform, while the second term produces the well-known oscillations associated with transient propagation in a waveguide [6]. These are primarily due to the cutoff effect rather than classic dispersion, and aren't observed with TEM structures such as a dispersive transmission line.

The physical interpretation of $R_{sc}(t)$ may now be readily explained. The first term in (15) represents the initial reflection from the air-dielectric interface. Because no information about the second interface yet exists, it is exactly the interfacial reflection coefficient $\Gamma(t)$. This is followed a time $2d/v$ later by a waveform that is transmitted through the first interface, propagates to the conductor, reflects, propagates back to the interface, and then is transmitted through in the opposite direction. The third term follows a time $4d/v$ from the initial reflection, and comprises the first multiple bounce inside the dielectric. Each subsequent term represents an additional bounce within the material.

It now becomes clear that the key quantity for describing the transient reflected field is the time-domain interfacial reflection coefficient $\Gamma(t)$. Once this is found, $T^{\pm}(t) = \delta(t) \pm \Gamma(t)$ follows immediately, and $R_{sc}(t)$ can be computed, as can $R_{oc}(t)$ and $T(t)$. To find $\Gamma(t)$, the Laplace variable $s = j\omega$ is defined, and the formulas for $Z_1$ and $Z_2$ are substituted into (10). This gives

$$\Gamma(s) = \frac{\sqrt{s^2 + s_1^2} - \sqrt{\epsilon_r}\sqrt{s^2 + s_2^2}}{\sqrt{s^2 + s_1^2} + \sqrt{\epsilon_r}\sqrt{s^2 + s_2^2}} \tag{18}$$

where $s_1 = k_c c$ and $s_2 = k_c v$. Since $\lim_{s\to\infty} \Gamma(s) \neq 0$, there is an impulsive component to $\Gamma(t)$ (which Butrym et al. [17] call the *singular* component), and it useful to isolate it. Noting that

$$\lim_{s\to\infty} \Gamma(s) = \Gamma_\infty = \frac{1 - \sqrt{\epsilon_r}}{1 + \sqrt{\epsilon_r}}, \tag{19}$$

it follows that

$$\begin{aligned} \tilde{\Gamma}(s) &= \Gamma(s) - \Gamma_\infty \\ &= \frac{2\sqrt{\epsilon_r}}{1 + \sqrt{\epsilon_r}} \frac{\sqrt{s^2 + s_1^2} - \sqrt{s^2 + s_2^2}}{\sqrt{s^2 + s_1^2} + \sqrt{\epsilon_r}\sqrt{s^2 + s_2^2}}. \end{aligned} \tag{20}$$

Here $\tilde{\Gamma}$ is called the *reduced reflection coefficient* (which is called the *regular* component of the reflection coefficient by Butrym et al. [17]).

To simplify (20), the radicals in the denominator may be cleared by multiplying the numerator and denominator by the factor

$$\sqrt{s^2+s_1^2}-\sqrt{\epsilon_r}\sqrt{s^2+s_2^2}. \tag{21}$$

This produces

$$\tilde{\Gamma}(s)=\frac{2\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}}\frac{N(s)}{D(s)}. \tag{22}$$

The denominator of this expression is

$$D(s) = s^2+(k_c c)^2-\epsilon_r s^2-\epsilon_r(k_c v)^2 \tag{23}$$

$$= (1-\epsilon_r)s^2. \tag{24}$$

Thus, (22) becomes

$$\tilde{\Gamma}(s)=KF(s)G(s), \tag{25}$$

where

$$K=\frac{2\sqrt{\epsilon_r}}{(1+\sqrt{\epsilon_r})(1-\epsilon_r)} \tag{26}$$

and

$$F(s) = \frac{1}{s}\left[\left(\sqrt{s^2+s_1^2}-s\right)-\left(\sqrt{s^2+s_2^2}-s\right)\right] \tag{27}$$

$$G(s) = \frac{1}{s}\left[\left(\sqrt{s^2+s_1^2}-s\right)-\sqrt{\epsilon_r}\left(\sqrt{s^2+s_2^2}-s\right)-(\sqrt{\epsilon_r}-1)\,s\right]. \tag{28}$$

Most of the terms in $F(s)$ and $G(s)$ take the generic form

$$W(x,s)=\frac{1}{s}\left(\sqrt{s^2+x^2}-s\right). \tag{29}$$

This function can be inverted using the Laplace transform pairs [21]

$$\sqrt{s^2+x^2}-s \leftrightarrow \mathrm{u}(t)\frac{x}{t}\mathrm{J}_1(xt) \tag{30}$$

$$\frac{1}{s}F(s) \leftrightarrow \mathrm{u}(t)\int_0^t f(\tau)d\tau \tag{31}$$

where $F(s)\leftrightarrow f(t)$. Combining these gives

$$W(x,s)\leftrightarrow w(x,t)=\mathrm{u}(t)x\int_0^{xt}\frac{\mathrm{J}_1(u)}{u}du. \tag{32}$$

This integral is tabulated in [22], and is used to produce

$$w(x,t)=x\left[\mathcal{J}_0(xt)-\mathrm{J}_1(xt)\right]\mathrm{u}(t). \tag{33}$$

Here $\mathscr{J}_0(x)$ is the Bessel function integral

$$\mathscr{J}_0(x) = \int_0^x \mathrm{J}_0(u)du, \tag{34}$$

which has well-known properties and many standard routines for its computation [23].

With $w(x,t)$ known, the reduced reflection coefficient may be evaluated from (25) using the convolution theorem. This gives

$$\begin{aligned}\tilde{\Gamma}(t) &= K\left[w(s_1,t) - w(s_2,t)\right] \\ &\quad * \left[w(s_1,t) - \sqrt{\epsilon_r}w(s_2,t) - (\sqrt{\epsilon_r}-1)\delta(t)\right]\end{aligned} \tag{35}$$

or

$$\begin{aligned}\tilde{\Gamma}(t) &= K\left\{w(s_1,t)*w(s_1,t) - (\sqrt{\epsilon_r}+1)w(s_1,t)*w(s_2,t)\right. \\ &\quad \left. +\sqrt{\epsilon_r}w(s_2,t)*w(s_2,t) - (\sqrt{\epsilon_r}-1)\left[w(s_1,t)-w(s_2,t)\right]\right\}.\end{aligned} \tag{36}$$

The first and third convolutions in (36) are evaluated analytically in Appendix A. Using (A6), $\tilde{\Gamma}(t)$ becomes finally

$$\tilde{\Gamma}(t) = \frac{2\sqrt{\epsilon_r}}{\epsilon_r - 1}\left[-\frac{s_1^2 t\mathrm{u}(t)}{\sqrt{\epsilon_r}} + w(s_1,t) + w(s_2,t) + w(s_1,t)*w(s_2,t)\right]. \tag{37}$$

Note that the expression (37) is exact, and is applicable to all lossless dielectric materials.

## 3. NUMERICAL VALIDATION

Equation (37) is the final form of the time-domain interfacial reflection coefficient. It may be validated numerically by comparing it to the inverse transform of the frequency-domain interfacial reflection coefficient $\tilde{\Gamma}(\omega)$ computed using the FFT.

There is no closed-form expression for the one remaining convolution in (36), so it must be evaluated numerically. This may be done using discrete convolution or the FFT. An alternative is to use the exponential approximation for Bessel functions given in [24]. Write

$$\mathrm{J}_0(x) = \frac{1}{2}\sum_{n=1}^{2N} a_n \mathrm{e}^{z_n x}, \quad 0 \le x \le 50, \tag{38}$$

where $\{a_n\}$ and $\{z_n\}$ are given in [24]. Differentiating (38) gives

$$\mathrm{J}_1(x) = \frac{1}{2}\sum_{n=1}^{2N} b_n \mathrm{e}^{z_n x}, \tag{39}$$

where $b_n = -z_n a_n$. Similarly, integrating (38) from 0 to $t$ gives

$$\mathcal{J}_0(x) = \frac{1}{2}\sum_{n=1}^{2N+1} c_n \mathrm{e}^{z_n x}, \tag{40}$$

where $z_{2N+1} = 0$ and

$$c_n = \frac{a_n}{z_n}, \quad c_{2N+1} = -\sum_{n=1}^{2N} \frac{a_n}{z_n}. \tag{41}$$

Combining these gives

$$w(x,t) = x\mathrm{u}(t)\frac{1}{2}\sum_{n=1}^{2N+1} d_n \mathrm{e}^{z_n x t}, \tag{42}$$

where $d_n = c_n - b_n$ and $b_{2N+1} = 0$. Thus,

$$\begin{aligned} w(s_1,t) * w(s_2,t) &= \mathrm{u}(t)\frac{1}{4}\int_0^t \sum_{n=1}^{2N+1} d_n \mathrm{e}^{z_n s_1 \tau} \sum_{m=1}^{2N+1} d_m \mathrm{e}^{z_m s_2 (t-\tau)} d\tau \\ &= \mathrm{u}(t)\frac{s_1 s_2}{4}\sum_{n=1}^{2N+1}\sum_{m=1}^{2N+1} d_n d_m \frac{\mathrm{e}^{z_n s_1 t} - \mathrm{e}^{z_m s_2 t}}{z_n s_1 - z_m s_2}. \end{aligned} \tag{43}$$

As an example, consider a sample of acrylic placed into an X-band waveguide (WR-90) with dimensions 0.9 by 0.4 inches (2.286 by 1.016 cm). Acrylic is a low-loss plastic with a dielectric constant of approximately 2.6 over all of X-band. For this example, a value of $\epsilon_r = 2.64$ was chosen, to match the results from the experiment described in Section 4. The frequency domain reflection coefficient (10) was then evaluated. A plot of $\Gamma(\omega)$ is shown in Figure 2. Three distinct frequency ranges can be seen. Below $f_{c1} = c/(2a\sqrt{\epsilon_r}) = 4.04\,\mathrm{GHz}$ the wave is evanescent in both the air and material regions, and the magnitude of the reflection coefficient is less than unity. Between $f_{c1}$ and $f_{c2} = c/(2a) = 6.557\,\mathrm{GHz}$ the wave is evanescent in the material region, but propagates in the air region, and the magnitude of the reflection coefficient is unity (total reflection). Above $f_{c2}$ the wave propagates in both regions, and the magnitude of $\Gamma$ reduces from unity to $\Gamma_\infty = -0.238$ as $\omega \to \infty$. The phase of $\Gamma$ is shown in Figure 3.

To obtain $\tilde{\Gamma}(\omega)$, the value of $\Gamma_\infty$ is subtracted from $\Gamma(\omega)$, with the resulting magnitude shown in Figure 4. It can be seen that $|\tilde{\Gamma}|$ is quite small at the high frequency boundary of X-band, which allows the inverse FFT to be employed without windowing. The time-domain reduced reflection coefficient $\tilde{\Gamma}(t)$ may be computed by taking the inverse transform of $\tilde{\Gamma}(\omega)$. The result is shown in Figure 5. Also shown in Figure 5 is the time-domain reduced reflection coefficient found by evaluating the time-domain formula (37) using the expansion (43). The results are nearly identical, providing a validation of the time-domain formula (37).

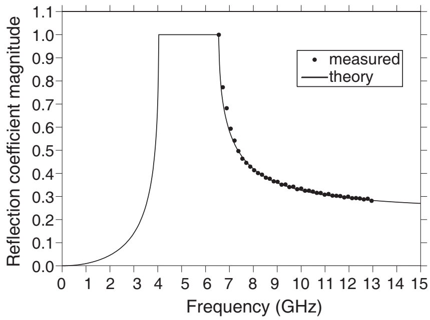


**Figure 2.** Magnitude of measured interfacial reflection coefficient $\Gamma$ compared to the theoretical value computed using $\epsilon_r = 2.64$.

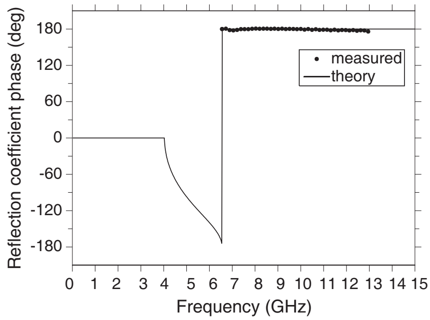


**Figure 3.** Phase of measured interfacial reflection coefficient $\Gamma$ compared to the theoretical value computed using $\epsilon_r = 2.64$.

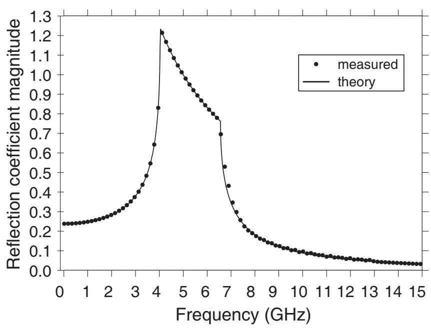


**Figure 4.** Magnitude of measured reduced interfacial reflection coefficient $\tilde{\Gamma}$ compared to the theoretical value computed using $\epsilon_r = 2.64$. Below cutoff, the measured data is synthesized from the extracted value of $\epsilon_r$.

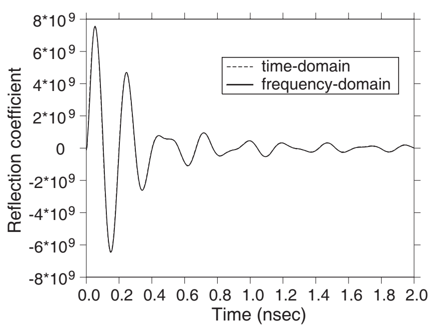


**Figure 5.** Time-domain reduced interfacial reflection coefficient computed by taking the FFT of the frequency-domain formula compared to that obtained using the time-domain formula.

## 4. EXPERIMENT

To provide an experimental validation of the results of Section 3, an acrylic sample of thickness $d = 0.498\,\mathrm{cm}$ (0.196 inches) was inserted into the end of a section of WR-90 waveguide. The opposite end was connected to an HP 8510C network analyzer and the reflection coefficient $S_{11}$ was measured under two terminating conditions. In the first situation a metal plate was placed against the material sample, providing the reflection coefficient $R_{sc}$. In the second case, a matched load was attached to the waveguide section, providing the reflection coefficient $R_{oc}$. Calibration was performed using a short/load/thru method, which placed the reference plane at the the front face of the material sample. Measurement of $S_{11}$ was made in the band 6.557 GHz to 13 GHz. Note that $R_{sc}$ and $R_{oc}$ could not be measured below waveguide cutoff since the attenuation of the evanescent wave was too severe to produce accurate results using the calibration procedure. Measurements above 13 GHz became inaccurate due to excitation of higher-order waveguide modes.

It is difficult to measure $\Gamma$ directly, since this would require zero reflected field in the material region (in essence, isolating $\Gamma(t)$ in the series (15)). Using an air-filled waveguide matched load is ineffective since it introduces a material/air interface that causes a reflected wave, and using a material-filled matched load is impractical. However, $\Gamma$ can be determined using the measured values of $R_{sc}$ and $R_{oc}$. Since in each of these measurements the propagation factor $P(\omega)$ is identical, both (7) and (11) may be solved for $P$ and the results equated to give

$$P = \frac{\Gamma - R_{oc}}{\Gamma - \Gamma^2 R_{oc}} = \frac{\Gamma - R_{sc}}{1 - \Gamma R_{sc}}. \tag{44}$$

Rearranging gives a cubic equation for $\Gamma$,

$$\Gamma^3 - A\Gamma^2 + A\Gamma - 1 = 0, \tag{45}$$

where $A = (R_{sc}R_{oc} + R_{sc} + 1)/R_{oc}$. The cubic may be factored into the form $(\Gamma - 1)(\Gamma^2 - [A-1]\Gamma + 1) = 0$. Since the solution $\Gamma = 1$ is unphysical, the two possible values for $\Gamma$ are

$$\Gamma = \frac{A-1}{2} \pm \sqrt{\left(\frac{A-1}{2}\right)^2 - 1}. \tag{46}$$

For a lossless material, only one of these solutions has $|\Gamma| < 1$.

To compare the measured values of $\Gamma$ to theory, it is necessary to know the value of $\epsilon_r$. This can be determined experimentally using the measured reflection coefficients $R_{sc}$ and $R_{oc}$. With the measured value of $\Gamma$ known, $P$ can be found from (44). These two terms allow

$\mu_r$ and $\epsilon_r$ to be computed as a function of frequency. From (9), the propagation constant in the material region is

$$k_{z2} = \frac{\ln P \pm j2n\pi}{-jd} \tag{47}$$

where $n$ determines the branch of the square root function, and is chosen so that both $\Re\{\mu_r\} > 0$ and $\Re\{\epsilon_r\} > 0$. Then, using

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{\mu_r k_{z1} - k_{z2}}{\mu_r k_{z1} + k_{z2}} \tag{48}$$

gives immediately

$$\mu_r = \frac{k_{z2}}{k_{z1}} \frac{1+\Gamma}{1-\Gamma}. \tag{49}$$

Next, the wavenumber in region 2 is

$$k_2^2 = \mu_r \epsilon_r k_0^2 = k_{z2}^2 + k_c^2, \tag{50}$$

and thus

$$\mu_r \epsilon_r = 1 + \frac{k_{z2}^2 - k_{z1}^2}{k_0^2} \tag{51}$$

which gives $\epsilon_r$. Figure 6 shows the extracted values of $\mu_r$ and $\epsilon_r$ found using (49) and (51). Over the band 8.2–12.4 GHz (X-band), the average values are $\epsilon_r = 2.640 - j0.02472$ and $\mu_r = 0.9816 + j0.002796$. The result of $\mu_r \approx 1$ gives some confidence that the extraction technique is working properly.

Figures 2 and 3 show the measured magnitude and phase of $\Gamma$ found using (46). Also shown are the theoretical values found using (10) with $\epsilon_r = 2.64$. It is seen that the measured values match very well with the theoretical values, with the largest differences occurring near the measurement band edges. The experimental reduced reflection coefficient may be found using the measured value of $\epsilon_r$. Figure 4 shows this value compared to the theoretical value, with both found using $\epsilon_r = 2.64$. The inverse transform of the measured $\tilde{\Gamma}(\omega)$ gives the time-domain reduced reflection coefficient. However, this can't be found directly, since data for $f < f_{c2} = 6.557$ GHz is not available. To allow for a comparison, the missing data may be replaced by the theoretical data found using the experimental value of $\epsilon_r$. When this data is inverse transformed using the FFT, the result shown in Figure 7 is obtained. The comparison to the time-domain expression (37) is excellent, further validating the time-domain formula.

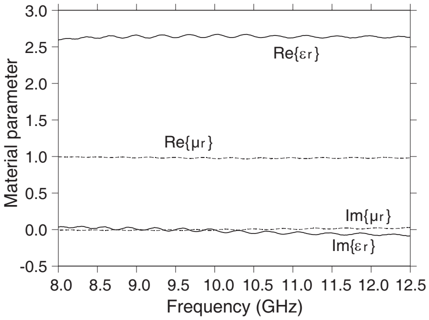


**Figure 6.** Values of $\epsilon_r$ and $\mu_r$ extracted from the measured reflection coefficients $R_{sc}$ and $R_{oc}$.

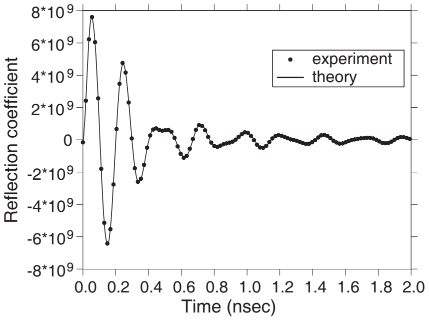


**Figure 7.** Time-domain reduced interfacial reflection coefficient obtained from an inverse FFT of the measured $\Gamma$, compared to that obtained using the time-domain formula with $\epsilon_r = 2.64$.

## 5. CONCLUSION

A simple closed form expression is introduced to compute the time-domain reflection coefficient for a transient $TE_{10}$ mode wave incident on a dielectric step discontinuity in a rectangular waveguide. An exponential series approximation is provided for efficient computation of the reflected and transmitted field waveforms. Validation of the formula is accomplished both numerically and experimentally. The closed-form expression should prove useful in time-domain material characterization systems that use waveguide applicators.

## APPENDIX A. DERIVATION OF A CONVOLUTION FORMULA

Consider the convolution

$$w(x,t) * w(x,t) = \left[\mathrm{u}(t) * \left\{\mathrm{u}(t)\frac{x}{t}\mathrm{J}_1(xt)\right\}\right] * \left[\mathrm{u}(t) * \left\{\mathrm{u}(t)\frac{x}{t}\mathrm{J}_1(xt)\right\}\right]$$
$$= \mathrm{u}(t) * \mathrm{u}(t) * \left\{\mathrm{u}(t) x^2 \int_0^t \frac{\mathrm{J}_1(x\tau)}{\tau}\frac{\mathrm{J}_1(x[t-\tau])}{t-\tau} d\tau\right\}. \quad \text{(A1)}$$

The integral may be evaluated [22] to give

$$\int_0^t \frac{\mathrm{J}_1(x\tau)}{\tau}\frac{\mathrm{J}_1(x[t-\tau])}{t-\tau} d\tau = 2\frac{\mathrm{J}_2(xt)}{t}. \quad \text{(A2)}$$

This gives

$$w(x,t) * w(x,t) = x^2 \mathrm{u}(t) * \mathrm{u}(t) \int_0^t 2 \frac{\mathrm{J}_2(x\tau)}{\tau} d\tau. \quad \text{(A3)}$$

This integral may be evaluated [22] to give

$$\int_0^t 2 \frac{\mathrm{J}_2(x\tau)}{\tau} d\tau = 1 - 2 \frac{\mathrm{J}_1(xt)}{xt}. \quad \text{(A4)}$$

Thus

$$w(x,t) * w(x,t) = x^2 \mathrm{u}(t) \left[ t - 2x \int_0^{xt} \frac{\mathrm{J}_1(\tau)}{\tau} d\tau \right] \quad \text{(A5)}$$

or

$$w(x,t) * w(x,t) = x^2 t \mathrm{u}(t) - 2w(x,t). \quad \text{(A6)}$$

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