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SCATTERING OF ELECTROMAGNETIC RADIATION BY A COATED PERFECT ELECTROMAGNETIC CONDUCTOR SPHERE

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Abstract—An analytic theory for the electromagnetic scattering from a coated perfect electromagnetic conductor (PEMC) sphere is developed. The sphere is characterized by its M parameter, and the coating material by its permittivity and permeability, which may attain arbitrary values, including negative ones. The theory is applied to the calculation of various scattering cross sections. It is found that the scattered fields contain cross polarized components, which do not exist in the case of a coated perfect electric conductor (or perfect magnetic conductor) sphere. Symmetry properties of the solutions, which reflect a generalized form of electric-magnetic duality, are demonstrated.

1. INTRODUCTION

The perfect electromagnetic conductor has recently been introduced as a generalization of the perfect electric conductor (PEC) and the perfect magnetic conductor (PMC) [1]. At the surface of a PEMC the boundary conditions are

$$\vec{n} \times \left(\vec{H} + M\vec{E}\right) = 0 \tag{1}$$

$$\vec{n} \cdot \left(\vec{D} - M\vec{B}\right) = 0 \tag{2}$$

where $\vec{n}$ is the unit normal, and the admittance like parameter M characterizes the PEMC. For $M = 0$, the PMC case is retrieved, and the limit $M \to \pm\infty$ corresponds to the PEC case. A PEMC material acts as a perfect reflector of electromagnetic waves, but differs from the PEC and the PMC in that the reflected wave has a cross-polarized component. The implications of this non-reciprocal effect have been demonstrated for the planar geometry [1, 2], for a cylinder [3] and

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for a sphere [2, 4, 5]. The problem of the scattering by a PEMC sphere was first treated analytically in the small size limit [2], and later a general analytic Mie type solution for arbitrary sphere size was developed [4]. A numerical method for calculating the scattering properties of a PEMC sphere, using the electric field integral equation, has been applied by Sihvola et al. [5].

Here we investigate how the scattering properties of a PEMC sphere are modified when it is coated with a dielectric layer. The permittivity and the permeability of the coating material can have arbitrary values, including negative ones. Thus, the theory presented here can be applied to plasmonic or metamaterial coatings. These types of coatings have recently been the subject of many investigations concerning the possibility of achieving electromagnetic cloaking of scattering objects [6–8].

2. SCATTERING THEORY FOR COATED PEMC SPHERE

We consider the geometry of a PEMC sphere of radius a, which is coated by a spherical shell, so that the external radius is b. The permittivity of the shell material is $\varepsilon_1 = \varepsilon_r\varepsilon_0$ and its permeability is $\mu_1 = \mu_r\mu_0$, where ε_0 and μ_0 are the free space permittivity and permeability, respectively. The electromagnetic fields will be expanded in terms of the spherical vector wave functions [9]

$$\vec{M}^{(i)}_{\sigma mn} = \vec{\nabla} \times [\vec{r}\, Y_{\sigma mn}(\theta, \varphi) z_n(kr)] \tag{3}$$

$$\vec{N}^{(i)}_{\sigma mn} = \frac{1}{k}\vec{\nabla} \times \vec{M}^{(i)}_{\sigma mn} \tag{4}$$

Here spherical coordinates (r, θ, φ) are used, and the subscript σ stands for e (even) or o (odd), according to whether $\cos m\phi$ or $\sin m\phi$ is used when multiplying by the associated Legendre polynomial $P_n^m(\cos\theta)$ in order to obtain the spherical harmonic $Y_{\sigma mn}(\theta, \varphi)$. The wavenumber k is given by $k_1 = \omega\sqrt{\varepsilon_1\mu_1}$ inside the coating, and by $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$ outside it. The superscript i specifies the choice of the radial function $z_n(kr)$. For i=1 this is a spherical Bessel function $j_n(kr)$, for $i = 2$ a spherical Neumann function $n_n(kr)$, and for $i = 3$ a spherical Hankel function $h_n(kr)$. A plane wave of frequency ω, propagating in the z direction, with the electric field polarized in the x direction, is incident on the sphere. The expansion of the incident field is given by [10]

$$\vec{E}^i = E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(\vec{M}^{(1)}_{o1n} - i\vec{N}^{(1)}_{e1n}\right) \tag{5}$$

$$\vec{H}^i = -(E_0/\eta_0) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} (\vec{M}^{(1)}_{e1n} + i\vec{N}^{(1)}_{o1n}) \tag{6}$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$.

The scattered field in the region $r > b$ is expanded in the form

$$\vec{E}^s = E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(a^s_n \vec{M}^{(3)}_{o1n} + c^s_n \vec{M}^{(3)}_{e1n} - ib^s_n \vec{N}^{(3)}_{e1n} - id^s_n \vec{N}^{(3)}_{o1n}\right) \tag{7}$$

$$\vec{H}^s = -(E_0/\eta_0) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(b^s_n \vec{M}^{(3)}_{e1n} + d^s_n \vec{M}^{(3)}_{o1n} + ia^s_n \vec{N}^{(3)}_{o1n} + ic^s_n \vec{N}^{(3)}_{e1n}\right) \tag{8}$$

In the standard Mie type scattering theory only the coefficients a^s_n and b^s_n are needed in the scattered field expansion. Since in the PEMC boundary conditions (1) and (2) a mixing of $\vec{E}$ and $\vec{H}$ occurs, the coefficients c^s_n and d^s_n have to be added. These represent the cross-polarized components of the scattered field. The fields inside the coating are expanded in the form

$$\begin{aligned}\vec{E}^t = {} & E_0 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} (a^{t1}_n \vec{M}^{(1)}_{o1n} + a^{t2}_n \vec{M}^{(2)}_{o1n} + c^{t1}_n \vec{M}^{(1)}_{e1n} + c^{t2}_n \vec{M}^{(2)}_{e1n} \\ & -ib^{t1}_n \vec{N}^{(1)}_{e1n} - ib^{t2}_n \vec{N}^{(2)}_{e1n} - id^{t1}_n \vec{N}^{(1)}_{o1n} - id^{t2}_n \vec{N}^{(2)}_{o1n})\end{aligned} \tag{9}$$

$$\begin{aligned}\vec{H}^t = {} & -(E_0/\eta_1) \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} (b^{t1}_n \vec{M}^{(1)}_{e1n} + b^{t2}_n \vec{M}^{(2)}_{e1n} + d^{t1}_n \vec{M}^{(1)}_{o1n} + d^{t2}_n \vec{M}^{(2)}_{o1n} \\ & +ia^{t1}_n \vec{N}^{(1)}_{o1n} + ia^{t2}_n \vec{N}^{(2)}_{o1n} + ic^{t1}_n \vec{N}^{(1)}_{e1n} + ic^{t2}_n \vec{N}^{(2)}_{e1n})\end{aligned} \tag{10}$$

where $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$.

We now apply the boundary conditions at the interfaces. At $r = a$ the tangential field components have to satisfy the boundary condition

$$H^t_t + ME^t_t = 0 \tag{11}$$

At $r = b$ the usual Maxwell boundary conditions of the continuity of the tangential components of the electric and magnetic fields are applied:

$$E^i_t + E^s_t = E^t_t \tag{12}$$

$$H^i_t + H^s_t = H^t_t \tag{13}$$

Applying the boundary conditions (11)–(13) and using the orthogonality properties of the angular functions we obtain the system of six linear equations

$$b_n^{t1} j_n(k_1 a) + b_n^{t2} n_n(k_1 a) - c_n^{t1} M\eta_1 j_n(k_1 a) - c_n^{t2} M\eta_1 n_n(k_1 a) = 0 \quad (14)$$

$$b_n^{t1} M\eta_1 \left[k_1 a j_n(k_1 a)\right]' + b_n^{t2} M\eta_1 \left[k_1 a n_n(k_1 a)\right]' + c_n^{t1} \left[k_1 a j_n(k_1 a)\right]' + c_n^{t2} \left[k_1 a n_n(k_1 a)\right]' = 0 \quad (15)$$

$$c_n^{t1} j_n(k_1 b) + c_n^{t2} n_n(k_1 b) - c_n^s h_n(k_0 b) = 0 \quad (16)$$

$$b_n^{t1} \frac{1}{k_1 b} \left[k_1 b j_n(k_1 b)\right]' + b_n^{t2} \frac{1}{k_1 b} \left[k_1 b n_n(k_1 b)\right]' - b_n^s \frac{1}{k_0 b} \left[k_0 b h_n(k_0 b)\right]' = \frac{1}{k_0 b} \left[k_0 b j_n(k_0 b)\right]' \quad (17)$$

$$b_n^{t1} \eta_0 j_n(k_1 b) + b_n^{t2} \eta_0 n_n(k_1 b) - b_n^s \eta_1 h_n(k_0 b) = \eta_1 j_n(k_0 b) \quad (18)$$

$$c_n^{t1} \frac{\eta_0}{k_1 b} \left[k_1 b j_n(k_1 b)\right]' + c_n^{t2} \frac{\eta_0}{k_1 b} \left[k_1 b n_n(k_1 b)\right]' - c_n^s \frac{\eta_1}{k_0 b} \left[k_0 b h_n(k_0 b)\right]' = 0 \quad (19)$$

for the six coefficients b_n^{t1}, b_n^{t2}, c_n^{t1}, c_n^{t2}, b_n^s and c_n^s. Here the primes denote differentiation with respect to the arguments of the spherical functions.

Furthermore, the system of six linear equations

$$a_n^{t1} M\eta_1 j_n(k_1 a) + a_n^{t2} M\eta_1 n_n(k_1 a) - d_n^{t1} j_n(k_1 a) - d_n^{t2} n_n(k_1 a) = 0 \quad (20)$$

$$a_n^{t1} \left[k_1 a j_n(k_1 a)\right]' + a_n^{t2} \left[k_1 a n_n(k_1 a)\right]' + d_n^{t1} M\eta_1 \left[k_1 a j_n(k_1 a)\right]' + d_n^{t2} M\eta_1 \left[k_1 a n_n(k_1 a)\right]' = 0 \quad (21)$$

$$d_n^{t1} \eta_0 j_n(k_1 b) + d_n^{t2} \eta_0 n_n(k_1 b) - d_n^s \eta_1 h_n(k_0 b) = 0 \quad (22)$$

$$a_n^{t1} j_n(k_1 b) + a_n^{t2} n_n(k_1 b) - a_n^s h_n(k_0 b) = j_n(k_0 b) \quad (23)$$

$$a_n^{t1} \frac{\eta_0}{k_1 b} \left[k_1 b j_n(k_1 b)\right]' + a_n^{t2} \frac{\eta_0}{k_1 b} \left[k_1 b n_n(k_1 b)\right]' - a_n^s \frac{\eta_1}{k_0 b} \left[k_0 b h_n(k_0 b)\right]' = \frac{\eta_1}{k_0 b} \left[k_0 b j_n(k_0 b)\right]' \quad (24)$$

$$d_n^{t1} \frac{1}{k_1 b} \left[k_1 b j_n(k_1 b)\right]' + d_n^{t2} \frac{1}{k_1 b} \left[k_1 b n_n(k_1 b)\right]' - d_n^s \frac{1}{k_0 b} \left[k_0 b h_n(k_0 b)\right]' = 0 \quad (25)$$

for the six coefficients a_n^{t1}, a_n^{t2}, d_n^{t1}, d_n^{t2}, a_n^s and d_n^s is obtained.

Once the expansion coefficients are obtained by numerically solving the two linear systems (14)–(19) and (20)–(25), the various cross sections can be calculated from the radial component of the complex Poynting vector. Integrating over a large sphere, as in the standard Mie theory [10], the extinction cross section of the coated sphere, in units of the geometric cross section of the PEMC sphere, is found to be given by [4]

$$Q_e = -\frac{2}{\rho^2} \sum_{n=1}^{\infty} (2n+1) \mathrm{Re} \left(a_n^s + b_n^s \right) \tag{26}$$

where $\rho = k_0 a$ is the size parameter of the PEMC sphere. The scattering cross section is given by

$$Q_s = \frac{2}{\rho^2} \sum_{n=1}^{\infty} (2n+1) \left(|a_n^s|^2 + |b_n^s|^2 + |c_n^s|^2 + |d_n^s|^2 \right) \tag{27}$$

The following expressions are obtained for the radar cross section, and the forward scattering cross section, respectively:

$$\begin{aligned} \sigma(180^\circ) = \left(\frac{2}{\rho}\right)^2 \Bigg[& \left| \sum_{n=1}^{\infty} (-1)^n (n+\frac{1}{2})(b_n^s - a_n^s) \right|^2 \\ & + \left| \sum_{n=1}^{\infty} (-1)^n (n+\frac{1}{2})(c_n^s + d_n^s) \right|^2 \Bigg] \end{aligned} \tag{28}$$

$$\sigma(0^\circ) = \left(\frac{2}{\rho}\right)^2 \left[\left| \sum_{n=1}^{\infty} (n+\frac{1}{2})(b_n^s + a_n^s) \right|^2 + \left| \sum_{n=1}^{\infty} (n+\frac{1}{2})(c_n^s - d_n^s) \right|^2 \right] \tag{29}$$

In Eqs. (27)–(29) the sums containing a_n^s and b_n^s represent the contribution of the co-polarized scattered fields, while those containing c_n^s and d_n^s give the contribution of the cross polarized scattered fields.

3. NUMERICAL RESULTS AND DISCUSSION

We have performed a large number of numerical calculations for various sphere sizes, M values, coating thicknesses, and coating permittivity and permeability. From the results of these calculations some interesting conclusions concerning the polarization and symmetry properties of the solutions can be drawn.

(1) For real ε_r and μ_r the relations

$$\mathrm{Re}(a_n^s) = -(|a_n^s|^2 + |d_n^s|^2) \tag{30}$$

$$\mathrm{Re}(b_n^s) = -(|b_n^s|^2 + |c_n^s|^2) \tag{31}$$

are found to be always valid. From (26) and (27) it follows that the extinction cross section is equal to the scattering cross section. Thus, the absorption cross section, given by the difference between the two, is equal to zero. This is a manifestation of the fact that the PEMC sphere is totally reflecting, so that no losses occur inside it, and for real ε_r and μ_r no losses occur in the coating.

(2) The relation $c_n^s = d_n^s$ is found to be always valid. From (29), it follows that there exists no cross polarized component in the forward direction.

(3) For given sphere and coating sizes and an impedance matched coating, i.e., $\varepsilon_r = \mu_r$, it is found that the sum $a_n^s + b_n^s$ is independent of M. From (26) and (29), it follows that the extinction cross section and the forward scattering cross section are independent of M. Furthermore, it is found that the total radar (backscattering) cross section (RCS) is independent of M.

(4) The relation $a_n^s = b_n^s$ is found to hold when the following two impedance matching conditions are satisfied: (a) The coating material has the symmetry property $\varepsilon_r = \mu_r$; (b) The M parameter of the PEMC material is equal to $1/\eta_0$. From (28) it follows that in this case there exists no co-polarized component in the backward direction, but a cross-polarized component does exist. This is a generalization of the known theorem that the backscattering from an impedance matched dielectric sphere is equal to zero [11].

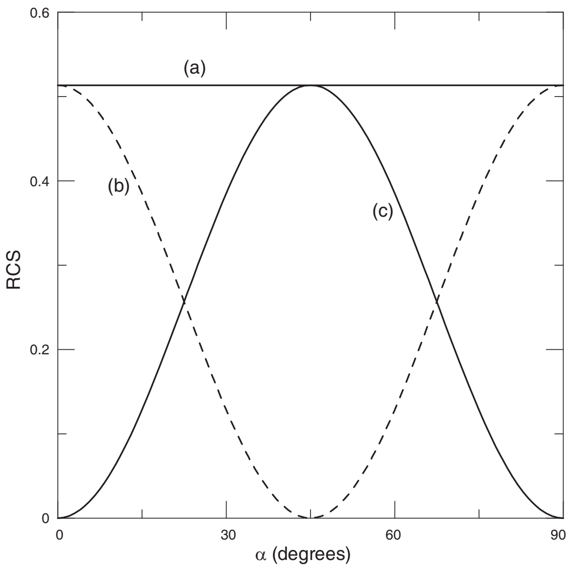

Figure 1. The RCS of a PEMC sphere of size parameter $\rho = 3$. (a) Total RCS; (b) Co-polarized contribution to the RCS; (c) Cross polarized contribution to the RCS.

We now present some numerical results which demonstrate the effect of a dielectric coating on the RCS. The RCS of an uncoated PEMC sphere of size parameter $\rho = 3$ is shown in Figure 1. This was calculated using the analytical method presented in [4], and shows that dependence of the total RCS and its co-polarized and cross polarized components on M. Instead of the parameter M, which extends over an infinite range, we use the dimensionless variable α, defined by

$$\tan\alpha = M\eta_0 \tag{32}$$

so that $\alpha = 0$ corresponds to the PMC case ($M = 0$) and $\alpha = 90^\circ$ corresponds to the PEC case ($M = \infty$). The total RCS does not depend on M, but the relative contributions of the co-polarized and cross polarized contributions depend on M. For $\alpha = 45^\circ$, i.e., $M\eta_0 = 1$, the backscattered wave is completely cross polarized, whereas for $\alpha = 0$ (PMC sphere) and $\alpha = 90^\circ$ (PEC sphere) it is completely co-polarized. Next we add a dielectric coating, such that the outer to inner size ratio is given by $b/a = 1.1$. We assume that the coating has a near zero index of refraction, as is the case for some recently produced metamaterials [12]. For such a coating a large reduction of the radar cross section may be achieved (although the total scattering

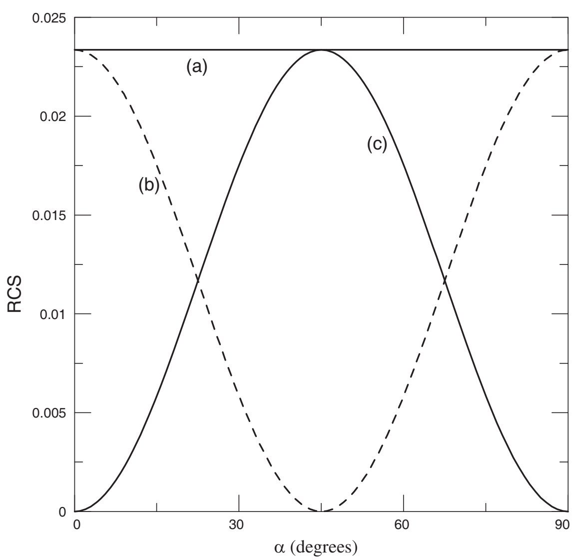

Figure 2. The RCS of a coated PEMC sphere. The core size parameter is $\rho = 3$ and the coating parameters are $b/a = 1.1$, $\varepsilon_r = \mu_r = 0.01$. (a) Total RCS; (b) Co-polarized contribution to the RCS; (c) Cross polarized contribution to the RCS.

cross section is not reduced). The cross sections calculated for an impedance matched coating with $\varepsilon_r = \mu_r = 0.01$ are shown in Figure 2. Comparing with Figure 1, we find that the addition of the coating reduces the total RCS by more than an order of magnitude. The total RCS does not depend on the value of M, whereas curves (b) and (c) exhibit a symmetry about the $\alpha = 45°$ point, for which $M\eta_0 = 1$. As in the case of the uncoated PEMC sphere (Figure 1) the co-polarized component vanishes and the cross polarized component attains its maximum at $\alpha = 45°$, i.e., for $M\eta_0 = 1$. These symmetry properties no longer hold when $\varepsilon_r \neq \mu_r$. This is demonstrated by Figure 3, which was calculated for $\varepsilon_r = 0.01$ and $\mu_r = 0.05$.

Now the total RCS varies with M, and the cross polarized component attains its maximum at $\alpha = 28°$. We next examine the dual case, with the values of ε_r and μ_r interchanged, so that $\varepsilon_r = 0.05$ and $\mu_r = 0.01$. The results for this case are shown in Figure 4. The curves of Figure 4 are reflected about the $\alpha = 45°$ line in comparison with the corresponding curves of Figure 3. From the definition of α, Eq. (32), it follows that such a reflection corresponds to replacing $M\eta_0$ by $(M\eta_0)^{-1}$.

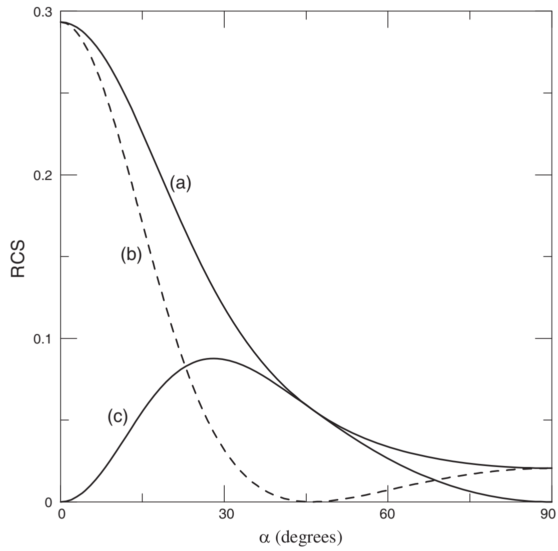

Figure 3. The RCS of a coated PEMC sphere. The core size parameter is $\rho = 3$ and the coating parameters are $b/a = 1.1$, $\varepsilon_r = 0.01$, $\mu_r = 0.05$. (a) Total RCS; (b) Co-polarized contribution to the RCS; (c) Cross polarized contribution to the RCS.

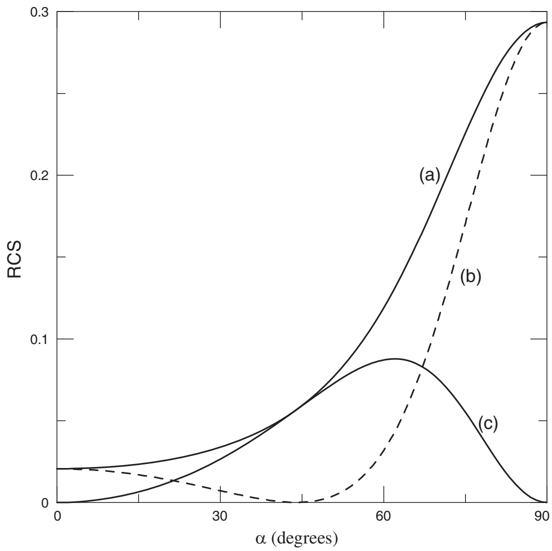

Figure 4. The RCS of a coated PEMC sphere. The core size parameter is $\rho = 3$ and the coating parameters are $b/a = 1.1$, $\varepsilon_r = 0.05$, $\mu_r = 0.01$. (a) Total RCS; (b) Co-polarized contribution to the RCS; (c) Cross polarized contribution to the RCS.

4. CONCLUSION

We have presented an analytical theory for the scattering of an electromagnetic plane wave by a coated perfect electromagnetic conductor sphere. This was achieved by extending the classical scattering theory for a coated sphere so as to allow for the appearance of cross polarized components in the scattered fields. The theory is valid for arbitrary sphere size and coating thickness. Also, the permittivity and the permeability of the coating material may attain arbitrary values, and may also be negative, as in metamaterials. The theory provides systems of linear equations of order six, from which the scattering coefficients are readily obtained numerically. Once these coefficients are known, the various scattering cross sections can be calculated. Numerical examples of cross section calculations were presented, and the symmetry properties of the solutions were pointed out. Thus, the solutions obey the following generalized electric-magnetic duality relation. The cross sections of a coated PEMC sphere with a given value of $M\eta_0$ and coating parameters $\varepsilon_r = A$, $\mu_r = B$ are equal to those of a PEMC sphere of the same size, with a coating of the same thickness, but with $M\eta_0$ replaced by $(M\eta_0)^{-1}$, and coating parameters $\varepsilon_r = B$, $\mu_r = A$.

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