VALIDITY CHECK OF MUTUAL INDUCTANCE FORMULAS FOR CIRCULAR FILAMENTS WITH LATERAL AND ANGULAR MISALIGNMENTS

S. I. Babic, F. Sirois, and C. Akyel

Département de Génie Électrique
École Polytechnique
C. P. 6079, Succ. Centre Ville, QC H3C 3A7, Montréal, Canada

Abstract—In this paper we derived the formula for calculating the mutual inductance between circular filaments with lateral and angular misalignment by using the approach of the magnetic vector potential. The results obtained correspond to those of F. W. Grover, although the latter used the general formula given by the Neumann integral instead of a vector potential approach. However, the major purpose of this paper is to clarify some confusion introduced in previous works regarding the mutual inductance calculation between thin filamentary circular coils with parallel axes in air. This problem has been solved by Kim et al. [8] using the magnetic vector potential, but unfortunately it leads to erroneous results, even for slight misalignments of the coils' center axes. This is why we chose to use the approach of the magnetic vector potential to show that, when properly derived, the results must indeed reduce to the well known F. W. Grover's formulas.

1. INTRODUCTION

In this paper, we use the approach of the magnetic vector potential to prove F. W. Grovers' formulas obtained by integration of the Neumann formula, and also to clarify some previously published results on mutual inductance calculation between circular filaments with parallel axes in air. The mutual inductance calculation between coaxial circular filaments has been thoroughly treated by a number of authors since the time of Maxwell, and an accuracy exceeding anything required in practice is nowadays possible [1–7]. A formula for two circles whose axes intersect was first given by Maxwell [1]. Formulas for

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Corresponding author: S. I. Babic (slobodan.babic@polymtl.ca).

circular loops with parallel axes have been given by Butterworth [2] and Snow [3]. Unfortunately, these formulas were slowly convergent and not useable with a wide range of parameters. Using Butterworth's formula [2], Grover developed a general method to calculate the mutual inductance between circular filaments located at any position with respect to each other [4, 5]. Today, with powerful numerical methods, such as the Finite Element Method (FEM) and the Boundary Element Method (BEM), it is possible to calculate accurately and rapidly this important electrical parameter. However, there is still an interest to address this problem using semi-analytical methods, as it considerably simplifies the mathematical procedures and associated programming. The computation time is also generally significantly reduced. One such approach has been presented in [8], in which the mutual inductance between circular filaments with parallel axes has been calculated by using the approach of the magnetic vector potential. Obviousely, this approach should lead to the same formulas as those obtained by Grover [4, 5]. However, a quick comparison shows that this is not the case, which leads to misleading results. Therefore, in this paper, we review this case in details, and we show the right way to retrieve Grover's formula from a magnetic vector potential approach. An application example is provided in the last section, in order to prove our assertions.

## 2. REVIEW OF BASIC EXPRESSIONS

2.1. Formula for Circular Coils with Both Lateral and Angular Misalignments

In [4] and [5], Grover presented a formula for computing the mutual inductance M between two filamentary circular coils with inclined axes (e.g., see Fig. 1). The first coil has a radius R_P, and the second coil has a radius R_S. The distance between the coils' centers is c, and the distance between their axes is d. The resulting expression proposed by Grover for M is:
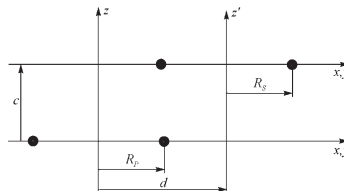
$$M = \frac{2\mu_0}{\pi} \sqrt{R_P R_S} \int\limits_0^\pi \frac{[\cos\theta - \frac{d}{R_S}\cos\phi]\Psi(k)}{k\sqrt{V^3}} d\phi \qquad (1)$$

where

$$\alpha = \frac{R_S}{R_P}, \ \beta = \frac{c}{R_P}, \quad V = \sqrt{1 - \cos^2\phi\sin^2\theta - 2\frac{d}{R_S}\cos\phi\cos\theta + \frac{d^2}{R_S^2}}$$
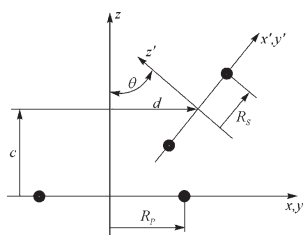
Figure 1. Filamentary circular coils with angular and lateral misalignment (axes intersect but not at the center of either coil).

Figure 2. Filamentary circular coils with lateral misalignment (z and z' axes parallel).

$$k^2 = \frac{4\alpha V}{(1 + \alpha V)^2 + \xi^2}, \ \xi = \beta - \alpha \cos\phi \sin\theta$$

$$\text{and } \Psi(k) = \left(1 - \frac{k^2}{2}\right) K(k) - E(k)$$

In the last equations, $K(k)$ and $E(k)$ are respectively the complete elliptic integrals of the first kind and the second kind, defined as

$$K(k) = \int_0^\pi \frac{1}{\sqrt{1 - k^2 \sin^2\theta}} d\theta \quad \text{and} \quad E(k) = \int_0^\pi \sqrt{1 - k^2 \sin^2\theta} d\theta \quad (2)$$

These elliptic integrals can be evaluated efficiently according to the classical algorithm presented in [9]. One has to be careful as the elliptic integral functions implemented in most computation software such as *Matlab* often use the parameter $m = k^2$ as the input argument, instead of the modulus k. One should note that the above formula corresponds to the general case when both lateral and angular misalignments are present, although the z and z' axes must lie in the same plane for (1) to be valid.

2.2. Formula for Circular Coils with Parallel Axes but Lateral Misalignment (Angle $\theta = 0$)

This situation, which is depicted in Fig. 2, is a direct simplification of the case presented above. The mutual inductance formula can therefore be obtained directly from (1), by setting $\theta = 0$. We end up with the

formula below, also provided in [5]:

$$M = \frac{2\mu_0}{\pi} \sqrt{R_P R_S} \int\limits_0^\pi \frac{(1 - \frac{d}{R_S} \cos\phi)\Psi(k)}{k\sqrt{V^3}} d\phi \tag{3}$$

where

$$\alpha = \frac{R_S}{R_P}, \ \beta = \frac{c}{R_P}, \ V = \sqrt{1 + \frac{d^2}{R_S^2} - 2\frac{d}{R_S}\cos\phi}, \ k^2 = \frac{4\alpha V}{(1 + \alpha V) + \beta^2}$$

and $\Psi(k) = \left(1 - \frac{k^2}{2}\right) K(k) - E(k).$

Another solution to this problem (circular coils with parallel axes but lateral misalignment) has been proposed by Kim et al. [8]. In this case, the solution was obtained by using the magnetic vector potential. The final expression for the mutual inductance, expressed in the notation of the current paper, takes the form given below:

$$M = \frac{2\mu_0 R_S}{\pi} \int\limits_0^{2\pi} \frac{\sqrt{(R_P + r)^2 + c^2}}{r} \Psi(k) d\phi \tag{4}$$

where

$$r = \sqrt{(d + R_S \cos(\phi))^2 + (R_S \sin(\phi))^2},$$
$$k^2 = \frac{4R_P r}{(R_P + r)^2 + c^2}, \ \Psi(k) = \left(1 - \frac{k^2}{2}\right) K(k) - E(k)$$

However, a simple numerical evaluation for identical parameters shows that Grover's and Kim's expressions, i.e., Equations (3) and (4), do not lead to the same value of mutual inductance, which suggests that one of them is erroneous. Such a numerical example is presented in Section 4 of this paper.

In order to clarify this, we carried out the entire derivation of the mutual inductance formula, using the method of the magnetic vector potential, as done by Kim et al. [8]. All details of this derivation are provided in Section 3.

3. ANALYTICAL CALCULATION OF MUTUAL INDUCTANCE

Let's consider two circular filaments, as showed in Fig. 3. This case corresponds to the one presented in Fig. 1, although additional
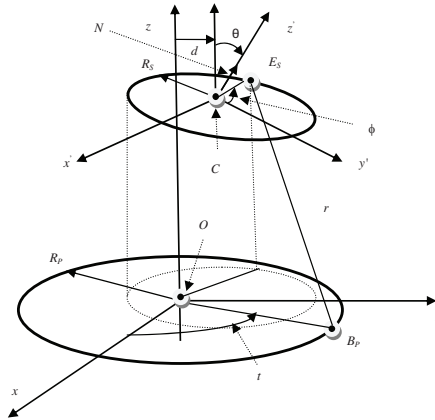
Figure 3. Filamentary circular coils with angular and lateral misalignment (axes intersect but not at the center of either coil).

variables were defined to clarify the analytical calculation presented below. Variables c, d, R_P and R_S are the same as in Figs. 1 and 2.

Before carrying on with the full derivation, it is useful to precise the content of Fig. 3.

1) The primary coil of radius R_P lies in the plane $XOY (z = 0)$, with the center at point $O(0, 0, 0)$.

2) The secondary coil of radius R_S lies in a plane whose unit normal vector $\mathbf{N} = \langle N_x, N_y, N_z \rangle$ is given by $N_x(x - x_C) + N_y(y - y_C) + N_z(z - z_C) = 0$, and the center of the coil is located at point $C(x_C, y_C, z_C)$.

3) The secondary axes system (x', y', z') is NOT the same as in Figs. 1 and 2.

4) Given the local coordinate system defined in Fig. 3, the angle between the unit vector N and axis z (or z') is θ (see Fig. 1) so that $N = \langle 0, \sin(\theta), \cos(\theta) \rangle$, and $C(0, y = d, z = c)$ (in terms of the global coordinate system).

5) B_P is an arbitrary point of the primary coil, whose coordinates are $(R_P \cos(t), R_P \sin(t), 0)$, $0 < t < 2\pi$.

6) E_S is an arbitrary point of the secondary coil, whose coordinates are $E_S(-R_S \sin(\phi),\ d + R_S \cos(\theta)\cos(\phi),\ c - R_S \sin(\theta)\cos(\phi))$, $0 < \phi < 2\pi$.

It should be emphasized that in all expressions provided below, the radius R_P of the primary coil must be larger or equal than the radius

R_S of the secondary coil. In the opposite case, the same procedure is applicable after setting the coil with larger radius as the primary coil.

The magnetic vector potential A at point E_S, produced by a circular current loop of radius R_P carrying the current I_P (See Fig. 3), is given by

$$\vec{A} = \frac{\mu_0}{4\pi} \int\limits_{l_P} \frac{I_P d\vec{l}_P}{r} \tag{5}$$

where

$$
\begin{aligned}
\mu_0 &= 4\pi \times 10^{-7}\,\text{H/m (magnetic permeability of vacuum)} \\
d\vec{l}_P &= (-\vec{i}\sin(t) + \vec{j}\cos(t))R_P dt \quad t \in (0; 2\pi) \\
&\vec{i}, \vec{j}, \vec{k} \text{ are the unit vectors of axes } x, y \text{ and } z \text{ respectively,} \\
r^2 &= \left|\vec{E}_S - \vec{B}_P\right|^2 = (R_P\cos t + R_S\sin\phi)^2 \\
&\quad + (R_P\sin t - R_S\cos\theta\cos\phi - d)^2 + (c - R_S\sin\theta\cos\phi)^2 \\
&= A + B\cos t + C\sin t \\
A &= R_P^2 + R_S^2\sin^2\phi + R_S^2\cos^2\theta\cos^2\phi + d^2 \\
&\quad + 2dR_S\cos\theta\cos\phi + (c - R_S\sin\theta\cos\phi)^2 \\
B &= 2R_P R_S\sin\phi \\
C &= -2R_P(d + R_S\cos\theta\cos\phi)
\end{aligned}
$$

Using Stokes's theorem, the flux through the secondary circuit due to a current in the primary circuit is

$$\Phi = \iint\limits_{S_S} \vec{B}\, d\vec{S}_S = \iint\limits_{S_S} (\nabla \times \vec{A})d\vec{S}_S = \int\limits_{l_S} \vec{A}d\vec{l}_S \tag{6}$$

where
$$d\vec{l}_S = (-\vec{i}\cos\phi - \vec{j}\cos\theta\sin\phi + \vec{k}\sin\theta\sin\phi)R_S d\phi$$

The mutual inductance between the secondary and primary coils is given by:

$$M = \frac{\Phi}{I_P} \tag{7}$$

From (5), (6) and (7), we obtain:

$$M = \frac{\mu_0}{4\pi}R_P R_S \int\limits_0^{2\pi}\int\limits_0^{2\pi} \frac{[\sin t\cos\phi - \cos t\cos\theta\sin\phi]\,d\phi dt}{r} \tag{8}$$

where
$$r = \sqrt{A + B\cos t + C\sin t}$$

Equation (8) should be amenable to the same form as Equation (1), i.e., Grover's formula. As a first step, we integrate with respect to the variable t, corresponding to the substitution $t = 2\psi + \alpha$, [9]. The following notations are useful in order to carry out the integration:

$$\tan\alpha = \frac{C}{B} \quad p = \sqrt{B^2 + C^2}$$

with:

$$\tan\alpha = -\frac{d + R_S\cos\theta\cos\phi}{R_S\sin\phi}$$

$$p = 2R_P\sqrt{d^2 + R_S^2 + 2dR_S\cos\theta\cos\phi - R_S^2\sin^2\theta\cos^2\phi}$$

and:

$$p = 2R_P R_S\sqrt{1 - \sin^2\theta\cos^2\phi + \frac{d^2}{R_S^2} + 2\frac{d}{R_S}\cos\theta\cos\phi} = 2R_P R_S V$$

where

$$V = \sqrt{1 - \sin^2\theta\cos^2\phi + \frac{d^2}{R_S^2} + 2\frac{d}{R_S}\cos\theta\cos\phi}$$

$$\text{and } \cos\alpha = \pm\frac{\sin\phi}{V}, \sin\alpha = \mp\frac{\frac{d}{R_S} + \cos\theta\cos\phi}{V} \tag{9}$$

The first integration then gives:

$$I_1 = \int_0^{2\pi} \frac{[\sin t\cos\phi - \cos t\cos\theta\sin\phi]\,dt}{r} = 2\cos\phi\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}} \frac{\sin(2\psi+\alpha)d\psi}{\sqrt{A + p\cos(2\psi)}}$$

$$-2\cos\theta\sin\phi\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}} \frac{\cos(2\psi+\alpha)d\psi}{\sqrt{A + p\cos(2\psi)}} = \frac{4\cos\phi\cos\alpha}{\sqrt{A+p}}\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}} \frac{\sin\psi\cos\psi d\psi}{\Delta}$$

$$+\frac{2\cos\phi\sin\alpha}{\sqrt{A+p}}\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}} \frac{\cos(2\psi)d\psi}{\Delta} - \frac{2\cos\theta\sin\phi\cos\alpha}{\sqrt{A+p}}\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}} \frac{\cos(2\psi)d\psi}{\Delta}$$

$$+\frac{4\cos\theta\sin\phi\sin\alpha}{\sqrt{A+p}}\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}}\frac{\sin\psi\cos\psi d\psi}{\Delta}$$

$$=\frac{4(\cos\phi\cos\alpha+\cos\theta\sin\phi\sin\alpha)}{\sqrt{A+p}}\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}}\frac{\sin\psi\cos\psi d\psi}{\Delta}$$

$$+\frac{2(\sin\phi\cos\alpha\cos\theta-\cos\phi\sin\alpha)}{\sqrt{A+p}}\int_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}}\frac{(2\sin^2\psi-1)d\psi}{\Delta}$$

$$=-\frac{4(\cos\phi\cos\alpha+\cos\theta\sin\phi\sin\alpha)}{\sqrt{A+p}}\frac{\Delta}{k^2}\Big|_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}}$$

$$+\frac{2(\sin\phi\cos\alpha\cos\theta-\cos\phi\sin\alpha)}{\sqrt{A+p}}\left\{\frac{2}{k^2}\left[F(\psi,k^2)-E(\psi,k^2)\right]\right.$$

$$\left.-F(\psi,k^2)\right\}_{-\frac{\alpha}{2}}^{\pi-\frac{\alpha}{2}}$$

where

$$\alpha=\frac{R_S}{R_P},\ \beta=\frac{c}{R_P},\ z=c-R_S\sin\theta\cos\phi,$$

$$k^2=\frac{4p}{A+p}=\frac{4R_PR_SV}{(R_P+R_SV)^2+z^2}=\frac{4\alpha V}{(1+\alpha V)^2+\xi^2}$$

$$\xi=\beta-\alpha\cos\phi\sin\theta,\Delta=\sqrt{1-k^2\sin^2\psi}$$

Using the following transformations [9, 10]:

$$F(\pi-\tfrac{\alpha}{2},k)=2K(k)-F(\tfrac{\alpha}{2},k),\quad E(\pi-\tfrac{\alpha}{2},k)=2E(k)-E(\tfrac{\alpha}{2},k)$$
$$F(-\tfrac{\alpha}{2},k)=-F(\tfrac{\alpha}{2},k),\qquad\qquad E(-\tfrac{\alpha}{2},k)=-E(\tfrac{\alpha}{2},k)$$

we obtain,

$$I_1=\frac{8(\cos\theta\sin\phi\cos\alpha-\cos\phi\sin\alpha)}{k^2\sqrt{A+p}}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right]\qquad(10)$$

From (8) and (10), we obtain

$$M=\frac{\mu_0}{\pi}\sqrt{R_PR_S}\int_0^{2\pi}\frac{(\cos\theta\sin\phi\cos\alpha-\cos\phi\sin\alpha)\Psi(k)}{k\sqrt{V}}d\phi\qquad(11)$$

It is possible to simplify (11) by using the transformation (9), i.e.,

$$\cos\theta\sin\phi\cos\alpha - \cos\phi\sin\alpha = \pm\frac{\cos\theta + \frac{d}{R_S}\cos\phi}{V}$$

which allows rewriting the mutual inductance expressed by (11) as:

$$M = \frac{\mu_0}{\pi}\sqrt{R_P R_S}\int_0^{2\pi}\frac{(\cos\theta + \frac{d}{R_S}\cos\phi)\Psi(k)}{k\sqrt{V^3}}d\phi \qquad (12)$$

Equation (12) can be evaluated on the two intervals $(0, \pi)$ and $(\pi, 2\pi)$. For the second interval we introduce the substitution $\phi = 2\pi - \varphi$, which makes the integral identical to the first one. Therefore, we can rewrite (12) on the interval $(0, \pi)$, i.e.,

$$M = \frac{2\mu_0}{\pi}\sqrt{R_P R_S}\int_0^{\pi}\frac{(\cos\theta + \frac{d}{R_S}\cos\varphi)\Psi(k)}{k\sqrt{V^3}}d\varphi \qquad (13)$$

Finally, the substitution $\phi = \pi - \varphi$ transforms (13) into well known Grover's formula (1):

$$M = \frac{2\mu_0}{\pi}\sqrt{R_P R_S}\int_0^{\pi}\frac{(\cos\theta - \frac{d}{R_S}\cos\phi)\Psi(k)}{k\sqrt{V^3}}d\phi \qquad (14)$$

where

$$\alpha = \frac{R_S}{R_P},\ \beta = \frac{z}{R_P},\ k^2 = \frac{4\alpha V}{(1+\alpha V)^2 + \xi^2}$$

$$V = \sqrt{1 - \sin^2\theta\cos^2\phi + \frac{d^2}{R_S^2} - 2\frac{d}{R_S}\cos\theta\cos\phi},$$

$$\xi = \beta - \alpha\cos\phi\sin\theta$$

$$\Psi(k) = \left(1 - \frac{k^2}{2}\right)K(k) - E(k)$$

By setting $\theta = 0$, (14) reduces to (3), i.e., the case of a lateral misalignment, as already shown in Section 2.2.

Thus, we analytically confirmed by the approach of the magnetic vector potential the Grover's formula for calculating the mutual inductance between circular filaments both with inclined and parallel axes.

Obviously, the formula obtained by Kim et al. [8], and corresponding to Equation (4) in this paper, should also be amenable to the same form as (3), since it was also derived from the magnetic vector potential. However, one can easily show that this is not the case. In addition, we easily show that a simple numerical application of (3) and (4) leads to different results, which is another reason to decline the result of Kim et al. in the calculation of the mutual inductance between circular filaments with parallel axes (lateral misalignment).

4. APPLICATION EXAMPLE

In this section, we compare the results obtained by Grover's [4, 5] and Kim's [8] formulas for a given application case. The problem consists in two circular coils of rectangular cross section (see Fig. 4), with the following dimensions:

1) *Primary Coil:* $R_P = 42.5\,\text{mm}$, $a = 10.0\,\text{mm}$, $h_P = 10.0\,\text{mm}$, $N_1 = N_P = 150$.

2) *Secondary Coil:* $R_S = 20.0\,\text{mm}$, $a = 10.0\,\text{cm}$, $b = 4.0\,\text{mm}$, $h_S = 4.0\,\text{mm}$, $N_2 = N_S = 50$.

The calculation of the mutual inductance of the proposed coil configuration will be realized by using the filament method [8, 11]. The dependence of the mutual inductance on the separation distance "c" (in [8] the separation distance is denoted "z") was calculated for several values of the lateral misalignment "d" (in [8] the off-center distance is "y"). See Table 1 for comparative results, in the case where the distance between the coils' centers is $c = 0$ (centers are coplanar), but variable lateral misalignment "d". We did not consider the cases for
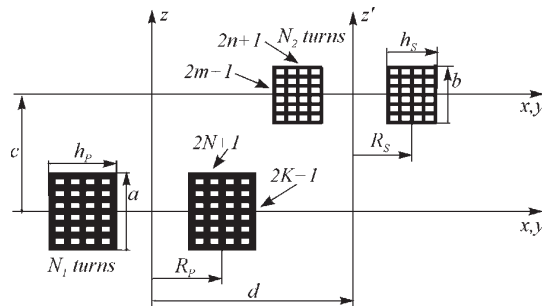
Figure 4. Geometric configuration considered in Section 4: Two circular coils of rectangular cross section with parallel axes (lateral misalignment).

Table 1. Mutual inductance calculation ($N = K = 2$, $n = m = 1$).

d (mm)	c (mm)	$M(10^{-4}\text{H})$ This work, Eq. (3)	$M(10^{-4}\text{H})$ [8], Eq. (20)	Discrepancy(%) Eq. (3) vs Eq. (20)
0.0	0.0	1.52875989	1.52875989	0.00
1.0	0.0	1.52967999	1.53063181	0.062
2.5	0.0	1.53453878	1.54049796	0.388
5.0	0.0	1.55229043	1.57627671	1.545
7.0	0.0	1.57597265	1.62337093	3.001
10.0	0.0	1.63029226	1.72877701	6.041
15.0	0.0	1.79299153	2.02559287	12.973
15.5	0.0	1.81647101	2.06644759	13.762
69.5	0.0	−0.36397730	1.57888396	−
80.0	0.0	−0.17767980	1.05074054	−
100.0	0.0	−0.07231342	0.60979570	−
200.0	0.0	−0.00718415	0.13810829	−
500.0	0.0	−0.00043535	0.02158082	−
1000.0	0.0	−0.00005401	0.00537760	−

which the coils intercepted each other, because they are not physical ones.

For the cases where the smaller coil is located inside the bigger coil ($d = 0$ to 15.6 mm), the discrepancy between the two results (3) and [8] changes from 0% (case of coaxial coils) to 13.762%. In the second region where the smaller coil is outside the bigger coil ($d = 69.5$ to 1000 mm) the mutual inductance obtained by [11] using Equation (3) (this work) presents a sign reversal, and then reaches a negative maximum and approaches zero for larger values of lateral misalignment "d". This behavior is the one that corresponds to the theory, as the flux lines linked by the secondary coil change their orientation outside the primary coil [4, 12]. On the other hand, the mutual inductance obtained by Equation (20) of reference [8] slowly approaches zero when the smaller coil is outside the bigger coil, and never reach negative values, which is not correct according to theory. This therefore proves that Equation (20) of reference [8] is erroneous. Actually, it is exact only when the coils' axes correspond with each other.

5. CONCLUSION

In this paper, we confirmed the validity of Grover's formula for calculating the mutual inductance between circular filaments with

lateral and angular misalignment by using the approach of the magnetic vector potential. We clarified some previous work on the same subject, in which a wrong formula was presented for the calculation of the mutual inductance between circular filaments with parallel axes in air.

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