Progress In Electromagnetics Research B, Vol. 18, 83–99, 2009

DIRECTION OF ARRIVAL ESTIMATION BASED ON FOURTH-ORDER CUMULANT USING PROPAGATOR METHOD

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Abstract—In this paper direction-of-arrival estimation (DOA) of multiple narrow-band sources, based on higher-order statistics using propagator, is presented. This technique uses fourth-order cumulants of the received array data instead of second-order statistics (auto-covariance) and then the so-called propagator approach is used to estimate the DOA of the sources. The propagator is a linear operator which only depends on the array steering vectors and which can be easily extracted from the received array data. But it does not require any eigendecomposition of the cumulant matrix of the received data like MUSIC algorithm. Computer simulations are carried out to compare the performance of the proposed method to those of methods based on auto-covariance using MUSIC and propagator algorithms.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation ranks as one of the most important problems of array signal processing. Considerable research efforts have been and continue to be made for developing efficient and effective algorithms for DOA estimation as evident from the volume of journal publications on this problem in the signal processing literature over the last three decades. DOA estimation is important in many applications such as radar, sonar and electronic surveillance. Recent applications include array processing for wireless communications at the base station for increasing the capacity and quality of the systems. In all these systems the time complexity and the capability of resolving two closely spaced sources plays an important role in deciding the performance of the systems.

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There are many array models and algorithms [1–7] are available in literature for estimating the DOA of the narrowband sources. Among these, MUltiple SIgnal Classification (MUSIC) [7, 8] is the high resolution algorithm based on the eigendecomposition of the auto-covariance of the received data and has been widely used. But, in applications where the large array size is required, the use of such method is unattractive owing to their intensive computational complexity. A possible alternative to the MUSIC method for source bearing estimation with arrays consisting of a large number of sensors is the propagator method (PM). Marcos et al. [9] have proposed the so-called 'propagator, method for array signal processing without any eigen-decomposition. The propagator is a linear operator based on a partition of the steering vectors, and was found to be a very effective tool for estimating the DOAs. PM has a lower computational complexity at the expense of negligible performance loss. In [10–16], the DOA estimation methods have been developed based on higher-order statistics instead of second-order statistics using generalized eigen structure analysis. Porat and Friedlander [12] proposed the DOA estimation method based on fourth-order cumulant to eliminate the effect of Gaussian noise from the non-Gaussian signals. Using fourth-order cumulant, a physical array size could be increased to larger size virtual array [14–16] and allows estimating large number of sources. In this paper, DOA estimation of multiple narrowband sources based on fourth-order statistics using propagator is presented.

The paper is organized as follows: In Section 2, the data model is discussed and the MUSIC algorithm for DOA estimation based on fourth-order cumulant is discussed in Section 3. The proposed cumulant propagator method is presented in Section 4. The statistical performance analysis of both cumulant propagator and cumulant music are presented in Section 5. In Section 6, the numerical simulations that illustrate the root mean-square error reduction and resolution improvement achieved by the cumulant propagator method as compared to the propagator method based on second-order statistics are presented. Finally, Section 7 concludes the paper.

Throughout this paper, vectors are denoted by lowercase bold letters and matrices by uppercase bold letters. The superscripts *, T and H denote respectively complex conjugation, transposition and conjugate transposition.

2. DATA MODEL

Consider a uniform linear array consists of L sensors, with equal inter-sensor spacing d, on which M plane wave signals impinge ($L >$

M). It is assumed that the signal emitted by the mth source is a zero-mean non-Gaussian signal. We assume that the M source signals $s_m(t)$, $m = 1, 2, \ldots, M$, are statistically independent. We further assume that the noises $v_l(t)$, $l = 1, 2, \ldots, L$, at various sensor outputs, are uncorrelated zero-mean white Gaussian processes which are independent to the signals. The received noise corrupted signal $x_l(t)$, during the observation interval, at the lth sensor output is sampled. The samples at all the sensor outputs can be represented as

$$\mathbf{x}(t) = \mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{v}(t) \text{ for } t = 1, 2, \ldots, N, \tag{1}$$

where

$\mathbf{x}(t)\underline{\Delta}[x_1(t)\ x_2(t)\ \ldots\ x_L(t)]^T$ is $L \times 1$ observation vector (snap-shot vector),

$\mathbf{A}(\theta)\underline{\Delta}[\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2)\ \ldots\ \mathbf{a}(\theta_M)]^T$ is $L \times M$ array manifold matrix,

$\mathbf{s}(t)\underline{\Delta}[s_1(t)\ s_2(t)\ \ldots\ s_M(t)]^T$ is $M \times 1$ signal vector,

and $\mathbf{v}(t)\underline{\Delta}[v_1(t)\ v_2(t)\ \ldots\ v_L(t)]^T$ is $L \times 1$ noise vector,

and where

$$\begin{aligned}\mathbf{a}(\theta_m) \ = \ & [1 \exp\{j(2\pi d/\lambda)\cos\theta_m\}\ \exp\{j(2\pi d/\lambda)2\cos\theta_m\}\ldots \\ & \exp\{j(2\pi d/\lambda)(L-1)\cos\theta_m\}]^T\end{aligned}$$

where λ is the wave length.

The problem is that given the array output $\{\mathbf{x}(t),\ t = 1, 2, \ldots, N\}$, where N denotes the number of snap-shots, estimate the DOA parameter θ_m, $m = 1, 2, \ldots, M$, of the impinging signals. For this problem, we shall work with a fourth-order cumulant of the received array output which can be expressed in matrix notation as

$$\begin{aligned}\mathbf{C} \ = \ & E\left[(\mathbf{x}\otimes\mathbf{x}^*)(\mathbf{x}\otimes\mathbf{x}^*)^H\right] - E[(\mathbf{x}\otimes\mathbf{x}^*)]\,E\left[(\mathbf{x}\otimes\mathbf{x}^*)^H\right] \\ & -E\left[\mathbf{x}\mathbf{x}^H\right]\otimes E\left[\left(\mathbf{x}\mathbf{x}^H\right)^*\right]\end{aligned} \tag{2}$$

where $\otimes$ denotes the Kronecker product and the dimension of the cumulant matrix $\mathbf{C}$ is $L^2 \times L^2$. Since the fourth-order cumulants of Gaussian noise are identically zero, using (1) in (2) we get

$$\mathbf{C} = (\mathbf{A}\otimes\mathbf{A}^*)\,\mathbf{C}_s\,(\mathbf{A}\otimes\mathbf{A}^*)^H \tag{3}$$

where

$$\mathbf{C}_s\underline{\Delta}E\left[(\mathbf{s}\otimes\mathbf{s}^*)(\mathbf{s}\otimes\mathbf{s}^*)^H\right] - E[(\mathbf{s}\otimes\mathbf{s}^*)]\,E\left[(\mathbf{s}\otimes\mathbf{s}^*)^H\right] - E\left[\mathbf{s}\mathbf{s}^H\right]\otimes E\left[\left(\mathbf{s}\mathbf{s}^H\right)^*\right]$$

is the fourth-order cumulants of $\mathbf{s}$. Since the sources are statistically independent, the fourth-order cumulants of the signal vector $\mathbf{s}$ is a diagonal matrix so (3) can be written as

$$\mathbf{C} = \sum_{m=1}^{M} [\mathbf{a}(\theta_m) \otimes \mathbf{a}^* (\theta_m)] \, \mu_m \, [\mathbf{a}(\theta_m) \otimes \mathbf{a}^* (\theta_m)]^H \tag{4}$$

where μ_m is the fourth-order cumulants of s_m for $m = 1, 2, \ldots, M$. Define

$$\mathbf{b}(\theta_m) \underline{\Delta} \mathbf{a}(\theta_m) \otimes \mathbf{a}^*(\theta_m), \; m = 1, 2, \ldots, M,$$
$$\mathbf{B}(\theta) \underline{\Delta} [\mathbf{b}(\theta_1) \; \mathbf{b}(\theta_2) \; \ldots \; \mathbf{b}(\theta_M)]$$

and

$$\mathbf{D} \underline{\Delta} \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_M\}$$

Now (4) can be written as

$$\mathbf{C} = \mathbf{B}\mathbf{D}\mathbf{B}^H \tag{5}$$

Since $\mathbf{B}$ is comprised of $\mathbf{b}(\theta_m) = \mathbf{a}(\theta_m) \otimes \mathbf{a}^*(\theta_m), \; m = 1, 2, \ldots, M,$ which are linearly independent, it has full column rank, and $\mathbf{D}$ is non-singular (since sources are independent).

3. MUSIC LIKE ALGORITHM

The range of $\mathbf{C}$, which is a L^2-dimensional space, can be decomposed into two orthogonal subspaces: (i) An M-dimensional subspace, called signal subspace, which is spanned by the eigenvectors corresponding to the M largest eigenvalues and (ii) the complementary $(L^2 - M)$-dimensional subspace called noise subspace. The signal subspace and the noise subspace can be found by eigendecomposition of the covariance matrix $\mathbf{C}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{L^2}$ denote the eigenvalues of $\mathbf{C}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{L^2}$ the corresponding eigenvectors. Then, the eigenvectors satisfy the equations

$$\mathbf{C}\mathbf{v}_k = \lambda_k \mathbf{v}_k \text{ for } k = 1, 2, \ldots, L^2, \tag{6}$$

From elementary linear algebra, it can be shown that $(L^2 - M)$ eigenvalues of $\mathbf{C}$ will be equal to zero. Without loss of generality, the eigenvalues can be arranged in nondecreasing order as

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M > \lambda_{M+1} = \lambda_{M+2} = \ldots = \lambda_{L^2} = 0$$

Let $\mathbf{Q} \; \underline{\Delta}[\mathbf{v}_1, \, \mathbf{v}_2, \, \ldots, \mathbf{v}_M \,]$ as a $L^2 \times M$ matrix, $\mathbf{V} \; \underline{\Delta}[\mathbf{v}_{M+1}, \mathbf{v}_{M+2}, \ldots, v_{L^2}]$ as a $L^2 \times (L^2 - M)$ matrix, $\mathbf{\Sigma} \underline{\Delta} \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_M)$ as $M \times M$ matrix and $\mathbf{\Lambda} \underline{\Delta} \mathrm{diag}(\lambda_{M+1}, \lambda_{M+2}, \ldots, \lambda_{L^2})$ as a $(L^2 - M) \times (L^2 - M)$ zero matrix. From Equations (5) and (6), we get

$$\mathbf{C}\mathbf{V} \; = \mathbf{V}\mathbf{\Lambda} = \mathbf{0} = \mathbf{B}\mathbf{D}\mathbf{B}^H V \tag{7}$$

Since the matrix $\mathbf{BD}$ has full column rank, (7) implies that

$$\mathbf{B}^H\mathbf{V} = \mathbf{0} \tag{8}$$

This means that the eigenvectors, associated with the eigenvalue zero of multiplicity $(L^2 - M)$, are orthogonal to the vectors $\mathbf{b}(\theta_m) = \mathbf{a}(\theta_m) \otimes \mathbf{a}^*(\theta_m)$, $m = 1, 2, \ldots, M$. This implies that the vectors $\mathbf{b}(\theta_m)$, $m = 1, 2, \ldots, M$, are orthogonal to the column space of $\mathbf{V}$. Since the vectors $\mathbf{b}(\theta_m)$, $m = 1, 2, \ldots, M$, are orthogonal to the eigenvectors corresponding to the zero eigenvalues, the true DOA parameters θ_m, $m = 1, 2, \ldots, M$, of the sources are the unique solutions of the equation

$$\mathbf{b}^H(\theta)\mathbf{V}\mathbf{V}^H\mathbf{b}(\theta) = 0 \tag{9}$$

In practice, we do not know the actual eigenvalues and eigenvectors of the cumulant matrix $\mathbf{C}$ and must estimate them from the received snap-shot vectors. We denote the estimated cumulant matrix by $\hat{\mathbf{C}}$. It can be computed from

$$\hat{\mathbf{C}} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{y}(t)\mathbf{y}^H(t) - \frac{1}{N^2}\sum_{t=1}^{N}\mathbf{y}(t)\sum_{t=1}^{N}\mathbf{y}^H(t) - \left(\hat{\mathbf{R}} \otimes \hat{\mathbf{R}}^*\right) \tag{10}$$

where $y(t)\underline{\Delta}\mathbf{x}(t) \otimes \mathbf{x}(t)^*$ and $\hat{\mathbf{R}}\underline{\Delta}\frac{1}{N}\sum\limits_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^H(t)$. Thus, one can estimate the vectors $\mathbf{b}(\theta_m)$, $m = 1, 2, \ldots, M$, by finding the vectors which are most nearly orthogonal to the eigenvectors corresponding to the $(L^2 - M)$ eigenvalues of $\hat{\mathbf{C}}$ that are approximately zero. Let $\hat{\mathbf{V}}$ denote the matrix defined similarly to $\mathbf{V}$, but made from the eigenvectors of $\hat{\mathbf{C}}$. Then, the DOA of the multiple source signals can be estimated by locating the M largest peaks of the spatial spectrum given by

$$F(\theta) = \frac{1}{\mathbf{b}^H(\theta)\hat{\mathbf{V}}\hat{\mathbf{V}}^H\mathbf{b}(\theta)}, \quad \theta \in [0, \pi] \tag{11}$$

4. PROPOSED ALGORITHM

Now we apply the concept of propagator method for estimating the DOA of the sources. We first give the definition of the propagator. Since the matrix $\mathbf{B}$ is a full column rank matrix, M rows of $\mathbf{B}$ are linearly independent. The other rows can be expressed as a linear combination of these M rows. Hereafter, we will assume that the first M rows of $\mathbf{B}$ are linearly independent. Let $\mathbf{B}_1$ denote the $M \times M$ sub-matrix of the $L^2 \times M$ matrix $\mathbf{C}$ comprising the first M rows. Denoting by $\mathbf{B}_2$ the $(L^2 - M) \times M$ sub-matrix of $\mathbf{C}$ comprising the remaining

L^2-M rows, we may write $\mathbf{B}_2 = \mathbf{P}^H\mathbf{B}_1$ where $\mathbf{P}^H$ is the $(L^2-M)\times M$ matrix whose entries are coefficients of the linear combinations. The Hermitian $\mathbf{P}$ of $\mathbf{P}^H$ is called propagator matrix. Thus, we have the partition

$$\mathbf{B} = \left[\mathbf{B}_1^H : \mathbf{B}_2^H\right]^H = \left[\mathbf{B}_1^H : \mathbf{B}_1^H\mathbf{P}\right]^H \tag{12}$$

Using (12), we can partition $\mathbf{C}$ in (5) as

$$\mathbf{C} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{D}_1\mathbf{P} \\ \mathbf{P}^H\mathbf{D}_1 & \mathbf{P}^H\mathbf{D}_1\mathbf{P} \end{bmatrix} \tag{13}$$

where $\mathbf{D}_1 = \mathbf{B}_1\mathbf{D}\mathbf{B}_1^H$. Defining

$$\mathbf{C_1}\underline{\Delta}\begin{bmatrix} \mathbf{D}_1 \\ \mathbf{P}^H\mathbf{D}_1 \end{bmatrix} \text{ and } \mathbf{C}_2\underline{\Delta}\begin{bmatrix} \mathbf{D}_1\mathbf{P} \\ \mathbf{P}^H\mathbf{D}_1\mathbf{P} \end{bmatrix} = \mathbf{C}_1\mathbf{P}$$

(13) can be expressed as

$$\mathbf{C} = [\mathbf{C}_1 : \mathbf{C}_2] \tag{14}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are matrices of dimension $L^2\times M$ and $L^2\times(L^2-M)$ respectively. From (12) we can write that

$$\mathbf{U}^H\mathbf{B}\underline{\Delta}\left[\mathbf{P}^H : -\mathbf{I}_{L^2-M}\right]\mathbf{B} = \mathbf{0} \tag{15}$$

where $\mathbf{U}^H = [\mathbf{P}^H : -\mathbf{I}_{L^2-M}]$ is the matrix of dimension $L^2\times(L^2-M)$ and $\mathbf{0}$ is the $(L^2-M)\times M$ matrix of zeros. The relation (15) means that the vectors $\mathbf{b}(\theta_m)$, $m = 1, 2, \ldots, M$, are orthogonal to the columns of $\mathbf{U}$. This means that the column space of the matrix $\mathbf{U}$ is included in the column space of $\mathbf{V}$. Since the matrix $\mathbf{U}$ contains the $(L^2-M)\times(L^2-M)$ identity matrix as sub-matrix, its L^2-M columns are linearly independent. Therefore, the column space of $\mathbf{U}$ and the column space of $\mathbf{V}$ is equal. This follows that the propagator defines the subspace as does the matrix $\mathbf{V}$ of the eigenvectors corresponding to the zero eigenvalues of the fourth-order cumulant matrix $\mathbf{C}$. Therefore, the true DOA parameters θ_m, $m = 1, 2, \ldots, M$, of the sources are the unique solutions of the equation

$$\mathbf{b}^H(\theta)\mathbf{U}\mathbf{U}^H\mathbf{b}(\theta) = 0 \tag{16}$$

Considering (14) to define a partition of the fourth-order cumulant matrix $\mathbf{C}$ computed on the basis of (2), the relation

$$\mathbf{C}_2 = \mathbf{C}_1\mathbf{P} \tag{17}$$

between the sub-matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ may not be satisfied in practice since the actual fourth-order cumulant matrix $\mathbf{C}$ has to be approximated by the corresponding sample estimates of the cumulant matrix $\hat{\mathbf{C}}$ which is in turn estimated from a finite number of snap-shots according to (10). However, a least-square solution for the estimate $\hat{\mathbf{P}}$ of the propagator matrix $\mathbf{P}$ satisfying $\hat{\mathbf{C}}_2 = \hat{\mathbf{C}}_1\hat{\mathbf{P}}$ may be obtained by minimizing the cost function

$$J(\hat{\mathbf{P}}) = \left\| \hat{\mathbf{C}}_2 - \hat{\mathbf{C}}_1\hat{\mathbf{P}} \right\|_F^2$$

where $\hat{\mathbf{C}}_1$, $\hat{\mathbf{C}}_2$ are sub-matrices of $\hat{\mathbf{C}}$ and $\|.\|_F$ denotes the Frobenius norm. The cost function $J(\hat{\mathbf{P}})$ being quadratic (convex) function of $\hat{\mathbf{P}}$, may be minimized to give the unique least-square solution for $\hat{\mathbf{P}}$:

$$\hat{\mathbf{P}} = \left(\hat{\mathbf{C}}_1^H \hat{\mathbf{C}}_1 \right)^{-1} \hat{\mathbf{C}}_1^H \hat{\mathbf{C}}_2$$

The propagator matrix is thus obtained. The computational cost incurred for determining $\hat{\mathbf{P}}$ is very much less than that incurred in a search for the eigen-elements of a fourth-order cumulant matrix. Given the vectors $\mathbf{b}(\theta)$, the DOA of the source signals can be estimated by locating the M largest peaks of the function given by

$$\xi(\theta) = \frac{1}{\mathbf{b}^H(\theta)\hat{\mathbf{U}}\hat{\mathbf{U}}^H\mathbf{b}(\theta)}, \quad \theta \in [0, \pi] \tag{18}$$

where $\hat{\mathbf{U}} = [\hat{\mathbf{P}}^H : -\mathbf{I}_{L^2-M}]$.

The difference between $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$ is that the columns of $\hat{\mathbf{V}}$ are orthogonal (since they are different eigen vectors) but the columns of $\hat{\mathbf{U}}$ are not orthogonal. In order to introduce the orthonormalization, we can replace matrix $\hat{\mathbf{U}}$ by its orthonormalized version

$$\hat{\mathbf{U}}_o = \hat{\mathbf{U}} \left(\hat{\mathbf{U}}^H \hat{\mathbf{U}} \right)^{-1/2} \tag{19}$$

We then obtain the following function

$$\xi_o(\theta) = \frac{1}{\mathbf{b}^H(\theta)\hat{\mathbf{U}}_o\hat{\mathbf{U}}_o^H\mathbf{b}(\theta)}, \quad \theta \in [0, \pi] \tag{20}$$

5. PERFORMANCE ANALYSIS

In this section, the performance analysis of the both cumulant MUSIC and the cumulant propagator methods are discussed in terms of the root mean-square error of the DOA estimates. Assume that θ_k is the actual DOA of the kth source signal. From (9) and (10)

$$\mathbf{b}^H(\theta_m)\mathbf{W} = 0 \text{ for } m = 1, 2, \ldots, M, \tag{21}$$

where $\mathbf{W} = \mathbf{V}$ for cumulant MUSIC and $\mathbf{U}$ for cumulant propagator method. The null spectrum function associated with MUSIC and propagator method can be written as

$$F(\theta, \mathbf{W}) = \mathbf{b}^H(\theta)\mathbf{G}\mathbf{b}(\theta) \tag{22}$$

where $\mathbf{G} = \mathbf{W}\mathbf{W}^H$. Using (21), we can write

$$F(\theta_m, \mathbf{W}) = 0 \text{ for } m = 1, 2, \ldots, M, \tag{23}$$

Due to the noisy observations, the estimated DOA $\hat{\theta}_m$ of the kth source signal will be deviated from θ_m to some amount $\Delta\theta_m$. It can be written as

$$\hat{\theta}_m = \theta_m + \Delta\theta_m \text{ for } m = 1, 2, \ldots, M, \tag{24}$$

The Taylor series expansion of $F(\hat{\theta}_m, \hat{\mathbf{W}})$ about the actual DOA θ_m is

$$\begin{aligned} F\left(\hat{\theta}_m, \hat{\mathbf{W}}\right) &= F\left(\theta_m, \hat{\mathbf{W}}\right) + \Delta\theta_m F^{(1)}\left(\theta_m, \hat{\mathbf{W}}\right) \\ &\quad + (1/2)\left(\Delta\theta_m\right)^2 F^{(2)}\left(\theta_m, \hat{\mathbf{W}}\right) + \ldots \end{aligned} \tag{25}$$

where $F^{(k)}$ denotes the kth derivative of F with respect to θ. With the approximation of first order expansion and taking first derivative for (25), we get

$$F^{(1)}\left(\hat{\theta}_m, \hat{\mathbf{W}}\right) = F^{(1)}\left(\theta_m, \hat{\mathbf{W}}\right) + \Delta\theta_m F^{(2)}\left(\theta_m, \hat{\mathbf{W}}\right) \tag{26}$$

The left-hand side of (26) is zero since $F(\hat{\theta}_m, \hat{\mathbf{W}})$ attains minima at $\hat{\theta}_m$. Therefore, from (26)

$$\Delta\theta_m = -F^{(1)}\left(\theta_m, \hat{\mathbf{W}}\right) / F^{(2)}\left(\theta_m, \hat{\mathbf{W}}\right) \tag{27}$$

Substituting $\hat{\mathbf{W}} = \mathbf{W} + \Delta\mathbf{W}$ in $\hat{\mathbf{G}} = \hat{\mathbf{W}}\hat{\mathbf{W}}^H$ and approximating $\hat{\mathbf{G}}$ by first order deviation in $\Delta\mathbf{W}$ we get

$$\hat{\mathbf{G}} \approx \mathbf{G} + \Delta\mathbf{G}$$

where

$$\Delta\mathbf{G} \approx (\Delta\mathbf{W})\mathbf{W}^H + \mathbf{W}(\Delta\mathbf{W})^H \tag{28}$$

Taking the first and second derivatives of $F(\theta_m, \hat{\mathbf{W}})$ with respect to θ and using the fact that $\mathbf{b}^H(\theta_m)\mathbf{W} = 0$ for $m = 1, 2, \ldots, M$, we get

$$\begin{aligned} F^{(1)}\left(\theta_m, \hat{\mathbf{W}}\right) &= \mathbf{b}^{(1)H}(\theta_m)\mathbf{W}(\Delta\mathbf{W})^H\mathbf{b}(\theta_m) + \mathbf{b}^H(\theta_m)(\Delta\mathbf{W})\mathbf{W}^H\mathbf{b}^{(1)}(\theta_m) \\ &= 2\text{Real}\left\{\mathbf{b}^H(\theta_m)(\Delta\mathbf{W})\mathbf{W}^H\mathbf{b}^{(1)}(\theta_m)\right\} \end{aligned} \tag{29}$$

and

$$F^{(2)}\left(\theta_m, \hat{\mathbf{W}}\right) = 2\mathbf{b}^{(1)H}(\theta_m)\mathbf{G}\mathbf{b}^{(1)}(\theta_m) + \Delta F^{(2)} \tag{30}$$

where

$$\Delta F^{(2)} = 2\mathbf{b}^{(1)H}(\theta_m)(\Delta\mathbf{G})\mathbf{b}^{(1)}(\theta_m) + 2\text{Real}\left\{\mathbf{b}^{H}(\theta_m)(\Delta\mathbf{G})\mathbf{b}^{(2)}(\theta_m)\right\}$$

Now, substituting (29) and (30) in (27) and expanding (27) to only first order deviation yields

$$\begin{aligned}\Delta\theta_m \;=\; & -\text{Real}\left\{\mathbf{b}^{H}(\theta_m)(\Delta\mathbf{W})\mathbf{W}^{H}\mathbf{b}^{(1)}(\theta_m)\right\} \\ & /\left\{\mathbf{b}^{(1)H}(\theta_m)\mathbf{W}\mathbf{W}^{H}\mathbf{b}^{(1)}(\theta_m)\right\}\end{aligned} \tag{31}$$

In order to calculate the deviation error $\Delta\theta_m$ in the DOA estimate $\hat{\theta}_m$ given in (31) it is necessary to evaluate the deviation matrix $\Delta\mathbf{W}$ (that is, $\Delta\mathbf{V}$ for cumulant MUSIC method and $\Delta\mathbf{U}$ for cumulant propagator method).

5.1. Deviation $\Delta\mathbf{V}$ in Cumulant MUSIC Method

The eigendecomposition of the estimated cumulant matrix is

$$\hat{\mathbf{C}} = \hat{\mathbf{V}}\hat{\Lambda}\hat{\mathbf{V}}^{H} + \hat{\mathbf{Q}}\hat{\mathbf{\Sigma}}\hat{\mathbf{Q}}^{H} \tag{32}$$

Substituting $\hat{\mathbf{V}} = \mathbf{V} + \Delta\mathbf{V}$, $\hat{\mathbf{Q}} = \mathbf{Q} + \Delta\mathbf{Q}$, $\hat{\mathbf{\Sigma}} = \mathbf{\Sigma} + \Delta\mathbf{\Sigma}$ and $\hat{\mathbf{\Lambda}} = \mathbf{\Lambda} + \Delta\mathbf{\Lambda} = \Delta\mathbf{\Lambda}$ (since $\mathbf{\Lambda}$ is a zero matrix) in (32) and approximating to first order deviation

$$\hat{\mathbf{C}} = \mathbf{C} + \Delta\mathbf{C}$$

where

$$\Delta\mathbf{C} = \mathbf{Q}\mathbf{\Sigma}(\Delta\mathbf{Q})^{H} + \mathbf{Q}(\Delta\mathbf{\Sigma})\mathbf{Q}^{H} + (\Delta\mathbf{Q})\mathbf{\Sigma}\boldsymbol{Q}^{H} + \mathbf{V}(\Delta\mathbf{\Lambda})\mathbf{V}^{H} \tag{33}$$

Pre-multiplying (33) by $\mathbf{V}^{H}$ and using the fact that $\mathbf{V}^{H}\mathbf{Q} = \mathbf{0}$ and $\mathbf{V}^{H}\mathbf{V} = \mathbf{I}_{L^2-M}$ yields

$$\mathbf{V}^{H}(\Delta\mathbf{C}) = \mathbf{V}^{H}(\Delta\mathbf{Q})\mathbf{\Sigma}\boldsymbol{Q}^{H} + (\Delta\mathbf{\Lambda})\mathbf{V}^{H} \tag{34}$$

Post-multiplying (34) by $\mathbf{Q}$ and using the fact that $\mathbf{V}^{H}\mathbf{Q} = \mathbf{0}$ and $\mathbf{Q}^{H}\mathbf{Q} = \mathbf{I}_M$ yields

$$\mathbf{V}^{H}(\Delta\mathbf{C})\mathbf{Q} = \mathbf{V}^{H}(\Delta\mathbf{Q})\mathbf{\Sigma} \tag{35}$$

Define $\mathbf{Z}\underline{\Delta} - (\Delta\mathbf{Q})^{H}\mathbf{V}$, then from (35)

$$\mathbf{Z} = -\mathbf{\Sigma}^{-1}\mathbf{Q}^{H}(\Delta\mathbf{C})^{H}\mathbf{V} \tag{36}$$

Note that, $(\mathbf{V}+\mathbf{QZ})^H(\mathbf{V}+\mathbf{QZ}) \approx \mathrm{I}_{L^2-M}$ since $\mathbf{Z}^H\mathbf{Z}$ has second order term in $(\Delta\mathbf{C})$. Also it can be shown that columns of $(\mathbf{V}+\mathbf{QZ})$ spans the space spanned by the columns of $\mathbf{V}$. So we can write that

$$\hat{\mathbf{V}} = \mathbf{V} + \mathbf{QZ} \tag{37}$$

Comparing (37) with $\hat{\mathbf{V}} = \mathbf{V} + \Delta\mathbf{V}$ we see that we have

$$\Delta\mathbf{V} \ = \ \mathbf{QZ} \ = -\mathbf{Q}\boldsymbol{\Sigma}^{-1}\mathbf{Q}^H(\Delta\mathbf{C})^H\mathbf{V} \tag{38}$$

5.2. Deviation $\Delta\mathbf{U}$ in Cumulant Propagator Method

The estimate of the cumulant matrix $\hat{\mathbf{C}}$ can be partitioned as

$$\hat{\mathbf{C}} = [\hat{\mathbf{C}}_1 \ \hat{\mathbf{C}}_2] \text{ with } \hat{\mathbf{C}}_2 = \hat{\mathbf{C}}_1\hat{\mathbf{P}} \tag{39}$$

Substituting the $\hat{\mathbf{C}}_1 = \mathbf{C}_1 + \Delta\mathbf{C}_1$ and $\hat{\mathbf{C}}_2 = \mathbf{C}_2 + \Delta\mathbf{C}_2$ into (39) and expressing $\Delta\mathbf{C}$ as

$$\Delta\mathbf{C} = [\Delta\mathbf{C}_1 \quad \Delta\mathbf{C}_2]$$

Let $\Delta\mathbf{P}$ be the deviation in $\hat{\mathbf{P}}$ as $\hat{\mathbf{P}} = \mathbf{P}+\Delta\mathbf{P}$ and $\Delta\mathbf{U}$ be the deviation in $\hat{\mathbf{U}}$ as $\hat{\mathbf{U}} = \mathbf{U} + \Delta\mathbf{U}$. Therefore, from $\hat{\mathbf{C}}_2 = \hat{\mathbf{C}}_1\hat{\mathbf{P}}$ we can derive $\Delta\mathbf{P}$ by approximating to first order terms as

$$\Delta\mathbf{C}_2 + \mathbf{C}_2 = (\Delta\mathbf{C}_1 + \mathbf{C}_1)(\mathbf{P} + \Delta\mathbf{P}) \approx (\Delta\mathbf{C}_1)\mathbf{P} + \mathbf{C}_1\mathbf{P} + \mathbf{C}_1(\Delta\mathbf{P}) \tag{40}$$

This implies that

$$\Delta\mathbf{P} \ = \left(\mathbf{C}_1^H\mathbf{C}_1\right)^{-1}\mathbf{C}_1^H(\Delta\mathbf{C}_2 - \Delta\mathbf{C}_1\mathbf{P}) \tag{41}$$

Then

$$\begin{aligned}\Delta\mathbf{U} \ &= \ \hat{\mathbf{U}} - \mathbf{U} = \left[\hat{\mathbf{P}}^H : -\mathrm{I}_{L^2-M}\right]^H - \left[\mathbf{P}^H : -\mathrm{I}_{L^2-M}\right]^H \\ &= \ \left[(\Delta\mathbf{P})^H 0\right]^H = \mathbf{Y}(\Delta\mathbf{C})^H\mathbf{U} \end{aligned} \tag{42}$$

where

$$\mathbf{Y} \ = -[\mathbf{C}_1(\mathbf{C}_1^H\mathbf{C}_1)^{-1}\mathbf{0}]^H$$

Now substituting (38) and (42) in to (31) we get

$$\begin{aligned}\Delta\theta_m \ &= \ -\mathrm{Real}\left\{\mathbf{b}^H(\theta_m)\mathbf{Q}\boldsymbol{\Sigma}^{-1}\mathbf{Q}^H(\Delta\mathbf{C})^H\mathbf{V}\mathbf{V}^H\mathbf{b}^{(1)}(\theta_m)\right\} \\ &\quad / \left\{\mathbf{b}^{(1)H}(\theta_m)\mathbf{V}\mathbf{V}^H\mathbf{b}^{(1)}(\theta_m)\right\}\end{aligned} \tag{43a}$$

for cumulant MUSIC method and

$$\begin{aligned}\Delta\theta_m \ &= \ -\mathrm{Real}\left\{ \ \mathbf{b}^H(\theta_m)\mathbf{Y}(\Delta\mathbf{C})^H\mathbf{U}\mathbf{U}^H \ \mathbf{b}^{(1)}(\theta_m)\right\} \\ &\quad / \left\{\mathbf{b}^{(1)H}(\theta_m)\mathbf{U}\mathbf{U}^H\mathbf{b}^{(1)}(\theta_m)\right\}\end{aligned} \tag{43b}$$

for cumulant propagator method. Then the variance of the DOA estimate for cumulant MUSIC and cumulant propagator methods can be calculated, respectively, by

$$\mathrm{Var}(\Delta\theta_m)_{\mathrm{CM}} = \mathrm{Var}\left[\mathrm{Real}\left\{\mathbf{b}^H(\theta_m)\,\mathbf{Q}\mathbf{\Sigma}^{-1}\mathbf{Q}^H(\Delta\mathbf{C})^H\mathbf{V}\mathbf{V}^H\mathbf{b}^{(1)}(\theta_m)\right\}\right] \Big/ \left\|\mathbf{b}^{(1)H}(\theta_m)\mathbf{V}\right\|^4$$

and

$$\mathrm{Var}\,(\Delta\theta_m)_{\mathrm{CP}} = \mathrm{Var}\left[\mathrm{Real}\left\{\mathbf{b}^H(\theta_m)\mathbf{Y}(\Delta\mathbf{C})^H\mathbf{U}\mathbf{U}^H\mathbf{b}^{(1)}(\theta_m)\right\}\right] \Big/ \left\|\mathbf{b}^{(1)H}(\theta_m)\mathbf{U}\right\|^4$$

This can be further reduced by assuming the elements of $\Delta\mathbf{C}$ are uncorrelated zero-mean random variables with equal variance σ^2. Therefore, the square root of the variance (i.e., standard deviation) will be equal to root mean-square error (RMSE) and can be simplified [17–19] to

$$\mathrm{RMSE}_{\mathrm{CM}} = \left(\sigma^2/2\right)\mathbf{b}^H(\theta_m)\mathbf{Q}\mathbf{\Sigma}^{-2}\mathbf{Q}^H\mathbf{b}(\theta_m)\Big/\left\|\mathbf{b}^{(1)H}(\theta_m)\mathbf{V}\right\|^2 \quad (44a)$$

and

$$\mathrm{RMSE}_{\mathrm{CP}} = \left(\sigma^2/2\right)\left[\mathbf{b}^H(\theta_m)\,\mathbf{Y}\mathbf{Y}^H\mathbf{b}(\theta_m)\right]\left[\mathbf{b}^{(1)H}(\theta_m)\mathbf{U}\mathbf{U}^H\mathbf{U}\mathbf{U}^H\mathbf{b}^{(1)}(\theta_m)\right]\Big/\left\|\mathbf{b}^{(1)H}(\theta_m)\mathbf{U}\right\|^4 \quad (44b)$$

If the matrix $\mathbf{U}$ is replaced by its orthonormalized version $\mathbf{U}_o$, then (44b) is reduced to

$$\mathrm{RMSE}_{\mathrm{CP}} = \left(\sigma^2/2\right)\left[\mathbf{b}^H(\theta_m)\mathbf{Y}\mathbf{Y}^H\mathbf{b}(\theta_m)\right]\Big/\left\|\mathbf{b}^{(1)H}(\theta_m)\mathbf{U}_o\right\|^2 \quad (45)$$

In the noise free case, it can be shown that $\mathbf{U}_o = -\mathbf{V}$ as discussed in [9, 20]. So the RMSE in (44a) and (45) differ only by the numerator value. Using the definition of the propagator in (12) and the matrix $\mathbf{Y}$ defined in (42), the numerators of (44a) and (45) can be written as

$$\mathbf{b}^H(\theta_m)\;\mathbf{Q}\mathbf{\Sigma}^{-2}\mathbf{Q}^H\mathbf{b}(\theta_m) = \mathbf{b}_1^H(\theta_m)\tilde{\mathbf{U}}^H\mathbf{Q}\mathbf{\Sigma}^{-2}\mathbf{Q}^H\tilde{\mathbf{U}}\mathbf{b}_1(\theta_m) \quad (46a)$$

$$\mathbf{b}^H(\theta_m)\mathbf{Y}\mathbf{Y}^H\mathbf{b}(\theta_m) = \mathbf{b}_1^H(\theta_m)\left(\mathbf{C}_1^H\mathbf{C}_1\right)^{-1}\mathbf{b}_1(\theta_m) \quad (46b)$$

where $\tilde{\mathbf{U}} = [\mathbf{I}_M : \mathbf{P}]^H$ is an $L^2 \times M$ matrix orthogonal to $\mathbf{U}$ and $\mathbf{b}_1(\theta_m)$ is an $M \times 1$ vector containing the first M elements of $\mathbf{b}(\theta_m)$. Using the fact that $\mathbf{V}$ and $\mathbf{Q}$ are orthogonal, the orthonormalized version of $\tilde{\mathbf{U}}$ is

$$\tilde{\mathbf{U}}_o = \tilde{\mathbf{U}}\left(\tilde{\mathbf{U}}^H\tilde{\mathbf{U}}\right)^{-1/2} = -\mathbf{Q} \quad (47)$$

In noise free case, the cumulant matrix $\mathbf{C}$ can be written as

$$\mathbf{C} = [\mathbf{C}_1 : \mathbf{C}_2] = \left[\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}_1^H : \ \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}_2^H\right]$$

where $\mathbf{Q}_1$ is a $M \times M$ matrix containing the first M rows of $\mathbf{Q}$. Therefore,

$$\mathbf{C}_1^H\mathbf{C}_1 = \ \mathbf{Q}_1\boldsymbol{\Sigma}\mathbf{Q}_1^H \tag{48}$$

So from (47) and the definition of $\mathbf{Q}_1$ that

$$\mathbf{Q}_1 = -(\tilde{\mathbf{U}}^H\tilde{\mathbf{U}})^{-1/2} \text{ and } \mathbf{Q} = \tilde{\mathbf{U}}\mathbf{Q}_1 \tag{49}$$

Now using (49), the right hand side of (46a) can be written as

$$\begin{aligned}
&\mathbf{b}_1^H(\theta_m)\tilde{\mathbf{U}}^H\mathbf{Q}\boldsymbol{\Sigma}^{-2}\mathbf{Q}^H\tilde{\mathbf{U}}\mathbf{b}_1(\theta_m) \\
= \ &\mathbf{b}_1^H(\theta_m)\left(\tilde{\mathbf{U}}^H\tilde{\mathbf{U}}\right)^{H/2}\boldsymbol{\Sigma}^{-2}\left(\tilde{\mathbf{U}}^H\tilde{\mathbf{U}}\right)^{1/2}\mathbf{b}_1(\theta_m) \\
= \ &\mathbf{b}_1^H(\theta_m)\mathbf{Q}_1^{-H/2}\boldsymbol{\Sigma}^{-2}\mathbf{Q}_1^{-1/2}\mathbf{b}_1(\theta_m) = \mathbf{b}_1^H(\theta_m)\left(\mathbf{C}_1^H\mathbf{C}_1\right)\mathbf{b}_1(\theta_m)
\end{aligned}$$

Now, we showed that under noise free case the numerators are also equal. So the cumulant propagator method performs similar to cumulant MUSIC for high and moderate signal-to-noise ratios(SNRs). It is worth noting that the computational complexity of the cumulant propagator method is much less than the cumulant MUSIC.

For the above analysis, the first order derivative $\mathbf{b}^{(1)}(\theta_m)$ of $\mathbf{b}(\theta_m)$ is necessary. The derivative of $\mathbf{b}(\theta_m)$ with respect to the actual DOA θ_m is

$$\mathbf{b}^{(1)}(\theta_m) = \mathbf{a}^{(1)}(\theta_m) \otimes \ \mathbf{a}^*(\theta_m) + \ \mathbf{a}(\theta_m) \otimes \ \mathbf{a}^{(1)*}(\theta_m)$$

where

$$\mathbf{a}^{(1)}(\theta_m) = -j(2\pi d/\lambda)\sin\theta_m[\mathbf{h} \odot \mathbf{a}(\theta_m)]$$

and where $\odot$ denotes Hadamard product and $\mathbf{h} = [0, 1, 2, \ldots, (L-1)]^T$.

6. SIMULATION RESULTS

In this section, computer simulations are presented to demonstrate the performance of the fourth-order cumulant-propagator method. The following performance measures are used to investigate the performance of the DOA estimation technique: (i) Root-mean square error (RMSE) (ii) resolution capability (iii) detection probability (iv) capability to resolve two nearby sources and (v) minimum resolvable angular separation. Computer simulations have been carried out using Matlab 7.0.4 to evaluate the performance of the both methods. A uniform linear array consisting of 9 sensors and 100 snapshots are used for all the computer simulation.

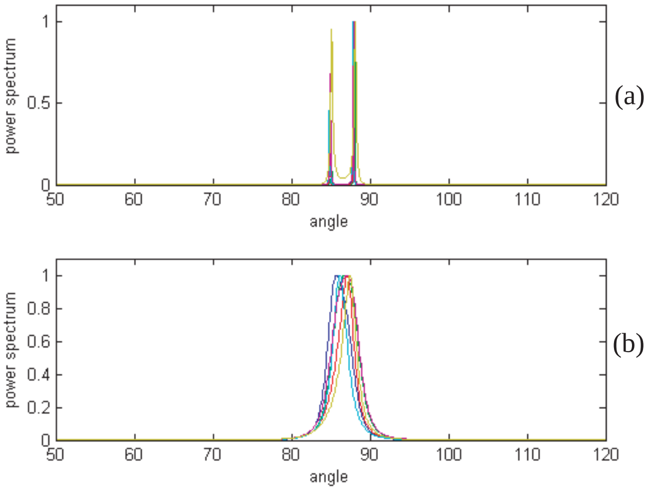

Figure 1. Power spectrum plots of two sources from the directions $\theta = 85°$ and $88°$ at an equal SNR of 0 dB. (a) For the proposed method and (b) for the propagator based on auto-covariance method.

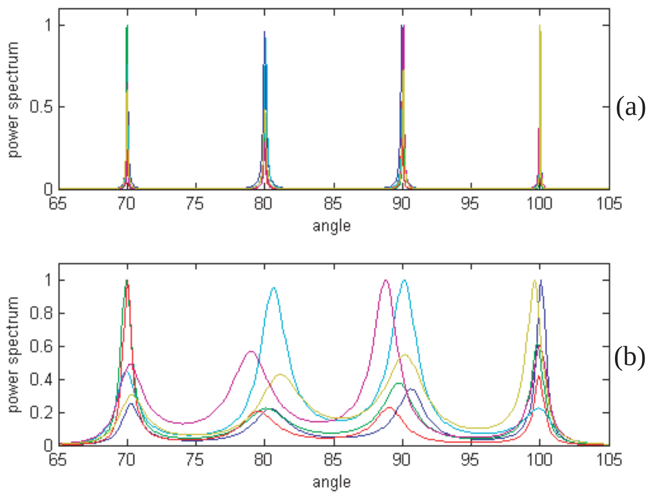

Figure 2. Power spectrum plots of 4 sources from the directions $\theta = 70°, 80°, 90°$ and $100°$ at an equal SNR of 0 dB. (a) For the proposed method and (b) for the propagator based on auto-correlation method.

First, the resolution capabilities of the propagator method based on fourth-order statistics and second-order statistics for the case of two signal sources separated by 3° at an equal SNR of 0 dB are considered. The power spectrum plots, when the direction of both sources lie in the range [35°, 145°], are plotted for 6 independent trials and are shown in Fig. 1(a) for the fourth-order cumulant propagator method and in Fig. 1(b) for the propagator method based on second order statistics. The power spectrum plots show that the fourth-order cumulant propagator method resolves the two sources consistently in all the 6 trials while the propagator method based on second order statistics does not resolve consistently in every trial.

Next, the resolution capabilities of both methods for four signal sources from the directions $\theta = 70°,\ 80°,\ 90°$ and $100°$ at an equal SNR of 0 dB are considered and the power spectrum graphs are plotted for 6 independent trials and are shown in Figs. 2(a) and 2(b) respectively for the fourth-order cumulant propagator method and second-order cumulant propagator method. The plots attest to the consistent performance of the proposed method.

Figure 3 shows the detection probability of second source at an angular separation that is varied between 1° and 9° in the vicinity of first source. The detection probability plots show that the cumulant propagator method is capable of resolving two sources separated by an angle of 3° successfully but the propagator method based on second-order statistics is not able to do so.

Furthermore, the robustness with respect to the number of sources is also analyzed through simulations. Fig. 4 shows the performance with respect to the number of distinct sources at an SNR of 0 dB for each source at an angular separation of 10° between adjacent sources. The detection probability is plotted against the number of distinct

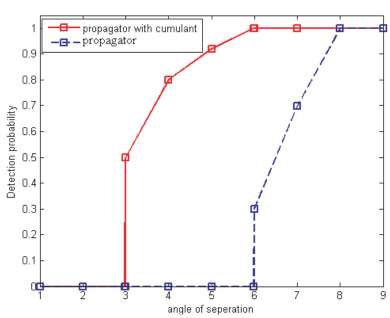

Figure 3. Detection probability versus angular separation.

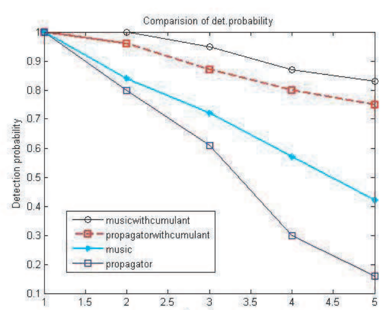

Figure 4. Detection probability versus number of distinct sources.

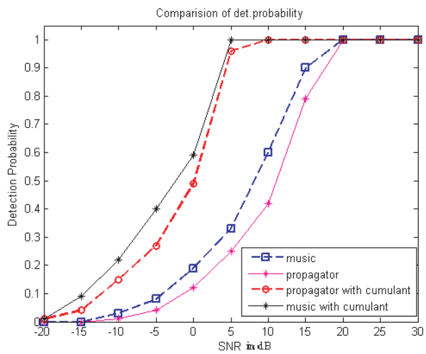

Figure 5. Detection probability versus SNRs.

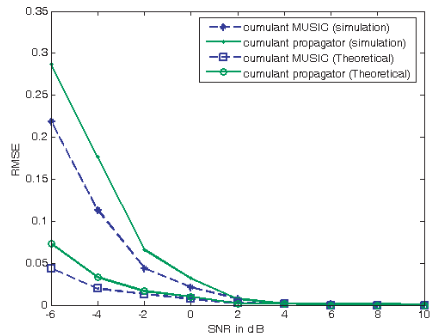

Figure 6. Root-mean square error versus SNR.

sources and is shown in Fig. 4, which clearly shows that the proposed method is capable of detecting up to 5 sources separated by 10° with high reliability. In contrast, the propagator method based on second-order statistics is able to detect only up to 2 sources. Fig. 5 illustrates the performance of both the methods at different SNR values. It depicts the detection probability at various SNRs from -20 dB to 30 dB. The detection probability for all three cases is computed over 250 independent trails by detecting the source(s) within an interval of $\pm 0.1°$ around the actual DOA.

In the final experiment, one source with direction-of-arrival at 30° is considered and simulations are been performed over 250 independent trails for cumulant propagator method based on simulation for various SNRs and the RMSE is computed using the formula

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\theta - \hat{\theta}_i\right)^2}$$

where n is number of trials and $\hat{\theta}_i$ is the DOA estimate of the actual direction θ for the ith trial. The RMSE is also computed for cumulant propagator method based on the theoretical results obtained in Equation (44b) for various SNRs. The RMSE versus SNR is plotted for both the results and the plots are shown in Fig. 6. It is observed from the plots that the simulation results are close to the theoretical results for moderate and high SNR values.

7. CONCLUSION

In this paper, a cumulant propagator method, which has better resolution capability than the propagator method based on second-order statistics, is presented. The propagator method is applied to the fourth-order cumulant matrix. Simulation studies reveal that the performance of the cumulant propagator method is superior to that of the propagator method based on second-order statistic in resolving distinct-power sources as well as in detecting the number sources. It is also seen from the simulation results that the cumulant propagator method delivers estimates with less RMSE than the propagator method based on second-order statistics.

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