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LINEARIZATION ERROR IN ELECTRICAL IMPEDANCE TOMOGRAPHY

N. Polydorides

Energy Initiative
Massachusetts Institute of Technology
77 Massachusetts Ave. 26-305, MA 02139, USA

Abstract—In borehole electromagnetic tomography and resistivity survey a linearized model approximation is often used, in the context of regularized regression, to image the conductivity distribution in a domain of interest. Due to the error introduced by the simplified model, quantitative image reconstruction becomes challenging without implementing a nonlinear algorithm. We derive a closed form expression of the linearization error in electrical impedance tomography based on the complete electrode model. The error term is expressed in an integral form involving the gradient of the perturbed electric potential in the interior of the domain and renders itself readily available for analytical or numerical computation. For real isotropic conductivity inhomogeneities with piecewise uniform characteristic functions the perturbed potential field can be shown to satisfy Poisson's equation with Robin boundary conditions and interior point sources positioned at the interfaces of the inclusions. Simulation experiments using a finite element method have been performed to validate these results.

1. INTRODUCTION

The complete electrode model in electrical impedance tomography is derived from Maxwell's time harmonic equations at the quasi-static limit and describes the electric potential field in the closure of a conductive domain with known electrical properties and impressed boundary excitation conditions. The model has been extensively discussed, analyzed and implemented in numerous publications, including Somersalo et al. [12] who prove existence and uniqueness of the complete electrode model solution, Paulson et al. [7] who

Corresponding author: N. Polydorides (nickpld@mit.edu).

analyze and compare the model to other more abstract models with emphasis on the profile of the boundary current density, and Pidcock et al. [8] who provide some analytic solutions for domains with regular geometries. The model has been implemented numerically and distributed under general public licence by Vauhkonen et al. [13] and by Polydorides et al. in [9] for two and three-dimensional problems respectively. A comprehensive discussion on the electrical imaging models, as indeed the technology and applications of impedance imaging can be found in the reviews by Borcea [2] and the textbooks by Kaipio et al. [6] and Holder [5]. Also note that the similar problem of electrical capacitance tomography [16] involves the same governing equation with somewhat different boundary conditions and thus most of the discussion ahead becomes relevant to this problem as well. For the scope of this study we shall briefly outline the complete electrode model equations based on which the linearization error will subsequently be derived.

Let $\Omega \subset \Re^d$, $d \in \{2,3\}$ be a simply connected, bounded, conductive domain with Lipschitz boundary $\partial\Omega$, and consider electrodes e_ℓ, $\ell = 1,\ldots,L$ attached on the boundary such that $\Gamma_e = \cup_{\ell=1}^{L} e_\ell$ denotes the part of the boundary underneath the electrodes and $\Gamma_o = \partial\Omega \backslash \Gamma_e$ the rest of the surface. If $\mathbf{r} \in \overline{\Omega}$ denotes the d-dimensional position vector in the closure of the domain, at the quasi-static limit the electric potential u satisfies the elliptic partial differential equation

$$\nabla \cdot \left[\sigma(\mathbf{r})\nabla u(\mathbf{r})\right] = 0, \quad \mathbf{r} \in \Omega \tag{1}$$

where σ is the real, isotropic electrical conductivity. The impressed boundary currents are expressed by the Neumann conditions

$$\int_{e_\ell} \mathrm{d}s\, \sigma(\mathbf{r})\nabla u(\mathbf{r}) \cdot \mathbf{n} \,=\, I_\ell, \quad \mathbf{r} \in \Gamma_e,\ \ell = 1,\ldots,L \tag{2}$$

$$\sigma(\mathbf{r})\nabla u(\mathbf{r}) \cdot \mathbf{n} \,=\, 0, \quad \mathbf{r} \in \Gamma_o \tag{3}$$

with $\mathbf{n}$ the outward unit normal on the boundary. The voltage measurement recorded at the ℓ'th electrode with contact impedance z_ℓ is given by the Robin boundary condition

$$V_\ell = u(\mathbf{r}) + z_\ell\, \sigma(\mathbf{r})\nabla u(\mathbf{r}) \cdot \mathbf{n} \quad \mathbf{r} \in \Gamma_e,\ \ell = 1,\ldots,L \tag{4}$$

assuming that the characteristic function of the contact impedance is uniform on each electrode and $\Re\{z_\ell\} > 0$. In their landmark paper [12] Somersalo et al. show that the model admits a unique solution upon enforcing the charge conservation principle on the applied current patterns and a choice of ground is made

$$\sum_{\ell=1}^{L} I_\ell = 0, \qquad u(\mathbf{r}_g) = 0 \quad \mathbf{r}_g \in \overline{\Omega} \tag{5}$$

In a functional analysis treatise of the model, as demonstrated in [12] and [2], one typically considers infinite dimensional Hilbert spaces for the conductivity and electric potential functions. In particular, a unique solution $(u^*, V^*) \in H^1(\Omega) \oplus \Re^L$ exists for $\sigma \in L^\infty(\Omega)$, in which case the Neumann to Dirichlet mapping

$$\Lambda_\sigma \; \sigma \nabla u \cdot \mathbf{n}(\partial\Omega) = u(\partial\Omega) \tag{6}$$

is self-adjoint and positive definite with $u \in H^1(\Omega)$ in the interior of the domain and $u \in H^{1/2}(\partial\Omega)$ at the boundary. The inverse EIT problem is then to reconstruct σ from some knowledge of Λ_σ. The study of this problem was popularized after the seminal paper of Calderon [4] who proved injectivity of the Dirichlet to Neumann mapping Λ_σ^{-1} in a more abstract setting, and showed that for a conductivity sufficiently closed to a known constant the linearized problem could yield an inverse solution with an error bounded in L^∞ norm, an idea later generalized by Silvester et al. in [11].

2. THE LINEARIZATION ERROR

A local perturbation in the conductivity of the domain relates to the induced changes in the boundary voltage measurements via the Jacobian of the forward mapping. This approximate linear relation holds true subject to a linearization error when the perturbations are small in magnitude. Evidently, for arbitrarily large conductivity changes the linear approximation fails as the linearization error dominates. The contribution of this paper is the derivation of a closed form expression for this error as indeed to provide an analytic formula relating the differential boundary voltage data to their corresponding conductivity perturbations. In our approach we extend the results of Breckon [3] and Polydorides et al. [9] who consider the perturbations in the electrical power of the domain.

From the divergence theorem for a scalar field w and a vector field $\mathbf{A}$

$$\int_\Omega \mathrm{d}v \; \mathbf{A} \cdot \nabla w + \int_\Omega \mathrm{d}v \; w \nabla \cdot \mathbf{A} = \int_{\partial\Omega} \mathrm{d}s \; w \mathbf{A} \cdot \mathbf{n}$$

substituting for $\mathbf{A} = \sigma \nabla u$ and simplifying for $\nabla \cdot \mathbf{A} = 0$ by virtue of (1) yields

$$\int_\Omega \mathrm{d}v \; \sigma \nabla u \cdot \nabla w = \int_{\partial\Omega} \mathrm{d}s \; w \sigma \nabla u \cdot \mathbf{n}$$

where $\mathrm{d}v$ and $\mathrm{d}s$ are volume (or area in two dimensions) and surface metrics respectively. Setting $w = u$ and importing the boundary condition (4) we get

$$\begin{aligned}\int_\Omega \mathrm{d}v\, \sigma|\nabla u|^2 &= \sum_{\ell=1}^{L}\int_{\partial\Omega} \mathrm{d}s\, \big(V_\ell - z_\ell\sigma\nabla u\cdot\mathbf{n}\big)(\sigma\nabla u\cdot\mathbf{n}) \\ &= \sum_{\ell=1}^{L}\left(\int_{\partial\Omega}\mathrm{d}s\, V_\ell\sigma\nabla u\cdot\mathbf{n} - \int_{\partial\Omega}\mathrm{d}s\, z_\ell|\sigma\nabla u\cdot\mathbf{n}|^2\right)\end{aligned}$$

From the Robin conditions (2) the surface integrals vanish everywhere on the boundary aside Γ_e, hence the first surface integral above simplifies further to

$$\sum_{\ell=1}^{L}\int_{\partial\Omega}\mathrm{d}s\, V_\ell\sigma\nabla u\cdot\mathbf{n} = \sum_{\ell=1}^{L} V_\ell\int_{\Gamma_e}\mathrm{d}s\, \sigma\nabla u\cdot\mathbf{n} = \sum_{\ell=1}^{L} V_\ell I_\ell$$

from where we arrive at the *power conservation law*

$$\sum_{\ell=1}^{L} V_\ell I_\ell = \int_\Omega \mathrm{d}v\, \sigma|\nabla u|^2 + \sum_{\ell=1}^{L} z_\ell\int_{\Gamma_e}\mathrm{d}s\, |\sigma\nabla u\cdot\mathbf{n}|^2 \tag{7}$$

asserting that the power imported to the domain is either stored as electric potential or dissipated at the contact impedances of the electrodes [9].

Taking perturbations $\sigma\to\sigma+\delta\sigma$, $u\to u+\delta u$, yields a change in the normal component of the current density at the boundary as

$$j(\mathbf{r}) + \delta j(\mathbf{r}) = \big(\sigma+\delta\sigma\big)\nabla\big(u(\mathbf{r})+\delta u(\mathbf{r})\big)\cdot\mathbf{n}, \quad \mathbf{r}\in\Gamma_e \tag{8}$$

Keeping the applied current I_ℓ fixed and substituting into (7) the integral over the domain gives

$$\begin{aligned}&\int_\Omega \mathrm{d}v\, (\sigma+\delta\sigma)|\nabla(u+\delta u)|^2 = \int_\Omega \mathrm{d}v\,\sigma|\nabla u|^2 + \int_\Omega\mathrm{d}v\,\sigma|\nabla\delta u|^2 \\ &+2\int_\Omega\mathrm{d}v\,\sigma\nabla u\cdot\nabla\delta u + \int_\Omega\mathrm{d}v\,\delta\sigma|\nabla(u+\delta u)|^2\end{aligned} \tag{9}$$

while the perturbed surface current density integral now becomes

$$\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\, z_\ell|(\sigma+\delta\sigma)\nabla(u+\delta u)\cdot\mathbf{n}|^2 = \sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\, z_\ell|j+\delta j|^2 \tag{10}$$

Applying the perturbations on the boundary condition (4) assuming $\Re\{z_l\} > 0$ yields

$$\delta j = z_\ell^{-1}(\delta V_\ell - \delta u), \quad \ell = 1,\ldots,L \tag{11}$$

and substituting into the right hand side of (10) yields

$$\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\; z_\ell|j+\delta j|^2=\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\; z_\ell|j|^2$$
$$+\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\;\frac{1}{z_\ell}(\delta V_\ell-\delta u)^2+2\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\; j\,(\delta V_\ell-\delta u) \qquad (12)$$

Adding (9) and (12) and subtracting from the power conservation law yields the perturbation in total power in the closure of the domain

$$\begin{aligned}\sum_{\ell=1}^{L}I_\ell\delta V_\ell \;=\; &\int_\Omega \mathrm{d}v\,\sigma|\nabla\delta u|^2+2\int_\Omega \mathrm{d}v\,\sigma\nabla u\cdot\nabla\delta u\\ &+\int_\Omega \mathrm{d}v\,\delta\sigma|\nabla u+\nabla\delta u|^2+\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\;\frac{1}{z_\ell}(\delta V_\ell-\delta u)^2\\ &+2\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\,(\delta V_\ell-\delta u)\sigma\nabla u\cdot\mathbf{n} \qquad (13)\end{aligned}$$

From the weak formulation for $w=\delta u$ the second integral on the right hand side and the surface integrals simplify to yield the *perturbed power conservation law*

$$\begin{aligned}\sum_{\ell=1}^{L}I_\ell\delta V_\ell \;=\; &-\int_\Omega \mathrm{d}v\,\sigma|\nabla\delta u|^2-\int_\Omega \mathrm{d}v\,\delta\sigma|\nabla u+\nabla\delta u|^2\\ &-\sum_{\ell=1}^{L}z_\ell\int_{\Gamma_e}\mathrm{d}s|\delta j|^2 \qquad (14)\end{aligned}$$

where the last term expands to

$$\begin{aligned}\sum_{\ell=1}^{L}\int_{\Gamma_e}\mathrm{d}s\,z_\ell|\delta j|^2 =\; &\sum_{\ell=1}^{L}\frac{1}{z_\ell}\delta V_\ell^2+\sum_{\ell=1}^{L}\frac{1}{z_\ell}\int_{\Gamma_e}\mathrm{d}s\,\delta u^2\\ &-2\sum_{\ell=1}^{L}\frac{1}{z_\ell}\delta V_\ell\int_{\Gamma_e}\mathrm{d}s\,\delta u\end{aligned}$$

For a conductivity $\hat{\sigma}=\sigma+\delta\sigma$ the electric potential $\hat{u}=u+\delta u$ satisfies

$$\nabla\cdot\left[\hat{\sigma}(\mathbf{r})\;\nabla\hat{u}(\mathbf{r})\right]=0,\quad \mathbf{r}\in\Omega \qquad (15)$$

and and thus subtracting from (1) we obtain

$$\nabla\cdot\left[\sigma\,\nabla\delta u+\delta\sigma\,\nabla\hat{u}\right]=0 \qquad (16)$$

Applying the divergence theorem on (16) for $w = \delta u$ yields

$$\int_\Omega \mathrm{d}v\ \sigma|\nabla\delta u|^2 + \int_\Omega \mathrm{d}v\delta\sigma\nabla\hat{u}\cdot\nabla\delta u = \int_{\partial\Omega} \mathrm{d}s\ \delta u\,\sigma\,\nabla\delta u\cdot\mathbf{n} + \int_{\partial\Omega} \mathrm{d}s\ \delta u\,\delta\sigma\,\nabla\hat{u}\cdot\mathbf{n} \tag{17}$$

From the normal component of the boundary current density and the relation in (8) at any $\mathbf{r} \in \Gamma_e$

$$\delta j(\mathbf{r}) = \sigma(\mathbf{r})\nabla\delta u(\mathbf{r})\cdot\mathbf{n} + \delta\sigma(\mathbf{r})\nabla\hat{u}(\mathbf{r})\cdot\mathbf{n}$$

thus the weak form (17) becomes

$$\int_\Omega \mathrm{d}v\ \sigma|\nabla\delta u|^2 + \int_\Omega \mathrm{d}v\ \delta\sigma\ \nabla\hat{u}\cdot\nabla\delta u = \int_{\partial\Omega} \mathrm{d}s\ \delta u\,\delta j \tag{18}$$

Combining (18) with the power conservation law (7) yields

$$\begin{aligned}\sum_{\ell=1}^{L} I_\ell\delta V_\ell \ &= \ -\int_\Omega \mathrm{d}v\ \delta\sigma|\nabla u|^2 - \int_\Omega \mathrm{d}v\ \delta\sigma\nabla u\cdot\nabla\delta u \\ &\quad - \int_{\partial\Omega} \mathrm{d}s\ \delta u\,\delta j - \sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} \mathrm{d}s\ |\delta j|^2\end{aligned} \tag{19}$$

and by substituting from (11), the third term in the right hand side above develops to

$$\begin{aligned}\int_{\partial\Omega} \mathrm{d}s\ \delta u\,\delta j \ &= \ \int_{\Gamma_e} \mathrm{d}s\ \big(\delta V_\ell - z_\ell\delta j\big)\delta j \\ &= \sum_{\ell=1}^{L} \delta V_\ell \int_{\Gamma_e} \mathrm{d}s\ \delta j - \sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} \mathrm{d}s\ |\delta j|^2\end{aligned} \tag{20}$$

We now show that the perturbations in conductivity and electric potential give rise to a perturbation in boundary current density with vanishing integral,

$$\int_{\Gamma_e} \mathrm{d}s\ \delta j = 0 \tag{21}$$

From the Neumann boundary condition (2) the current applied at the ℓ'th electrode satisfies

$$I_\ell = \int_{\Gamma_e} \mathrm{d}s\ \sigma\nabla u\cdot\mathbf{n} = \int_{\Gamma_e} \mathrm{d}s\ j$$

and therefore keeping I_ℓ fixed for $\ell = 1,\ldots,L$ under $\hat{\sigma} = \sigma + \delta\sigma$, $\hat{u} = u + \delta u$ we have

$$I_\ell = \int_{\Gamma_e} \mathrm{d}s\ \hat{\sigma}\nabla\hat{u}\cdot\mathbf{n} = \int_{\Gamma_e} \mathrm{d}s\ (j + \delta j)$$

Splitting the last integral and equating the last two equations we obtain (21).

We are now ready to tabulate our main result. Using (21) the integral of the power perturbation at the boundary (20) simplifies to

$$\int_{\partial\Omega} \mathrm{d}s\ \delta u\, \delta j = -\sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} \mathrm{d}s\ |\delta j|^2$$

and combining with (19) we arrive at

$$\sum_{\ell=1}^{L} I_\ell \delta V_\ell = -\int_{\Omega} \mathrm{d}v\ \delta\sigma |\nabla u|^2 - \int_{\Omega} \mathrm{d}v\ \delta\sigma \nabla u \cdot \nabla \delta u \tag{22}$$

Clearly the second integral is a nonlinear function of $\delta\sigma$ due to the product of the conductivity perturbation with the electric potential field perturbation δu which itself relates nonlinearly to $\delta\sigma$ via Maxwell's laws. To formulate an expression for the perturbation on the voltage measurements we develop first the left hand side of (22). Assume a pair drive system, for which we denote the direct current pattern $I^d \in \Re^L$, the measurement pattern $I^m \in \Re^L$ and the combined pattern $I^c \in \Re^L$ satisfying the constraint (5) as

$$\begin{aligned} &I^d = [|I|, -|I|, 0, \ldots, 0], \quad I^m = [0, \ldots, 0, 1, -1] \\ &I^c = I^d + I^m \end{aligned}$$

so that a current of magnitude $|I| > 0$ is injected into the domain by the electrode pair (e_p, e_n) and a measurement is obtained as a potential difference between the electrode pair (e'_p, e'_n). We then compute the differential potential

$$\begin{aligned} P &= u\big(I^c\big) - u\big(I^d\big) - u\big(I^m\big) \\ &= \sum_{\ell=1}^{L} I_\ell^c \delta V_\ell^c - \sum_{\ell=1}^{L} I_\ell^d \delta V_\ell^d - \sum_{\ell=1}^{L} I_\ell^m \delta V_\ell^m \end{aligned} \tag{23}$$

By the linearity of the Neumann to Dirichlet map, for $\ell = 1, \ldots, L$ we have

$$\delta V_\ell^c = \delta V_\ell^d + \delta V_\ell^m$$

and the above develops further to

$$\begin{aligned} P &= |I| \left(\delta V_{e_p}^c - \delta V_{e_n}^c - \delta V_{e_p}^d + \delta V_{e_n}^d\right) + \delta V_{e'_p}^c - \delta V_{e'_n}^c - \delta V_{e'_p}^m + \delta V_{e'_n}^m \\ &= |I| \left(\delta V_{e_p}^m - \delta V_{e_n}^m\right) + \delta V_{e'_p}^c - \delta V_{e'_n}^c - \delta V_{e'_p}^m + \delta V_{e'_n}^m \\ &= |I| \left(\delta V_{e_p}^m - \delta V_{e_n}^m\right) + \delta V_{e'_p}^d - \delta V_{e'_n}^d \end{aligned}$$

Recalling that the mapping in (6) is self-adjoint, the reciprocity principle asserts that the measurement obtained at the electrode pair (e_p, e_n) under a current pattern applied at (e'_p, e'_n) equals the measurement captured at $\left(e'_p, e'_n\right)$ for a current pattern of the same magnitude applied to electrode pair (e_p, e_n). Noting the magnitude relation between current and measurement patterns we have

$$|I|\left(\delta V^m_{e_p} - \delta V^m_{e_n}\right) = \left(\delta V^d_{e'_p} - \delta V^d_{e'_n}\right)$$

and thus we arrive at

$$P = 2\left(\delta V^d_{e'_p} - \delta V^d_{e'_n}\right) \tag{24}$$

Consider now the evaluation of the bilinear form in the right hand side of (22)

$$Q_{\delta\sigma}(u, \delta u) = -\int_\Omega \mathrm{d}v\ \delta\sigma|\nabla u|^2 - \int_\Omega \mathrm{d}v\ \delta\sigma\nabla u \cdot \nabla\delta u \tag{25}$$

under the same conditions, i.e., for the potential and perturbed potential induced by the current pattern $I^c - I^d - I^m$,

$$P = Q_{\delta\sigma}\big(u(I^c), \delta u(I^c)\big) - Q_{\delta\sigma}\big(u(I^d), \delta u(I^d)\big) - Q_{\delta\sigma}\big(u(I^m), \delta u(I^m)\big)$$

which gives

$$\begin{aligned} P = &- 2\int_\Omega \mathrm{d}v\ \delta\sigma\nabla u\big(I^d\big) \cdot \nabla u\big(I^m\big) \\ &- \int_\Omega \mathrm{d}v\ \delta\sigma\Big(\nabla u\big(I^d\big) \cdot \nabla\delta u\big(I^m\big) + \nabla u\big(I^m\big) \cdot \nabla\delta u\big(I^d\big)\Big) \end{aligned}$$

By the reciprocity principle the second integral simplifies further to yield

$$P = -2\int_\Omega \mathrm{d}v\ \delta\sigma\Big(\nabla u\big(I^d\big) \cdot \nabla u\big(I^m\big) + \nabla u\big(I^d\big) \cdot \nabla\delta u\big(I^m\big)\Big) \tag{26}$$

and thus equating (24) and (26) we obtain a formula for the change in boundary voltage measurements due to $\delta\sigma$ as

$$\delta V_\ell = -\int_\Omega \mathrm{d}v\ \Big(\delta\sigma\nabla u\big(I^d\big) \cdot \nabla u\big(I^m\big) + \delta\sigma\nabla u\big(I^d\big) \cdot \nabla\delta u\big(I^m\big)\Big) \tag{27}$$

Neglecting the second term above and assuming that $\|\sigma + \delta\sigma\|_{L^\infty} \sim \|\sigma\|_{L^\infty}$ yields the linearized model

$$\delta V_\ell \simeq -\int_\Omega \mathrm{d}v\ \delta\sigma\ \nabla u\big(I^d\big) \cdot \nabla u\big(I^m\big) \tag{28}$$

from which the familiar formula for the Jacobian of the forward mapping [9], derived also by Brandstätter in [15], is extracted as

$$J^{m,d} = \frac{\partial V_m^d}{\partial \sigma_k} = -\int_{\Omega_k} \mathrm{d}v \, \nabla u\big(I^d\big) \cdot \nabla u(I^m) \tag{29}$$

Based on (27) and (28) for arbitrarily large, finite perturbations $\delta\sigma$ the linear approximation introduces a linearization error

$$\begin{aligned} \mathcal{E}(\delta\sigma) &= -\int_\Omega \mathrm{d}v \; \delta\sigma \nabla u\big(I^d\big) \cdot \nabla \delta u(I^m) \\ &= -\int_\Omega \mathrm{d}v \; \delta\sigma \nabla \delta u\big(I^d\big) \cdot \nabla u(I^m) \end{aligned} \tag{30}$$

where the expression equivalence is due to field reciprocity.

3. THE EQUIVALENT SOURCE TERM

The expression for the linearization error involves the gradient of the perturbed electric potential δu and thus some elementary analysis on this field becomes imperative. The intent is to associate δu caused by an arbitrarily large perturbation in conductivity to an equivalent 'inducing' source function in the closure of the domain. Recall from (16) that under the imposed boundary conditions

$$\nabla \cdot \big[\sigma \, \nabla \delta u + \delta\sigma \, \nabla \hat{u}\big] = 0$$

thus rearranging and applying a vector calculus identity yields

$$\nabla \cdot \sigma(\mathbf{r}) \nabla \delta u(\mathbf{r}) = -\nabla \delta\sigma(\mathbf{r}) \cdot \nabla \delta u(\mathbf{r}) - \delta\sigma(\mathbf{r}) \nabla \cdot \nabla \hat{u}(\mathbf{r}) \tag{31}$$

which can be interpreted as a Poisson's equation on a domain with a known conductivity σ, an impressed source function

$$q(\mathbf{r}) = -\nabla \delta\sigma(\mathbf{r}) \cdot \nabla \delta u(\mathbf{r}) - \delta\sigma(\mathbf{r}) \nabla \cdot \nabla \hat{u}(\mathbf{r}) \tag{32}$$

and boundary conditions

$$V_\ell = \delta u(\mathbf{r}) + z_\ell \, \sigma(\mathbf{r}) \nabla \delta u(\mathbf{r}) \cdot \mathbf{n} \quad \mathbf{r} \in \Gamma_e \tag{33}$$

$$\delta u(\mathbf{r}_g) = 0 \tag{34}$$

for $\ell = 1, \ldots, L$. The special case of a small local perturbation $\delta\sigma = \delta(\mathbf{r} - \mathbf{r}')$, $\mathbf{r}' \in \Omega$ was investigated in the thesis of Breckon [3] in the context of the first-order approximate model $\nabla \cdot \big[\sigma \, \nabla \delta u + \delta\sigma \, \nabla u\big] \approx 0$ who showed that the equivalent source term is

$$q(\mathbf{r}) \approx -\nabla \delta\sigma(\mathbf{r}) \cdot \nabla u(\mathbf{r}) - \delta\sigma(\mathbf{r}) \nabla \cdot \nabla u(\mathbf{r})$$

Moreover, for $\sigma = 1$ the last term vanishes leaving an equivalent source function in the form of an electric dipole $q(\mathbf{r}) \approx -\nabla \delta(\mathbf{r}') \cdot \nabla u(\mathbf{r})$ of moment $|\nabla u(\mathbf{r})|$ positioned at $\mathbf{r}'$, a result consistent with Yorkey's compensation method in electrical resistor networks [14].

Returning in (32) and relaxing the linearity restrictions on $\delta\sigma$ consider a homogeneous inclusion $\delta\sigma \in D \subset \Omega$ with a constant characteristic function and smooth interface ∂D. Taking the gradient of $\delta\sigma$ over the domain indicates that the first term in (32) is simply a sum of electric dipoles positioned at the interface ∂D, this being essentially equivalent to taking the gradients of a multivariate rectangular function. In order to evaluate the second term we have to account for the field $\hat{u}$ over $\overline{D}$. Let ∂D^+, ∂D^- denote the inner and outer sides of the interface ∂D respectively, then the continuity conditions for the potential and the current density

$$\hat{u}(\partial D^+) = \hat{u}(\partial D^-) \tag{35}$$

$$\hat{\sigma}\nabla\hat{u} \cdot \mathbf{n}(\partial D^+) = \sigma\nabla\hat{u} \cdot \mathbf{n}(\partial D^-) \tag{36}$$

hold. Now since D is closed and source free in its interior $\nabla \cdot \hat{\sigma}\nabla\hat{u} = 0$. As σ is uniform in the support of the inclusion this can be factored out of the divergence. Noticing that the characteristic function of $\delta\sigma$ is zero outside $\bar{D}$, if $\hat{\sigma} \neq \sigma$ the scaled divergence term $\delta\sigma\nabla \cdot \nabla\hat{u}$ vanishes everywhere apart from the interface ∂D by virtue of (36). In essence the two terms in the right hand side of (32) assert that a bounded, piecewise constant perturbation in conductivity, yields a perturbed potential field that can be expressed as a solution of a Poisson equation for an equivalent source term that vanishes everywhere apart from the interface of the perturbation. The numerical experiments presented in the next section support this claim.

The concept of finding an equivalent source that yields the impact of the conductivity perturbation on the boundary data has also been investigated by Assenheimer et al. [1] and later generalized in a rigorous mathematical framework by Seo et al. [10] for T-scan impedance tomography models. In their study the authors showed that the distortion in the electric field induced by an inhomogeneity immersed in an otherwise homogenous medium can be expressed as a distribution of dipole sources. In this context some characteristics of the inclusion can be extracted from those of the equivalent sources.

4. NUMERICAL RESULTS

In this section we present some numerical results to verify the linearization error formula (30) as well as the equivalent sources (32). The linearization error was tested using the three-dimensional implementation of the complete electrode model in EIDORS 3D [9], while for the equivalent Poisson's sources we have opted for the two-dimensional EIDORS 2D [13] implementation of the model in order to enhance the visualization clarity.

4.1. Linearization Error

In a homogeneous cylindrical domain with conductivity background of $\sigma = 1\,\mathrm{S/m}$ and $L = 32$ boundary electrodes we introduce a conductivity change of fixed topology encompassing 12% of the domain's volume and magnitude that takes the values

$$\delta\sigma = \{-0.9, -0.99, -0.999, 0.01, 0.1, 0.9, 1.1, 11, 111, 1111\}$$

The results tabulated in Table 1 show the magnitude of the conductivity change, the change in the boundary voltage measurement corresponding to an arbitrarily chosen pair of current and measurement patterns, the contribution of the linear integral term (29) in the measurement change and the value of the linearization error (30). The changes in the boundary data, assuming unit magnitude currents, have been computed by subtracting the forward problem solutions

$$\delta V = \Lambda_\sigma \sigma \nabla u \cdot \mathbf{n} - \Lambda_{\sigma+\delta\sigma}(\sigma + \delta\sigma)\nabla(u + \delta u) \cdot \mathbf{n} \tag{37}$$

obtained by approximating numerically the model equations using linear finite elements. The validity of our linearization error result is manifested by the equality of δV to the sum of the $J\delta\sigma$ and $\mathcal{E}(\delta x)$ for each value of $\delta\sigma$. In fact, comparing the results of the second column of the table to the sum of those in the last two columns we have observed deviations in the order of forward numerical precision, e.g., $O(|10^{-6}|)$.

Table 1. Numerical results indicating the linearization error computed with the closed form expression (30). As anticipated the error grows nonlinearly as $|\delta\sigma|$ deviates from the homogeneous background value of $1\,\mathrm{S/m}$. The inclusion occupies 12% of the domain's volume.

$\delta\sigma$	δV	Linear term $J\delta\sigma$	Linear error $\mathcal{E}(\delta\sigma)$
-0.9	-0.0606	-0.0213	-0.0393
-0.99	-0.0915	-0.0234	-0.0681
-0.999	-0.0971	-0.0237	-0.0734
0.01	0.2351×10^{-3}	0.2367×10^{-3}	-0.0016×10^{-3}
0.1	0.0022	0.0024	-0.0002
0.9	0.0133	0.0213	-0.0080
1.1	0.0150	0.0260	-0.0111
11	0.0312	0.2604	-0.2293
111	0.0345	2.6279	-2.5933
1111	0.0348	26.3025	-26.2676

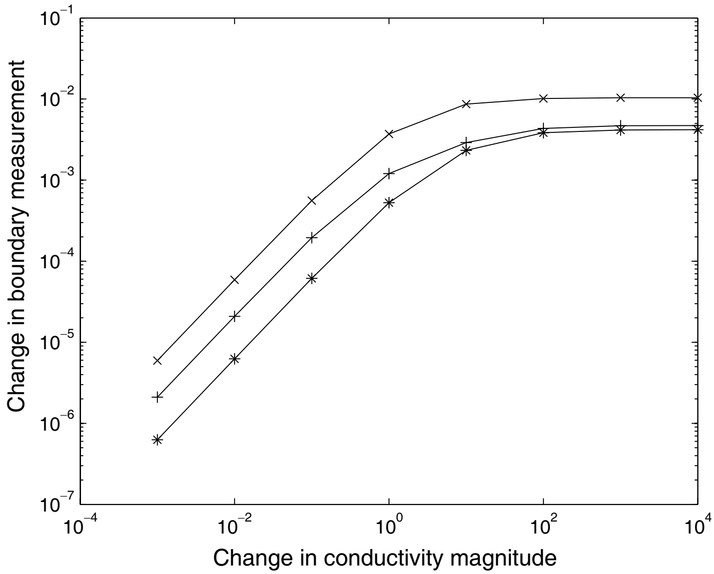

Figure 1. The change in three arbitrarily chosen boundary voltage measurements as a function of the perturbations in the interior conductivity assuming a three-dimensional cylindrical domain [9]. Notice the linearity at the left side of the curves and the 'saturation' on the right, indicating that the linearized models tend to over-estimate the boundary measurements at large conductivity changes.

Moreover, the results are indicative of the region where the linearized approximation can yield quantitatively accurate solutions to the inverse conductivity problem. Notice that the change in the boundary voltage data is always over-estimated by the linearized model, leaving a negative error term, which implies that the reconstruction of $\delta\sigma$ from a regularized cost function involving the norm $\|J\delta\sigma - \delta V\|$ will yield an under-estimated image of the conductivity. This remark is aligned to the effect of saturation observed in the profile of the boundary measurements with respect to changes in the interior conductivity, as suggested by the relevant graph in Figure 1. The same figure indicates also that for large conductivity changes the error $\mathcal{E}(\delta\sigma)$ tends to vary almost linearly with $\delta\sigma$.

4.2. Perturbed Potential Field and Equivalent Sources

We consider a homogeneous two-dimensional domain with circular geometry and background conductivity $\sigma = 1\,\mathrm{S/m}$, in which we

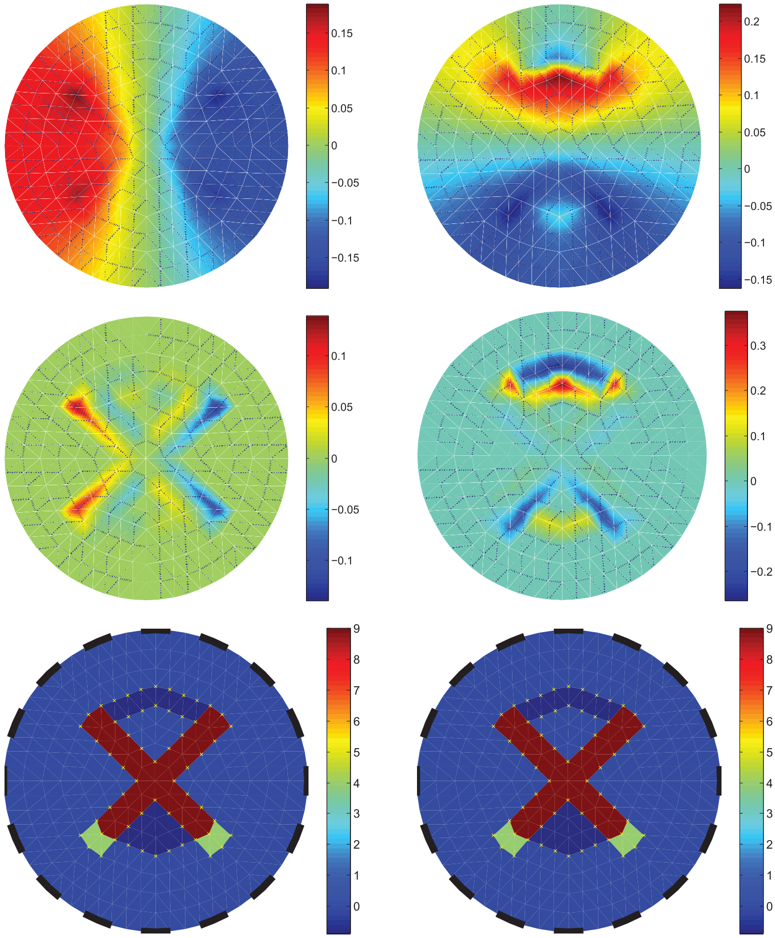

Figure 2. In each column from the top, plots of the approximated δu, q and $\delta\sigma$ on a finite element mesh indicating the position of the nonzero elements of the equivalent source q for two opposite pair drive current patterns. The markers ($\times$) indicate the position of the equivalent delta sources for each perturbed electric potential and the black arcs the boundary electrodes.

introduce three piecewise constant conductivity perturbations at magnitudes -0.9, 4 and $9\,\mathrm{S/m}$ as shown at the bottom of Figure 2. The domain is then excited by opposite pair drive current patterns injecting a current of unit magnitude and measuring the potential at the boundary through sixteen uniformly spaced electrodes, each having a contact impedance $z_\ell = 10\,\mathrm{Ohm}\Omega\,\mathrm{m}^2$. The perturbed electric potential field $\delta u = \hat{u} - u$ for two arbitrarily selected drive patterns appears at the top row of Figure 2. Solving the Poisson problem (32) for q with boundary conditions (33) we obtain the equivalent source fields as a superposition of delta functions, illustrated with some linear interpolation in the middle row of the same figure. To emphasize the exact location of these sources in relation to the interfaces of the immersed inhomogeneities we mark the finite element grid points where the q function attains a magnitude greater than the working numerical precision. To aid clarity these points are marked on the perturbed conductivity model for each δu.

5. CONCLUSION

We have derived a closed form expression for the linearization error in the complete electrode model for electrical impedance tomography. This contribution quantifies the suitability of the linear model to predict, and henceforth reconstruct, the boundary observations induced by conductivity perturbations on an otherwise known model. The error term is an integral involving the gradient of the perturbed potential field. Assuming piecewise constant conductivity profiles the perturbed potential field can be shown to satisfy Poisson's equation for a current density field comprised of point sources aligned at the interfaces of the conductivity perturbations.

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