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THE COMBINATION OF BCGSTAB WITH MULTI-FRONTAL ALGORITHM TO SOLVE FEBI-MLFMA LINEAR SYSTEMS ARISING FROM INHOMOGENEOUS ELECTROMAGNETIC SCATTERING PROBLEMS

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Abstract—The hybrid finite-element/boundary-integral method (FEBI) combined with the multilevel fast multipole algorithm (MLFMA) has been applied to model the three-dimensional scattering problems of inhomogeneous media. The stabilized Bi-conjugate gradient (BCGATAB) iterative solver based on the inner-looking algorithm is proposed to solve the final FEBI linear system, and the multifrontal algorithm combined with the approximate minimal degree permutation (AMD) is used for the LU decomposition of the FEM matrix. The accuracy and efficiency of the combined algorithm has been validated in the final of the paper. Numerical results show that the proposed method can greatly improve the efficiency of FEBI for scattering problems of inhomogeneous media.

1. INTRODUCTION

The finite element method (FEM) has gained great success in dealing with wave transmission problems owing to its strong ability for simulating arbitrary geometric structures and inhomogeneous media [1–4]. However, when dealing with scattering problems with FEM, the absorbing boundary conditions (ABC) [5, 6] or fictitious absorbers such as perfectly matched layers (PML) [7–9] should be adopted to truncate the computational area. In order to absorb the electromagnetic field efficiently, the absorbing boundary or the

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fictitious absorber should be placed far away from the scatterer, which will undoubtedly increase the discretization domain significantly. A larger discretization domain will in turn need finer meshes to suppress the dispersion errors. As a result, a great deal of unknowns will be produced. Moreover, the accuracy of the solutions obtained using ABC or PML is unpredictable as neither of them can completely absorb the incident wave from all incident angles [1]. As the boundary element method (BEM) [10–13] can express the radiation conditions accurately without increasing any discretization area, the combination of FEM and BEM (FEBI) is a good choice to solve the scattering problems of inhomogeneous media.

The disadvantage of FEBI is that the BEM matrix is dense, which greatly limits the application of FEBI in solving electrically-large problems. To overcome this problem, the multilevel fast multipole algorithm (MLFMA) [14, 15] or the adaptive integral method (AIM) [16] can be applied. In this paper, MLFMA is adopted as which is more widely used. A detailed description of the algorithm can be seen in [14]. After MLFMA accelerating, the storage needed by the FEBI matrix and the complexity of the matrix-vector multiplication in the FEBI method are reduced from $O(N+N_S^2)$ to $O(N+N_S \log N_S)$, which greatly increases the ability of FEBI in simulating electrically large problems.

Another disadvantage of the FEBI method is that the linear system is hard to solve using classical methods. When MLFMA is applied in the formation of the BEM matrix, the linear system is not given in an explicit form. Hence iterative methods [17–19] are the only option to solve the problem. However, the efficiency of different iterative solvers is quite different. In the following section, the solution strategy to the FEBI linear system is discussed in detail.

2. EFFICIENT ITERATIVE SOLVERS

Consider an arbitrarily inhomogeneous scattering object formed by one or several media in free space. The object may be formed by any media, for example, metal structures, isotropic dielectrics, and anisotropic dielectrics. For simplicity, the region occupied by the scatterer is assumed to be Ω, and the bounding surface is denoted as Γ, the normal vector of the outer surface is denoted as $\hat{n}$, the permittivity and permeability of the media are denoted as ε_r and μ_r respectively. The illustration of the problem is shown in Fig. 1.

Applying the FEBI process [20] to the above problem, a linear

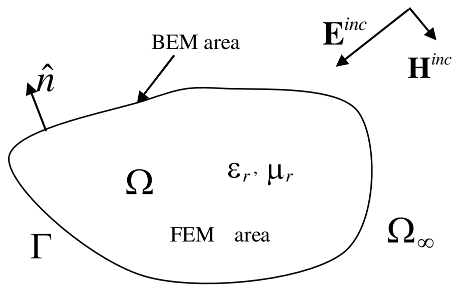

Figure 1. The illustration of FEBI computational domain.

system of the following form is generated:

$$\begin{bmatrix} K_{II} & K_{IS} & 0 \\ K_{SI} & K_{SS} & B \\ 0 & P & Q \end{bmatrix} \begin{Bmatrix} \mathbf{E}_I \\ \mathbf{E}_S \\ \bar{\mathbf{H}}_S \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ b \end{Bmatrix} \tag{1}$$

where $\mathbf{E}_I$ are the discrete electric fields inside Ω, and $\mathbf{E}_S\bar{\mathbf{H}}_S$ are the discrete electric and magnetic fields on Γ respectively. Denoting $\mathbf{W}$ the Whitney functions for tetrahedrons, $\mathbf{w}$ the Whitney functions for triangular patches, $\mathbf{g}$ the Rao-Wilton-Glisson basis functions, and

$$\mathbf{L}(\mathbf{X}) = jk_0 \iint_\Gamma \mathbf{X}(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}')\mathrm{d}\Gamma' + \iint_\Gamma \frac{j}{k_0}\nabla' \cdot \mathbf{X}(\mathbf{r}')\nabla G_0(\mathbf{r},\mathbf{r}')\mathrm{d}\Gamma' \tag{2}$$

$$\mathbf{K}(\mathbf{X}) = \iint_\Gamma \mathbf{X}(\mathbf{r}') \times \nabla G_0(\mathbf{r},\mathbf{r}')\mathrm{d}\Gamma' \tag{3}$$

The FEM matrix K_{II}, K_{IS}, $K_{SI}K_{SS}$ can be expressed with the following formulation:

$$K_{ij} = \iiint_\Omega \left[\frac{1}{\mu_r} (\nabla \times \mathbf{W}_i) \cdot (\nabla \times \mathbf{W}_j) - k_0^2 \varepsilon_r \mathbf{W}_i \cdot \mathbf{W}_j \right] d\Omega \tag{4}$$

And the matrix B, P, Q and b can be expressed as follows:

$$B_{mn} = jk_0 \int_\Gamma (\mathbf{w}_m \times \mathbf{w}_n) \cdot \hat{n} d\Gamma \tag{5}$$

$$P_{mn} = \int_\Gamma -\alpha \mathbf{g}_m \cdot \mathbf{K}(\mathbf{g}_n) + (1-\alpha)(n \times \mathbf{g}_m) \cdot \mathbf{L}(\mathbf{g}_n) d\Gamma \tag{6}$$

$$Q_{mn} = \int_\Gamma \mathbf{g}_m \cdot \mathbf{L}(\mathbf{g}_n) + (1-\alpha)\, n \times \mathbf{g}_m \cdot \mathbf{K}(\mathbf{g}_n) d\Gamma \tag{7}$$

$$b_m = \int_\Gamma \alpha \mathbf{g}_m \cdot \mathbf{E}^{inc} + (1-\alpha)\, n \times \mathbf{g}_m \cdot \bar{\mathbf{H}}^{inc} d\Gamma \tag{8}$$

In Eq. (1), the matrix K_{II}, K_{IS}, K_{SI}, K_{SS} and B is constructed with the FEM method, P and Q is constructed with the BEM method.

As can be seen from Eq. (1), the FEBI system matrix is formed by the FEM sparse matrix and the BEM dense matrix. The special structure of the FEBI matrix brings more difficulties for the solution of the linear system. As BEM matrix cannot be given in an explicit form when MLFMA is used in the formation of the BEM matrix, the FEBI linear system can only be solved with an iterative solver. However, with the existence of the FEM matrix, the whole FEBI matrix is highly ill-conditioned, which makes the linear system very difficult to solve. Conventional iterative methods [17], such as the conjugate gradient method (CG), converge very slowly. To overcome this difficulty, two methods are available. One method is to construct highly efficient preconditioners for conventional iterative algorithms (CA). The most effective preconditioner, which is proposed in [21], is constructed by applying the following absorbing boundary condition on the outer surface Γ:

$$n \times n \times \mathbf{E} - n \times \bar{\mathbf{H}} = n \times n \times \mathbf{E}^{inc} - n \times \bar{\mathbf{H}}^{inc} \tag{9}$$

Applying the FEM process to Eq. (9), a sparse FEM matrix M can be obtained, which is a good approximation of the BEM matrix in many cases. As a result, M can be used in constructing efficient preconditioners of Eq. (1).

Another efficient method is to construct effective iterative solvers. In [20], an iterative solver based on the Schur decomposition algorithm (DA) was proposed, which has been proven very effective to solve FEBI linear systems. Briefly speaking, the FEBI linear system can be rewritten as:

$$\begin{bmatrix} K & B' \\ P' & Q \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \bar{\mathbf{H}}_s \end{bmatrix} = \begin{Bmatrix} 0 \\ b \end{Bmatrix} \tag{10}$$

$$K = \begin{bmatrix} K_{II} & K_{IS} \\ K_{SI} & K_{SS} \end{bmatrix}, \quad B' = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad P' = \begin{bmatrix} 0 & P \end{bmatrix}$$

Applying the Schur decomposition, Eq. (10) can be decoupled into two linear systems:

$$\left[Q - P'K^{-1}B'\right] \bar{\mathbf{H}}_s = b \tag{11}$$

$$K\mathbf{E} = -B'\bar{\mathbf{H}}_s \tag{12}$$

As the electric unknowns E are eliminated, solving Eq. (11) is much easier than solving Eq. (1) with an iterative solver. After Eq. (11) is solved, E can be obtained by solving Eq. (12) with an appropriate solver.

In both of above methods, a multiplication of the inverse FEM matrix to a vector is involved in each iteration, which is equivalent to the solution of a FEM linear system. However, the FEM matrix involved in the DA-based method is smaller than that in the ABC preconditioned CA. What is more, as demonstrated in [20], the DA-based method outperforms the ABC preconditioned CA in convergence for many problems. For these reasons, the DA-based method is adopted in this paper, and the stabilized bi-conjugate gradient solver (BCGSTAB) [18] is applied for solving Eq. (11).

The BCGSTAB algorithm is an advanced version of the BiCG algorithm. It outperformed the CG algorithm for many problems. The deficiency of BCGSTAB is that the convergence behavior is erratic, and its convergence cannot be guaranteed in theory. For some very ill-conditioned linear systems, such as the edge-based FEM linear systems, it may dissipate. For this reason, many people prefer to use the CG algorithm. However, for some ill-conditioned linear systems, the CG algorithm is also hard to converge, though the convergence behavior of CG is monotonous. If some strategy are adopted to improve the condition number of the matrix, such as a suitable preconditioner is applied, the BCGSTAB algorithm rarely dissipate, and can converge rapidly to the accurate solution, as is the case especially in the solution of BEM linear systems [19]. In this paper, the Schur decomposition algorithm is used in combination with the BCGSTAB algorithm, which exhibits a superior convergence in the solution of scattering problems of inhomogeneous media, as shown in our numerical results.

One important factor that affects the efficiency of the DA-based method is the solution of the FEM linear system in iteratively solving Eq. (11). As the FEM matrix is highly ill-conditioned, the classical iterative methods, such as ICCG, SSORCG, etc., converge very slowly. What is more, the iterative process has to restart when the right-hand side is changed. In this paper, the multifrontal algorithm is applied to solve the FEM linear system, which is more efficient and more stable compared with iterative solvers.

The multifrontal algorithm is an advanced version of the frontal algorithm proposed by Irons [22], which was designed especially for large sparse FEM linear systems. In essence, this algorithm is a right-looking version of the LU decomposition algorithm. However, it partitions the whole factorization process into the factorization of a number of small dense frontal matrices, i.e., frontal matrices. During factorization, only the frontal matrix remains in the core memory. The factorized equations are stored in the out-of-core memory. Through this strategy, the memory needed can be reduced to minimal, and thus very large problem can be solved. Compared to the conventional

LU factorization algorithms, the multifrontal algorithm has several advantages: an efficient out-of-core scheme, effective vector processing on dense frontal matrices, and more liable to parallelize, etc. When applied in the solution of FEBI linear systems, the FEM matrix need to be factorized only once with the multifrontal algorithm. Only the forward substitution and backward substitution need to be performed during the iteration, which is really attractive compared to iterative solvers.

Based on the above algorithm, a general, accurate, and efficient code has been developed for scattering problems. In the following section, the accuracy of FEBI-MLFMA in different kinds of problems and the convergence property of the proposed DA-BCGSTAB algorithm are investigated in details by simulating several complex inhomogeneous targets.

3. NUMERICAL RESULTS

In this section, some numerical experiments are performed to investigate the performance of the described algorithm. In our tests, the residual error of iterative solvers is set to -40 dB in all examples. The first example is a perfectly conducting sphere with a diameter of 2λ, which is coated with a 0.017λ thick lossy dielectric layer. The relative dielectric constant of the coating is $\varepsilon_r = 4.0 - j$, and $\mu_r = 1.0$. In order to use FEBI-MLFMA, the problem is discretized into 14642 tetrahedrons, and the average mesh length is 0.075λ. As a result, 25131 unknowns were generated. In this problem, 3-level MLFMA is used, and the time used for filling the FEBI matrix is 17.65 s. The bi-static

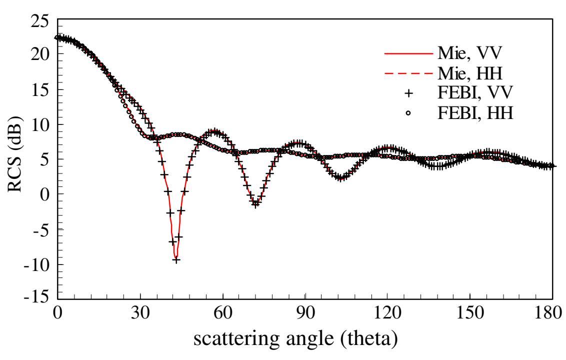

Figure 2. Bi-static radar cross sections of a dielectric-coated sphere with the relative permittivity $\varepsilon_r = 4 - j$.

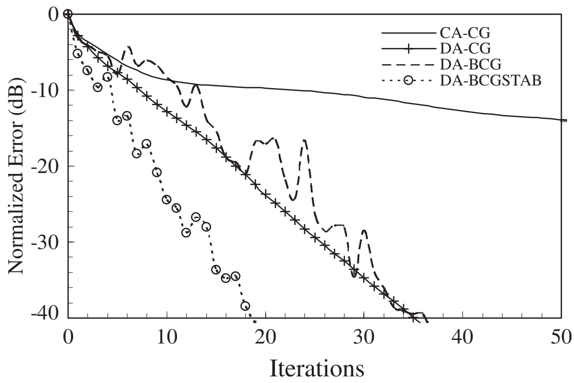

Figure 3. Convergence history of different solvers in the solution of scattering by the dielectric-coated sphere.

radar cross section (RCS) computed with FEBI-MLFMA is plotted in Fig. 2. The comparison with the exact Mie series solution is given and excellent agreement is found.

In order to test the efficiency of the suggested DA-BCGSTAB algorithm, the convergence characteristics of residual errors versus iteration number of DA-CG, DA-BCG, DA-BCGSTAB and the conventional CG algorithm for the dielectric-coated sphere are shown in Fig. 3, from which it can be seen that the DA-based iterative methods can converge much faster than conventional CG iterative methods. And the convergence number was further reduced when DA-BCGSTAB was used compared with DA-CG.

Table 1 displays the iteration number and CPU time used by the three different iterative methods. As can be seen from the table, the number of iterations and the CPU time used by the DA-BCGSTAB method are 19 and 4.26 s, respectively, 45 and 37.4 times smaller than the conventional CG algorithm, which shows the great advantage of the DA algorithm.

In the DA algorithm, the total LU factorization time used by the multifrontal method is 2.45 s. Before factorization, the total number of nonzero elements in the FEM matrix K is 197455. After factorization, the total number of nonzero elements is 2132294, about 9.8 times increased. However, the CPU time used per iteration by the DA method is about 0.2 s, only 22.5% to the conventional CG method. This is because the addition of nonzero elements is small compared with the nonzeros in the whole matrix.

In order to investigate the ability of FEBI-MLFMA in simulating inhomogeneous dielectric scatters, a spherical Luneburg lens with

Table 1. Iteration number and CPU time (s) needed by different solvers in the solution of scattering by the coated sphere.

Solver	iterations	Total CPU time (s)	CPU time (s) Per iteration
CA-CG	883	159.2	0.1803
DA-CG	35	7.55	0.2157
DA-BCG	37	7.58	0.2049
DA-BCGSTAB	19	4.26	0.2242

relative permittivity $\varepsilon_r = 2 - (r/a)^2$ was simulated, where r is the distance from the center of the sphere, a is the radius, $k_0 a = 5.0$. After discretization, a total of 70836 FEM unknown edges and 5679 BEM unknown edges are generated. In this problem, 3 level MLFMA is used. As the permittivity of the lens is not uniform, this problem has no analytical solutions. One method to validate the accuracy of the algorithm is to compare the results with the data given in [23]. However, another simple but accurate method is provided in this paper. As the permittivity is a slowly varied function of r, the sphere is firstly partitioned into 20 co-concentric spheres, denoted as S_i, $1 \leq i \leq 20$.

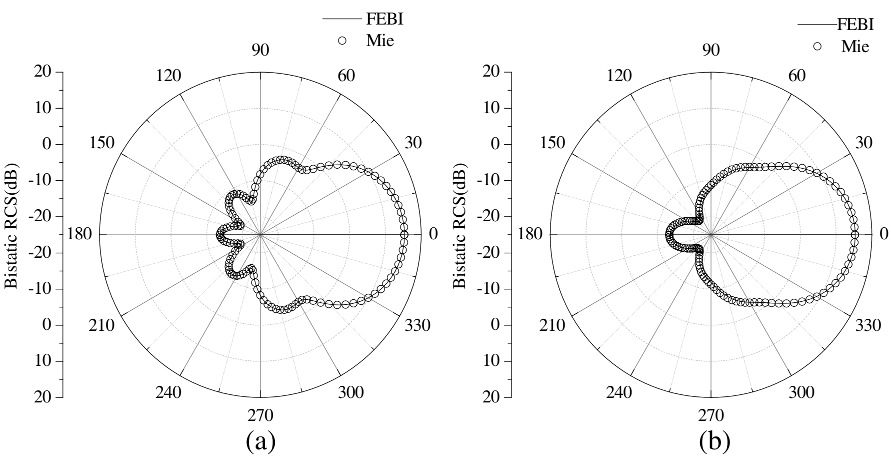

Figure 4. Bistatic RCS of a spherical Luneburg lens with relative permittivity $\varepsilon_r = 2 - (r/a)^2$, the circle represents data given by the Mie series solution. (a) HH polarization, (b) VV polarization.

The radius of S_i is determined by the relation: $r = a\sqrt{2 - \varepsilon_{ri}}$ and $\varepsilon_{ri} - \varepsilon_{ri-1} = 0.05$. In each layer between two adjacent sphere surfaces, the variation of the permittivity is no more than 0.05 and can be neglected. In this way, the Mie series solution can be used. The comparison of the bistatic RCS parameter given by FEBI-MLFMA and the Mie series solution was shown in Fig. 4. Both solutions agreed fairly well. Note that in Fig. 4, the Bistatic RCS are normalized to the geometrical cross section in order to keep consistence with the results in [23].

The convergence characteristics of residual errors versus iteration number of the DA-CG, DA-BCG and DA-BCGSTAB algorithm for the spherical Luneburg lens are displayed in Fig. 5, from which we can see that the DA-BCGSTAB algorithm still converges faster than DA-CG and DA-BCG for this problem. Comparatively, DA-BCGSTAB is more stable than DA-BCG. Table 2 depicted the iteration number and CPU time used by the various methods.

Table 2. Iteration number and CPU time (s) needed by different solvers in the solution of scattering by the spherical Luneburg lens.

Solver	iterations	Total CPU time (s)	CPU time (s) Per iteration
CA-CG	>1000	—	—
DA-CG	45	83.5	1.856
DA-BCG	51	92.7	1.818
DA-BCGSTAB	38	74.8	1.968

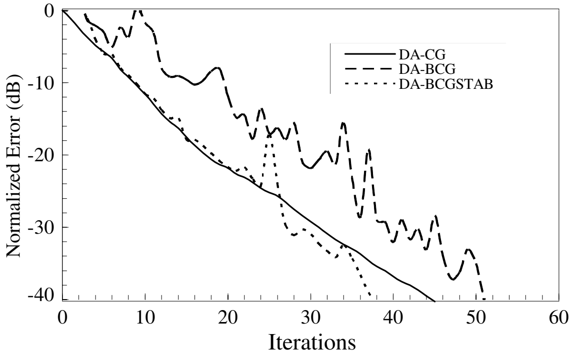

Figure 5. Convergence history of different solvers in the solution of scattering by the spherical Luneburg lens.

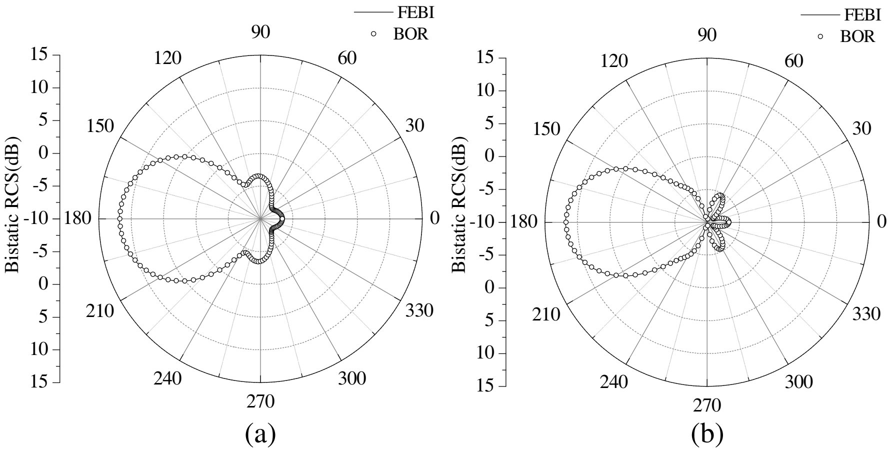

Figure 6. Bistatic RCS of the anisotropic sphere, the circle represents data given by the BOR-FEM method. (a) HH polarization, (b) VV polarization.

Next, the ability of FEBI-MLFMA in simulating the scattering of anisotropic dielectrics is examined. The problem is an anisotropic sphere with a diameter of 1λ, and the relative permittivity in the r, θ, ϕ direction is:

$$\varepsilon_r = \begin{bmatrix} 2.0-4.0j & & \\ & 2.5-5.0j & \\ & & 2.5-5.0j \end{bmatrix}$$

Note that in the x, y, z direction, the permittivity is a 3×3 full tensor, and is varied with the variation of θ and ϕ. To accurately simulate this dielectric, the 5 point Gauss integration was used in the formation of the FEM matrix. The problem is divided into 54715 tetrahedrons, with 66949 unknown FEM edges and 6102 unknown BEM edges. For validation, the bistatic RCS parameter computed by FEBI-MLFMA and by the BOR-FEM method was depicted in Fig. 6. As can be seen, both curves are totally coincident, which demonstrated the accuracy of FEBI-MLFMA in simulating anisotropic dielectrics. In this problem, the conventional CG algorithm used more than 1000 iterations to converge, the DA-CG algorithm used 46 iterations and 69.4 s CPU time, while DA-BCGSTAB used only 15 iterations and 23.4 s CPU time. Fig. 7 showed the convergence curves of the different algorithms.

The last example is the scattering of a metal-dielectric compound

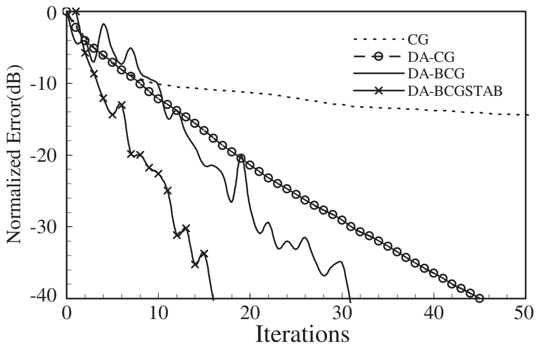

Figure 7. Convergence history of different solvers in the solution of scattering by an anisotropic sphere.

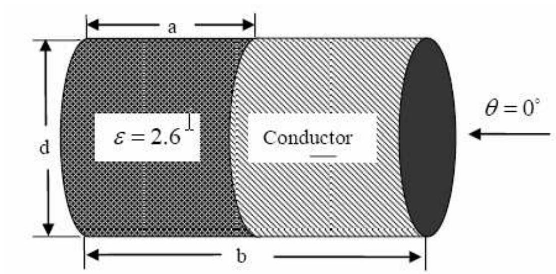

Figure 8. Configuration of the conducting-dielectric compound cylinder, $a = 5.08\,\text{cm}$, $b = 10.16\,\text{cm}$, $d = 7.62\,\text{cm}$.

cylinder. The dielectric part is formed by plexiglass ($\varepsilon_r = 2.6$), the metal part is aluminum. The configuration and dimension of the structure are illustrated in Fig. 8. In our FEBI-MLFMA simulation, aluminum is treated as PEC for simplicity. Firstly, the cylinder was meshed into 15408 tetrahedrons, with 21027 FEM edges and 4434 BEM edges generated. In this problem, 3 level MLFMA is used. The computed monostatic RCS parameter for this case and the measurement data are depicted in Fig. 9. In the figure, the two sets of data are nearly coincident except some small discrepancies, which may be because PEC is used to replace the aluminum in the FEBI-MLFMA simulation. However, our computed results using FEBI-MLFMA have excellent agreements with the computed results given in [24].

For comparison, the problem is computed repeatedly using DA-CG and DA-BCGSTAB with incident angles from $\theta = 0°$ to $\theta = 180°$, and the iteration numbers at different angles are recorded and depicted in Fig. 10. From the figure, DA-BCGSTAB showed a much

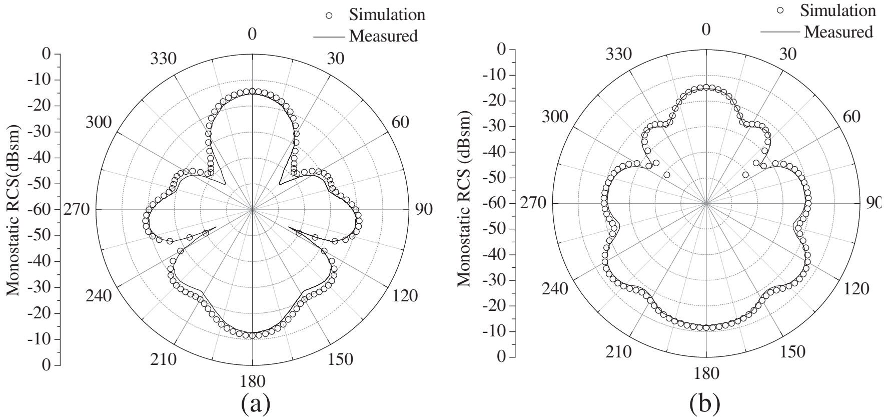

Figure 9. Computed and measured backscatter cross sections at 3.0 GHz for the inhomogeneous conducting-dielectric cylinder. (a) HH polarization, (b) VV polarization.

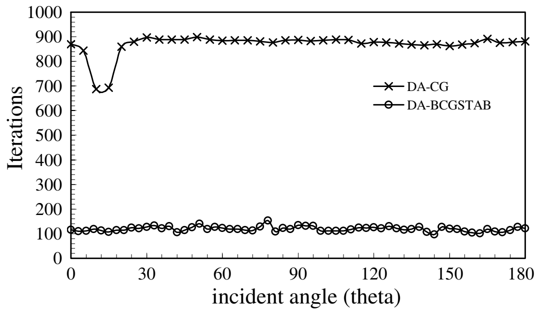

Figure 10. Number of iterations of DA-CG and DA-BCGSTAB for the inhomogeneous conducting-dielectric cylinder.

better convergence behavior than DA-CG at all angles. Once again, the efficiency of the DA-BCGSTAB algorithm in solving scattering problems of inhomogeneous media was demonstrated.

4. CONCLUSION

In this paper, the FEBI-MLFMA method is applied to model the scattering problems of inhomogeneous media. In order to improve efficiency, the discrete FEBI linear system is firstly decoupled with the Schur decomposition algorithm. Then the BCGSTAB iterative solver combined with the multifrontal solver is applied. Through the presented numerical results, it can be concluded that the FEBI-MLFMA method is very accurate and powerful in simulating scattering problems of inhomogeneous media, and the combination of DA-BCGSTAB with the multifrontal algorithm is a very efficient method for the FEBI-MLFMA solution of dielectric-scattering problems.

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