Progress In Electromagnetics Research M, Vol. 8, 131–141, 2009

INVERSE SOURCE PROBLEM FROM THE KNOWLEDGE OF RADIATED FIELD OVER MULTIPLE RECTILINEAR DOMAINS

F. Soldovieri

Istituto per il Rilevamento Elettromagnetico dell'Ambiente, CNR
Via Diocleziano 328, Napoli 80124, Italy

C. Mola, R. Solimene, and R. Pierri

Dipartimento di Ingegneria dell'Informazione
Seconda Università di Napoli
Via Roma 29, Aversa 81031, Italy

Abstract—This paper deals with an inverse source problem starting from the knowledge of the radiated field in Fresnel and near zone. In particular, here we are concerned with a 2D geometry characterized by a rectilinear magnetic source and measurement rectilinear domains in Fresnel and near zone. The effect of the added knowledge of the radiated field over a second observation domain is investigated via the Singular Values Decomposition of the radiation operator and we point out how the addition of a second observation domain allows us always to achieve a better noise rejection. Also, we determine conditions under which the knowledge of the field over the second domain increases the information content (as the number of singular values of the radiation operator before their asymptotic decay) for both the Fresnel and near zone cases. Finally reconstruction examples with noise-free and noisy data are presented.

1. INTRODUCTION

This work falls within the more general framework of the inverse problem of determining a radiating source starting from the knowledge of its radiated field [1–5]. In particular, within a two-dimensional and scalar geometry, we consider the canonical case of a bounded rectilinear magnetic source whose radiated field is collected over multiple bounded

Corresponding author: F. Soldovieri (soldovieri.f@irea.cnr.it).

rectilinear domains parallel to the source and located first in the Fresnel zone and after in the near zone [1, 2].

As well known, such a problem can be stated as a linear inverse one [6] and the main effort of the work is concerned with the analysis of the Number of Degrees of Freedom (NDF) of the radiated field and to establish the dependence of NDF on the geometrical features of the source and observation domains.

The analysis is performed by means of the Singular Values Decomposition (SVD) of the relevant linear integral operator (radiation operator). Here we focus the attention on the increase of the information content in dependence of the added knowledge of the radiated field over a second observation domain by pointing out the different behavior when the domains are in Fresnel or near zone [1, 2].

Therefore, the work is organized as follows. Section 2 gives the formulation of the inverse source problem and the results for a single observation domain are briefly recalled. Section 3 addresses the problem of the determination of the magnetic source starting from the knowledge of the field over two domains located in Fresnel zone and an estimation of the NDF behavior is provided by resorting to the results in [2]. Section 4 is concerned with the case of the observation domains in the near zone and reconstruction examples are presented with noise-free and noisy data. Finally, conclusions follow.

2. THE CASE OF THE SINGLE DOMAIN

This section is devoted to formulate the inverse problem of the determination of a rectilinear magnetic current, having a finite support, starting from the knowledge of the radiated field over rectilinear finite observation domains (see Fig. 1).

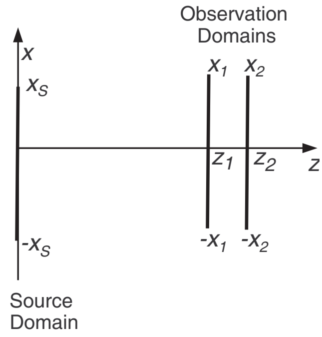

Figure 1. Geometry of the problem.

We assume that the magnetic source is directed and invariant along the y-axis and supported over $S = [-X_S, X_S]$ along the x-axis. Therefore, here a 2D problem is dealt with.

The knowledge of the tangential component (along the x-axis) of the radiated field E is assumed over a rectilinear domain $O_1 = [-X_1, X_1]$ at quota z_1. In this case, the relationship between tangential component of the electric field E and the magnetic current J_m, is given by [7]

$$E(x,z) = j\beta/4 \int_{-X_S}^{X_S} \frac{H_1^{(2)}(\beta r) z_1}{r} J_m(x')dx' \quad x \in O_1 \tag{1}$$

where $H_1^{(2)}(\cdot)$ is the Hankel function of second kind and first order, $\beta = 2\pi/\lambda$ is the wave-number and $r = \sqrt{(x-x')^2 + z_1^2}$ denotes the distance between the generic observation point and source point.

Equation (1) can be schematically seen as a linear transformation as

$$A: \; J_m \in L^2(S) \rightarrow E \in L^2(O_1) \tag{2}$$

Therefore, the problem of the reconstruction of the magnetic current J_m starting from the radiated field E can be stated as the inversion of the operator A.

The operator A is compact [1, 2, 6] and hence its singular values cluster to zero asymptotically (i.e., as their index tends to infinity). This means that the problem at hand is ill-posed. Accordingly, in order to obtain a stable inversion the singular spectrum should be suitably truncated by accounting for the noise level on data [6, 8].

For the case at hand, however, it has been shown that the singular values of A exhibit an almost step-like behavior [1, 2]. That is the singular values are almost constant before a knee after that they decay exponentially fast. Accordingly, two important consequences can be pointed out. First, the inversion of A is a severely ill-posed problem [6]. Second, the number of singular values to be retained in a regularized reconstruction scheme is essentially finite and generally weakly dependent on noise. This permits to identify the Number of Degrees of Freedom (NDF) of the radiated field just as the number of singular values above the asymptotic decay.

In the case of a single rectilinear observation domain, the problem of determining the NDF of radiated field has been dealt with analytically for the far-field and Fresnel zones [1]. In this case, the singular system of the operator is given in a closed form and the NDF is estimated as

$$N \cong \frac{4X_S X_1}{\lambda z_1} \tag{3}$$

while the singular functions are strictly related to the prolate spheroidal wave-functions [9]. Relationship (3) states an angular criterion for determining the NDF in far field and Fresnel zone, i.e., the NDF is related to the observation angle given by

$$\tan(\Theta) = \frac{X_1}{z_1} \tag{4}$$

In the same work [1], an analysis was provided also for the NDF of radiated field collected over a bounded rectilinear domain in near zone. These results will be recalled in the Section 4.

3. THE CASE OF TWO DOMAINS IN FRESNEL ZONE

This section is devoted to deal with the case of two observation domains located in the Fresnel zone, (see geometry of the problem depicted in Fig. 1). Besides the first observation domain $O_1 = [-X_1, X_1]$ at quota z_1, a second observation domain $O_2 = [-X_2, X_2]$ at quota z_2 is considered.

Under the Fresnel paraxial approximation, the relationship (1) between the magnetic source $J_m(x)$ and the tangential component of the electric field over the observation domain O_i can be approximated as:

$$E_i(x, z_i) = \frac{K \exp[-j\beta z_i]}{\sqrt{z_i}} \int_{-X_S}^{X_S} \exp\left[-j\frac{\beta(x-x')^2}{2z_i}\right] J_m(x')dx' \qquad x \in O_i \tag{5}$$

where $i = 1, 2$ and the quantity K accounts for factors not dependent on the geometry of the problem. The NDF of the radiated field can be estimated by analyzing the behavior of singular values of operator in Eq. (5) that can be rewritten as

$$B:\ J_m \in L^2(S) \rightarrow E = (E_1, E_2) \in L^2(O_1) \times L^2(O_2) \tag{6}$$

If we denote by $\{\sigma_n, u_n, v_n\}_{n=0}^{\infty}$, the singular system [6] of the operator B in Eq. (6), we have that the associated eigenvalue problem is

$$\sigma_n^2 u_n = B^+ B u_n \tag{7}$$

where B^+ is the adjoint operator [6] of B.

By resorting to the results in [2], Eq. (6) can be arranged, a part the unessential quantity outside the integral, as

$$B^+ B \cong 2\lambda P_S B_{\Omega_1} P_S + \lambda P_S B_{\Omega_2} P_S + \lambda P_S B_{\Omega_3} P_S \tag{8}$$

where P_S is a spatial limiting projector over S and B_{Ω_1}, B_{Ω_2}, B_{Ω_3} are band limiting projectors, respectively, over

$$\begin{aligned} \Omega_1 &= \left[-\min\left\{\frac{\beta X_1}{z_1}, \frac{\beta X_2}{z_2}\right\}, \min\left\{\frac{\beta X_1}{z_1}, \frac{\beta X_2}{z_2}\right\}\right] \\ \Omega_2 &= \left[\min\left\{-\frac{\beta X_1}{z_1}, -\frac{\beta X_2}{z_2}\right\}, \max\left\{-\frac{\beta X_1}{z_1}, -\frac{\beta X_2}{z_2}\right\}\right] \\ \Omega_3 &= \left[\min\left\{\frac{\beta X_1}{z_1}, \frac{\beta X_2}{z_2}\right\}, \max\left\{\frac{\beta X_1}{z_1}, \frac{\beta X_2}{z_2}\right\}\right] \end{aligned} \tag{9}$$

The eigensystem of each single operator defined on the basis of Eqs. (8) and (9) is known in closed form by resorting to the prolate spheroidal wave functions. For each operator, the behaviour of eigenvalues is strictly related to the spatial-bandwidth products $c_k = \frac{m\{S\}m\{\Omega_k\}}{4}$, $k \in \{1, 2, 3\}$, where $m\{\cdot\}$ denotes the extent of the interval it refers to. If the products c_k are sufficiently larger than one, the eigenvalues of the operator B^+B can be estimated according to the discussion reported in [2]. In general, the singular values will exhibit a two step-like behavior with the first $\left[\frac{2c_1}{\pi}\right]$ eigenvalues equal to 2λ, then, after such a first step, further $\left[\frac{2c_2}{\pi}\right] + \left[\frac{2c_3}{\pi}\right]$ eigenvalues will be equal to λ; after this second step an exponential asymptotic decay of the eigenvalues occurs. However, when $\left|\frac{2\beta X_S}{\pi}\left(\frac{X_1}{z_1} - \frac{X_2}{z_2}\right)\right|$ tends to be closer or even less than 1, in force of definition of Ω_i, the two-step behavior of the eigenvalues disappears and a single step occurs at the index $\frac{2\beta X_S}{\pi}\frac{X_1}{z_1} \cong \frac{2\beta X_S}{\pi}\frac{X_2}{z_2}$. Therefore, when $\frac{X_1}{z_1} = \frac{X_2}{z_2}$ (i.e., the two observation domains subtend the same angle Θ), eigenvalues behavior will have only one step at theoretical value $\frac{2\beta X_S}{\pi}\frac{X_1}{z_1} \cong \frac{2\beta X_S}{\pi}\frac{X_2}{z_2}$. However, also in this case the

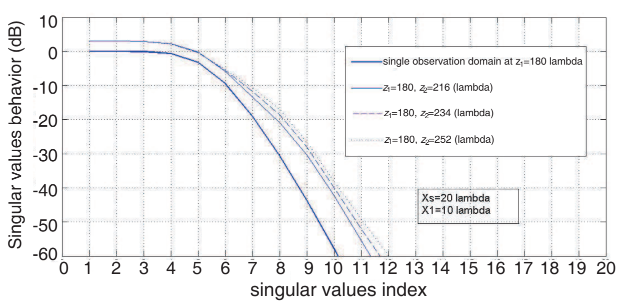

Figure 2. Singular value behavior for the domains in the Fresnel zone.

eigenvalues on the step are 2λ whereas in the case of single domain it is λ.

Figure 2 confirms this expected behaviour. In particular, we consider a magnetic current defined over a domain with $X_S = 20\lambda$. The first observation domain is at $z_1 = 180\lambda$ and with $X_1 = 10\lambda$. Three other cases have been considered where the quota of the second observation domain is $z_2 = 216\lambda$ and $z_2 = 234\lambda$ and $z_2 = 252\lambda$; the extent of these observation domains is chosen so that they subtend the same observation angle Θ (see Eq. (4)) of the first domain. The three singular value curves for the case of two domains are almost coincident and a slight difference is observed in the exponential decay, starting from about -10 dB. Also, the singular values corresponding to the case of single domain are at -3 dB below as compared to the ones of the two domains; this is perfectly consistent with the theory discussed above. Accordingly, we conclude that the case of two observation domains exhibits clear advantages in terms of noise rejection. What is more, such advantage can be further increased by adding more than one observation domain. However, this point is not discussed in this paper.

4. ANALYSIS IN THE NEAR ZONE

This section is devoted to present some numerical results for the case of two domains located in the near zone.

The case of a single observation domain in the near zone has been investigated in [1] and a criterion for the determination of the NDF has been provided as

$$N \cong \left[\frac{2}{\lambda}(R_1 - R_2)\right] \tag{10}$$

where $R_1 = \sqrt{(X_S + X_1)^2 + z_1^2}$ and $R_2 = \sqrt{(X_S - X_1)^2 + z_1^2}$.

Here, we aim at pointing out the differences between the case of Fresnel zone, where an angular criterion holds for the NDF determination, and the present case of the near zone.

To point out these differences, here we consider a source with extent $X_S = 20\lambda$. The first test case refers to an observation domain located at $z_1 = 2.5\lambda$ and with $X_1 = 10\lambda$ so that $\Theta \approx 2\pi/5$.

The solid line of Fig. 3 depicts the singular values behavior in the case of single domain for which estimate (10) returns $N \cong 38$ being strictly consistent with the number of singular values before the exponential decay shown in Fig. 3.

Three different cases have been considered with the addition of the second observation domain at quota $z_2 = 5\lambda$ and $z_2 = 10\lambda$ and

$z_2 = 15\lambda$, respectively. The extents of the second domains have been made so that all the observation domains subtend the same observation angle, i.e., $X_2 = 20\lambda$, $X_2 = 40\lambda$ and $X_2 = 60\lambda$, respectively.

Figure 3 depicts the singular behavior also for these three cases: two domains at $z_1 = 2.5\lambda$ and $z_2 = 5\lambda$; two domains at $z_1 = 2.5\lambda$ and $z_2 = 10\lambda$; two domains at $z_1 = 2.5\lambda$ and $z_2 = 15\lambda$.

By looking to the singular values it can be pointed out that, differently from the case of the observation domains located in the Fresnel zone, the second observation domain, even though subtends the

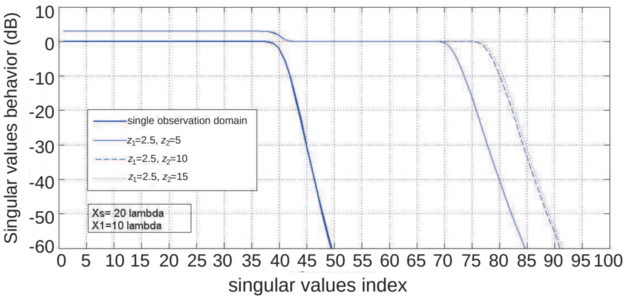

Figure 3. Singular values behavior for the domains in near zone in the case of $X_1 = 10\lambda$ and $z_1 = 2.5\lambda$. Solid thick line corresponds to the case of single domain; solid line corresponds to the case of two domain located at $z_1 = 2.5\lambda$ and $z_2 = 5\lambda$; dashed line corresponds to the case of $z_1 = 2.5\lambda$ and $z_2 = 10\lambda$; dotted line corresponds to the case $z_1 = 2.5\lambda$ and $z_2 = 15\lambda$.

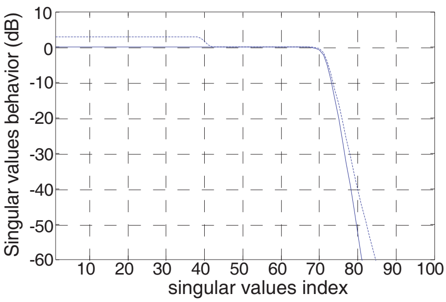

Figure 4. Comparison between the singular values in the case of the single domain at $z_2 = 5\lambda$ and two domains at $z_1 = 2.5\lambda$ and $z_2 = 5\lambda$. The case of single domain (solid line); the case of two domains (dotted line).

same angle Θ, permits to increase the NDF of the radiated field. This result is also different from the expectations based on pure diffraction arguments (as in Gabor's criterion) [10].

A more thorough analysis about the addition of the second observation domain, requires the comparison of the singular values behavior in the case of the two domains also to the case of only a single domain located at z_2. Such a comparison, for the case of two domains located at $z_1 = 2.5\lambda$ and $z_2 = 5\lambda$ and the case of a single domain at $z_2 = 5\lambda$ is reported in Fig. 4. From such a figure, it can be deduced that, in the case of two domains, the number of singular values before the knee is dictated by the second observation domain. In fact, estimate (10) evaluated for the second domain provides $N \cong 70$. The knowledge of the radiated field also over the first domain allows to achieve better performances in terms of noise rejection in a way similar to the one outlined for the Fresnel zone.

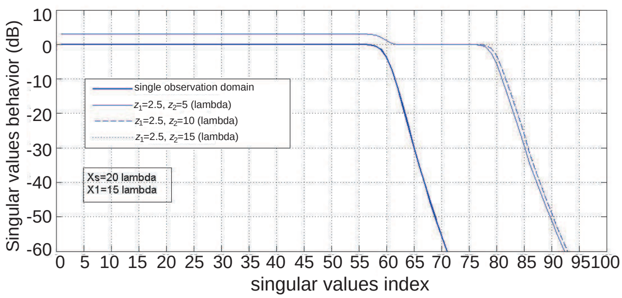

Figure 5. Singular values behavior for the domains in near zone in the case of $X_1 = 15\lambda$.

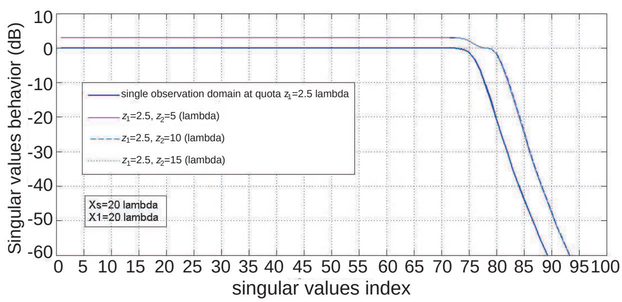

Figure 6. Singular values behavior for the domains in near zone in the case of $X_1 = 20\lambda$.

Results reported in Figs. 5 and 6 show, instead, what happens when the extent of the first observation domain (the closest one) increases.

Figures 5 depicts the singular behavior with the same source and the same quotas as the previous case reported in Fig. 3 but with $X_1 = 15\lambda$ so that $\Theta \approx 9\pi/20$. As can be seen, the increase of the information content (NDF) provided by the second observation domain (subtending the same observation angle as the first domain) becomes smaller as compared to the previous case. This is even more evident from Fig. 6 where $X_1 = 20\lambda$ so that $\Theta \approx 23\pi/50$ has been considered.

We can conclude that when the angle subtended by the first observation domain increases, the second observation domain provides a smaller increase in the number of the NDF whereas the number of significant singular values at 3 dB increases.

Let us now turn to consider some reconstruction results referring to the case of an uniform magnetic current distribution.

In particular, the reconstructions reported below are obtained by the Truncated SVD inversion scheme and by retaining only the singular functions corresponding to the singular values larger than 0.1 the maximum one.

We consider an uniform current distribution with extent $X_S = 20\lambda$ and first deal with the noise free-data.

For such a case, the reconstructions along with the actual current profile (red dotted line) are reported in Fig. 7. In particular, the blue dotted line refers to the case of only one observation domain at $z_1 = 2.5\lambda$ whereas the blue solid line refers to the case of a double observation domain at quotas $z_1 = 2.5\lambda$ and $z_2 = 5\lambda$, respectively. As

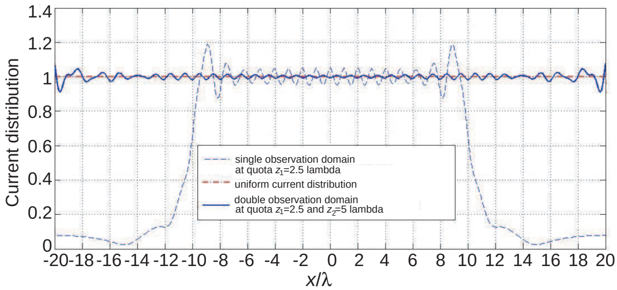

Figure 7. Source reconstruction for domains in near zone and noise free data. Red dotted line: actual source current; blue dotted line: Single domain reconstruction; blue solid line: Two-domains reconstruction.

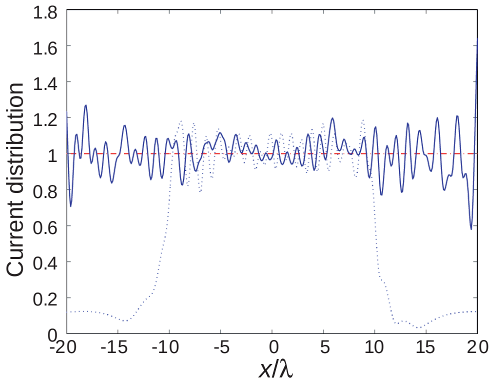

Figure 8. The same as Fig. 7 but with noisy data (SNR = 10 dB).

can be seen, the second case is characterized by a larger number of the singular values and ensures a better reconstruction which is very similar to the true one.

Finally, we report a reconstruction result with noisy data so that the SNR is equal roughly to 10 dB. Fig. 8 is the analogous to Fig. 7 and it allows to point out the good performances of the TSVD scheme in the rejection of the noise on data.

5. CONCLUSION

The paper has dealt with the study of the effect of multiple observation domains on the NDF of the radiated fields in the 2D geometry and for a magnetic source current.

Both the cases of domains in Fresnel and near zone have been presented and the analysis has been performed thanks to the Singular Values Decomposition of the relevant linear operator.

In particular, for the case of two domains in the Fresnel zone, we have shown that an angular criterion holds in order to determine the NDF of the radiated field. Moreover, it has been shown how the addition of a second observation domain subtending the same angle of the first one does not lead to an increase of the NDF while its effect consists in improving the noise rejection by increasing the values of the singular values of the operator.

In the case of domain in the near zone, we have pointed out that a criterion different from the angular one holds; by exploiting such a criterion we have observed that the NDF is dictated by the domain giving the greater estimate in (10).

As future developments, we plan to address the more realistic case of the 3D geometry where a planar source and planar observation domains will be considered.

REFERENCES

1. Pierri, R. and F. Soldovieri, "On the information content of the radiated fields in the near zone over bounded domains," *Inverse Problems*, Vol. 14, No. 2, 321–337, 1998.
2. Solimene, R. and R. Pierri, "Number of degrees of freedom of the radiated field over multiple bounded domains," *Opt. Lett.*, Vol. 32, 3113–3115, 2007.
3. Bucci, O. M., C. Gennarelli, G. Riccio, and C. Savarese, "Sampling representation of electromagnetic fields over three-dimensional domains," *Radio Science*, Vol. 34, 567–574, 1999.
4. Sten, J. C.-E. and E. A. Marengo, "Inverse source problem in an oblate spheroidal geometry," *IEEE Transactions on Antennas and Propagation*, Vol. 54, 3418–3428, 2006.
5. D'Agostino, F., F. Ferrara, C. Gennarelli, R. Guerriero, and M. Migliozzi, "Near field-far field transformation technique with helicoidal scanning for elongated antennas," *Progress In Electromagnetics Research B*, Vol. 4, 249–261, 2008.
6. Bertero, M. and P. Boccacci, *Introduction to Inverse Problems in Imaging*, Institute of Physics, Bristol, UK, 1998.
7. Balanis, C. A., *Advanced Engineering Electromagnetics*, John Wiley & Sons, Publishers, Inc., New York, 1989.
8. Barakat, R. and G. Newsam, "Algorithms for reconstruction of partially known, band-limited Fourier-transform pairs from noisy data," *J. Opt. Soc. Am. A*, Vol. 2, 2027–2039, 1985.
9. Frieden, B. R., "Evaluation, design and extrapolation methods for optical signals, based on use of the prolate functions," *Progress in Optics*, E. Wolf (ed.), Vol. 4, 311–407, North-Holland, Amsterdam, 1971.
10. Miller, D. A. B., "Communicating with waves between volumes: Evaluating orthogonal spatial channels and limits on coupling strengths," *Optics Letters*, Vol. 39, No. 11, 1681–1699, 2000.