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SPHERICAL RESONATOR WITH DB-BOUNDARY CONDITIONS

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Abstract—A novel set of boundary conditions requiring vanishing of the normal components of the **D** and **B** vectors at the boundary surface was introduced recently and labeled as the DB-boundary conditions. Basic properties of a resonator structure defined by the spherical DB boundary are studied in this paper. It is shown that the resonance modes polarized TE and TM with respect to the radial direction coincide with those of the respective PEC and PMC resonators. Modes in the DB resonator show higher degree of degeneracy than those of the PEC resonator which may find application in materials research.

1. INTRODUCTION

Electromagnetic field problems are generally defined by differential equations and boundary conditions. Considering a surface with unit normal vector **n**, typical boundary conditions impose two scalar restrictions for the electromagnetic field vectors tangential to the surface. For example, assuming the planar boundary $z = 0$, the PEC conditions require vanishing of the tangential components of the electric field, $E_x = E_y = 0$ while the PMC condition requires $H_x = H_y = 0$. The impedance condition [1, 2]

$$\mathbf{n} \times \left(\mathbf{E} - \overline{\overline{\mathsf{Z}}}_s \cdot (\mathbf{n} \times \mathbf{H}) \right) = 0, \qquad \mathbf{n} \cdot \overline{\overline{\mathsf{Z}}}_s = \overline{\overline{\mathsf{Z}}}_s \cdot \mathbf{n} = 0, \tag{1}$$

with the surface impedance dyadic $\overline{\overline{\mathsf{Z}}}_s$ is a generalization of both PEC and PMC.

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Another set of boundary conditions involving field components normal to the boundary surface was recently introduced [3–5] in terms of the vectors **D** and **B** as

$$\mathbf{n}\cdot\mathbf{D} = 0, \qquad \mathbf{n}\cdot\mathbf{B} = 0, \tag{2}$$

and labeled as the DB boundary conditions for brevity. The corresponding conditions for the **E** and **H** vectors depend on the medium in front of the boundary. Assuming a simple isotropic medium with permittivity ϵ and permeability μ, (2) is equivalent with the conditions

$$\mathbf{n}\cdot\mathbf{E} = 0, \qquad \mathbf{n}\cdot\mathbf{H} = 0. \tag{3}$$

The DB-boundary conditions (2) were originally arrived at as following from the conditions at the planar interface of a certain exotic material labeled as uniaxial IB (or skewon-axion) medium [5]. Another realization for the planar DB boundary can be obtained in terms of the interface of a uniaxially anisotropic medium defined by the permittivity and permeability dyadics [3–5]

$$\overline{\overline{\epsilon}} = \epsilon_z \mathbf{u}_z\mathbf{u}_z + \epsilon_t \overline{\overline{\mathsf{I}}}_t, \qquad \overline{\overline{\mu}} = \mu_z \mathbf{u}_z\mathbf{u}_z + \mu_t \overline{\overline{\mathsf{I}}}_t, \tag{4}$$

with the transverse unit dyadic defined by

$$\overline{\overline{\mathsf{I}}}_t = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y. \tag{5}$$

In fact, because of continuity of the normal components of **B** and **D**, the conditions (2) are obtained at the interface for vanishing axial parameters, $\epsilon_z \to 0$, $\mu_z \to 0$ while the transverse parameters ϵ_t, μ_t have no practical significance when the limits are attained. Such a medium has been subsequently labeled as ZAP (zero axial parameter) medium [6]. Media showing zero or almost zero parameter values have been recently under interest and their realization in terms of mixtures of metamaterials with positive and negative parameter values have been proposed [7, 8].

Plane-wave reflection from a planar DB boundary was analyzed in [5]. It was shown that the DB plane could be replaced by a PEC plane for fields polarized TE^z with respect to the normal (z) direction and, correspondingly, by PMC plane for TM^z fields. Thus, the analysis of planar DB boundaries can be reduced to that of PEC and PMC boundaries. For example, a parallel-plane waveguide with DB-boundary conditions on each plane supports modes consisting of plane waves reflecting from both planes. Splitting the modes in two groups, TE^z and TM^z with respect to the normal of both planes, the TE^z modes turn out to be the same as those corresponding to two PEC planes while the TM^z modes correspond to two PMC planes.

In this paper, we consider the resonator defined by DB conditions at a spherical boundary $r = a$ to find out whether a similar property is valid for the resonance modes polarized TE and TM with respect to the radial direction.

2. POTENTIAL REPRESENTATION OF FIELDS

It is known that any fields outside the sources in a homogeneous and isotropic medium can be decomposed in TE^r and TM^r parts with respect to the radial direction ($\mathbf{r}$) as [9]

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{\epsilon}\nabla \times (\mathbf{u}_r F(\mathbf{r})) - \frac{1}{j\omega\mu\epsilon}\nabla \times (\nabla \times (\mathbf{u}_r A(\mathbf{r}))), \tag{6}$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu}\nabla \times (\mathbf{u}_r A(\mathbf{r})) + \frac{1}{j\omega\mu\epsilon}\nabla \times (\nabla \times (\mathbf{u}_r F(\mathbf{r}))). \tag{7}$$

For $A(\mathbf{r}) = 0$ we obtain TE^r fields in terms of vector potential $\mathbf{u}_r F(\mathbf{r})$ while for $F(\mathbf{r}) = 0$ we obtain TM^r fields in terms of the vector potential $\mathbf{u}_r A(\mathbf{r})$.

Componentwise, dropping ($\mathbf{r}$) for brevity, the TE^r fields depend on the potentials as

$$E_r = 0, \tag{8}$$

$$E_\theta = -\frac{1}{\epsilon r \sin\theta}\partial_\varphi F, \tag{9}$$

$$E_\varphi = \frac{1}{\epsilon r}\partial_\theta F, \tag{10}$$

$$H_r = \frac{1}{j\omega\mu\epsilon}(\partial_r^2 + k^2)F, \tag{11}$$

$$H_\theta = \frac{1}{j\omega\mu\epsilon r}\partial_r\partial_\theta F, \tag{12}$$

$$H_\varphi = \frac{1}{j\omega\mu\epsilon r \sin\theta}\partial_r\partial_\varphi F, \tag{13}$$

while the TM^r fields are obtained in the form

$$H_r = 0, \tag{14}$$

$$H_\theta = \frac{1}{\mu r \sin\theta}\partial_\varphi A, \tag{15}$$

$$H_\varphi = -\frac{1}{\mu r}\partial_\theta A, \tag{16}$$

$$E_r = \frac{1}{j\omega\mu\epsilon}(\partial_r^2 + k^2)A, \tag{17}$$

$$E_\theta = \frac{1}{j\omega\mu\epsilon r}\partial_r\partial_\theta A, \tag{18}$$

$$E_\varphi = \frac{1}{j\omega\mu\epsilon r\sin\theta}\partial_r\partial_\varphi A. \tag{19}$$

Here $k = \omega\sqrt{\mu\epsilon}$. The radial vector potentials are known to satisfy the Helmholtz equations in the form [9]

$$(\nabla^2 + k^2)\frac{F}{r} = 0, \qquad (\nabla^2 + k^2)\frac{A}{r} = 0. \tag{20}$$

3. SPHERICAL RESONATOR

3.1. TE and TM Modes

The resonator modes satisfying (20) are obtained from potentials of the functional form

$$F_{mnp}(r,\theta,\varphi) = rj_n(kr)P_n^m(\cos\theta)\begin{pmatrix}\cos m\varphi \\ \sin m\varphi\end{pmatrix}, \tag{21}$$

$$A_{mnp}(r,\theta,\varphi) = rj_n(kr)P_n^m(\cos\theta)\begin{pmatrix}\cos m\varphi \\ \sin m\varphi\end{pmatrix}, \tag{22}$$

with $0 \le m \le n$. Here, $P_n^m(x)$ denotes the associated Legendre function [**?**]. The amplitude factors are normalized to unity. The index $p = 1, 2, 3 \ldots$ refers to the corresponding resonance wavenumber $k = k_{mnp}$. For $n = 0$, the potentials F and A become multiples of $rj_0(kr)$. From (8)–(19) we see that, in this case, all fields vanish and we can ignore that possibility.

The spherical Bessel function satisfies the differential equation

$$\partial_r\left(r^2\partial_r j_n(kr)\right) + \left[(kr)^2 - n(n+1)\right]j_n(kr) = 0, \tag{23}$$

which can also be written as

$$\left(r^2\partial_r^2 + 2r\partial_r + k^2r^2 - n(n+1)\right)j_n(kr) = 0. \tag{24}$$

Expanding

$$\left(\partial_r^2 + k^2\right)(rj_n(kr)) = \left(r\partial_r^2 + 2\partial_r + k^2 r\right)j_n(kr), \tag{25}$$

yields the relation

$$\left(\partial_r^2 + k^2\right)(rj_n(kr)) = \frac{n(n+1)}{r}j_n(kr). \tag{26}$$

Thus, the expressions (11) and (17) for the TE^r and TM^r radial field components can be rewritten as

$$H_r = \frac{n(n+1)}{j\omega\mu\epsilon r}F, \tag{27}$$

$$E_r = \frac{n(n+1)}{j\omega\mu\epsilon r}A, \tag{28}$$

for the mnp mode in question.

The DB boundary conditions (3) now require that the potentials satisfy

$$H_r(a,\theta,\varphi) = 0 \quad \Rightarrow \quad F(a,\theta,\varphi) = 0, \tag{29}$$

$$E_r(a,\theta,\varphi) = 0 \quad \Rightarrow \quad A(a,\theta,\varphi) = 0. \tag{30}$$

For the TE^r modes (30) is automatically satisfied. From (29), (9), (10) and

$$F(a,\theta,\varphi) = 0, \quad \Rightarrow \quad \partial_\theta F(a,\theta,\varphi) = 0, \quad \partial_\varphi F(a,\theta,\varphi) = 0 \tag{31}$$

we obtain

$$E_\theta(a,\theta,\varphi) = 0, \qquad E_\varphi(a,\theta,\varphi) = 0. \tag{32}$$

Because of the operator ∂_r in (12) and (13), the fields H_θ and H_φ do not vanish at the boundary. The condition (32) equals the PEC condition $\mathbf{n}\times\mathbf{E} = 0$. This leads to the conclusion that the TE^r modes in a spherical DB resonator equal those of the corresponding PEC resonator.

From the symmetry of equations we can conclude the dual case: TM^r modes in a spherical resonator equal those of the corresponding PMC resonator. Thus, the spherical DB resonator can be conceived as a kind of combination of PEC and PMC resonators.

Because the PEC or PMC boundary does not couple TE^r and TM^r modes, from the previous it follows that the DB boundary does not couple TE and TM modes. This is not obvious since the general impedance condition is known to couple TE and TM fields at the boundary surface and they cannot exist as independent modes.

3.2. Dominant Modes

The dominant mode of the spherical PEC resonator corresponding to the lowest resonance wavenumber ($ka = 2.744$) is labeled as TM$^r_{011}$ [10], but it does not exist in the DB resonator. The lowest TE^r mode in the PEC resonator is the one whose resonance wavenumber is obtained from

$$j_1(ka) = 0, \qquad ka = 4.493. \tag{33}$$

This serves also as the resonance wavenumber of the lowest TE^r mode in the DB resonator. From (21) we find three linearly independent potential functions (omitting amplitude factors)

$$F_{011}(r,\theta,\varphi) \;=\; rj_1(4.493r/a)P_1^0(\cos\theta), \tag{34}$$

$$F_{111}(r,\theta,\varphi) \;=\; rj_1(4.493r/a)P_1^1(\cos\theta)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}, \tag{35}$$

with

$$P_1^0(\cos\theta)=\cos\theta, \quad P_1^1(\cos\theta)=\sin\theta. \tag{36}$$

The TM^r modes in PMC resonators have the same wavenumbers and the potential functions (22) are

$$A_{011}(r,\theta,\varphi) \;=\; rj_1(4.493r/a)P_1^0(\cos\theta), \tag{37}$$

$$A_{111}(r,\theta,\varphi) \;=\; rj_1(4.493r/a)P_1^1(\cos\theta)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}, \tag{38}$$

which means that there is a six-fold degeneracy at the dominant resonance frequency.

The general resonance field is any sum of partial fields arising from these potential functions. Choosing the amplitude factors for the six potentials allows a lot of freedom to set conditions to the resonance field which may have application in the study of electromagnetic properties of materials. For example, with suitable excitations two resonance fields can be formed to satisfy

$$\mathbf{E}_+ = j\sqrt{\mu/\epsilon}\;\mathbf{H}_+, \quad \mathbf{E}_- = -j\sqrt{\mu/\epsilon}\;\mathbf{H}_-. \tag{39}$$

Because such fields have the form of wavefields [11], the eigenfields of a chiral medium, they can be applied in the measurement of chirality parameters of a material sample. In fact, it is known that a material sample is polarized differently in these two fields when the chirality parameter is not zero, whence the shift of the resonance frequency is different for the two eigenfields. The fields satisfying (39) are known to be self-dual, i.e., invariant in certain duality transformations [11]. It is easy to show that the DB boundary is self-dual, whence, unlike the PEC resonator, the DB resonator can be used to produce self-dual resonance fields.

4. CONCLUSION

Boundary conditions requiring vanishing of the components of **D** and **B** vectors normal to the boundary were recently introduced in [5] as arising at an interface of an exotic bi-anisotropic medium and labeled as DB-boundary conditions. As a continuation for studies concerning plane-wave reflections from the planar boundary, the spherical resonator defined by DB boundaries was analyzed in this paper. It was shown that for modes TE and TM with respect to the radial directions, the DB boundary can be replaced by respective PEC and PMC boundaries. Because all modes in a spherical resonator can be decomposed in TE^r and TM^r modes, the analysis of the DB resonator is thus reduced to that of PEC and PMC resonators. Since the DB boundary is invariant in certain duality transformations, this property can be used in defining resonance fields which cannot exist in PEC resonators and may have advantage, e.g., in materials research.

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