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RELATIVISTIC LAGUERRE POLYNOMIALS AND SPLASH PULSES

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Abstract—New solutions of the homogeneous wave equation of the type usually referred to as relatively undistorted waves are presented. Such solutions relate to the so-called "splash modes", from which indeed they can be generated by applying the Laguerre polynomial operator. Accordingly, the solutions here presented resort to the relativistic Laguerre polynomials — introduced about one decade ago within a purely mathematical context — which in fact appear as modulating factor of the basic "splash mode" waveform. Similar solutions of the homogeneous spinor wave equation are also suggested.

1. INTRODUCTION

The interest in the homogeneous scalar wave equation,

$$\left[\nabla^2 - c^{-2}\frac{\partial^2}{\partial t^2}\right] u(x, y, z, t) = 0, \tag{1}$$

is always alive. Here, $u(x, y, z, t)$ represents the scalar-valued wave field and c is the (constant) speed of propagation.

As is well known, with the characteristic variables $\tau = z - ct$ and $\sigma = z + ct$ Eq. (1) turns in the following

$$\left[\nabla_\perp^2 + 4\frac{\partial^2}{\partial\sigma\partial\tau}\right] u(x, y, \sigma, \tau) = 0, \tag{2}$$

with separable and non separable solutions (with respect to σ and τ) being looked for.

In particular, assuming the separable-variable solution

$$u(x, y, z, t) = e^{ik\sigma} w(x, y, \tau), \tag{3}$$

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one obtains for $w(x, y, \tau)$ the equation

$$\left[\nabla_{\perp}^2 + 4ik\frac{\partial}{\partial\tau}\right] w(x, y, \tau) = 0. \tag{4}$$

The latter is formally similar to the 2D paraxial wave equation, which is deduced for a monochromatic solution of (1) in the hypothesis of a slowly varying amplitude relative to the propagation variable z.

Of course, no approximations are implied in (2), whose solutions are then exact solutions of (1). In particular, one may obtain exact solutions of (1) as Hermite-Gaussian or Laguerre-Gaussian complex pulses just resorting to the well known Hermite-Gaussian or Laguerre-Gaussian solutions of the paraxial wave equation, with the obvious difference being in the presence of the variables τ and σ rather than the spatial coordinate z alone [1–7].

In this connection, we recall that a recent re-analysis of the paraxial wave equation in free space for both the rectangular and cylindrical geometry yielded a certain class of general separable-variable based solutions of it. Such solutions basically comprise a complex quadratic exponential modulated respectively by the Weber-Hermite function D_ν and the Whittaker first function $M_{\kappa,\mu}$ of suitable arguments [8–10]. Therefore, one may guess that the new results concerning the paraxial wave equation may in principle be transferred in the frame of Eq. (1), thus suggesting to introduce Weber-Hermite and Whittaker *pulses* or, following the terminology adopted in [8, 9], cartesian and cylindrical *pulses*. We may also note that, as is well known, solutions of the 2D wave equation can be used to yield solutions of the 3D wave equation. Therefore, Weber-Hermite solutions of the 1D paraxial wave equation might be used to obtain further solutions of the 3D wave equation. A primary hint in this sense can be found in [11] where, within the context of the bidirectional traveling plane wave representation of exact solutions of the wave equation, a generalization of the Gaussian pulse, involving the Weber-Hermite function of order -1 as modulating factor, was deduced.

The Hermite-Gaussian and Laguerre-Gaussian pulses are constructed from the fundamental (axially symmetric) Gaussian pulses [1–7]

$$G(x, y, z, t) = \frac{1}{\tau - iz_0} e^{ik\left(\sigma + \frac{x^2+y^2}{\tau - iz_0}\right)} \tag{5}$$

by repeated applications respectively of the Hermite and Laguerre polynomial operators [3–7]. The above can be associated with a source at the moving complex location ($x = y = 0, z = ct + iz_0$). Both k and z_0 are free parameters under the condition $kz_0 > 0$; their interplay may confer (5) a transverse plane-wave or packetlike character. Indeed, k yields the minimum frequency in the spectrum of (5), $\omega_{\min} = kc$, whereas z_0 determines the maximum frequency $\omega_{\max} = \frac{c}{z_0}$ [5–7].

The Gaussian pulse (5) is a particular example of what are usually reported as almost undistorted waves [12–16], which in general write as

$$u(x, y, z, t) = hf(\theta). \tag{6}$$

The *waveform* f is an arbitrary function of a single variable with continuous partial derivatives, whilst the *phase* function $\theta(x, y, z, t)$ and the *attenuation* (or *distortion*) factor $h(x, y, z, t)$ are fixed functions, the former obeying the characteristic equation

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 - c^{-2}\left(\frac{\partial\theta}{\partial t}\right)^2 = 0.$$

In particular, in (5) θ takes the form of the so-called Bateman-Hillion axisymmetric phase [12, 14–16]

$$\theta(x, y, z, t) = \sigma + \frac{x^2 + y^2}{\tau - iz_0}, \tag{7}$$

whilst the attenuation factor is simply $h(x, y, z, t) = \frac{1}{\tau - iz_0}$. Evidently, the choice for $f(\theta) = e^{ik\theta}$ makes (5) a separable-variable solution of (2).

A variety of solutions of (1) for both axisymmetric and non-axisymmetric phase θ has been recently suggested in [17–20]. See also [21] for a review.

As an exact non separable-variable solution of (2), we recall the "splash pulse" [5, 22–25], originally discussed in [5] as the first example of the class of finite energy solutions of the wave equation constructed from properly weighted superpositions over the free parameter k of the Gaussian pulses (5), used as basis functions. In [5], the "splash mode" was introduced as the real composite pulse

$$u_{\mathrm{sp}}(x, y, z, t) \propto u(x, y, z, t) - u(x, y, z, -t), \tag{8}$$

obtained from the difference of progressive and regressive solutions of the wave equation like

$$u(x, y, z, t) = \frac{1}{z_0 + i\tau}\frac{1}{-i\theta + a}, \tag{9}$$

with in particular $z_0 = a \equiv \gamma$. In [5] the case $\gamma = 1$ was considered in detail.

Generalizations of (9) to arbitrary exponents, i.e.,

$$u_q(x,y,z,t) = \frac{1}{z_0 + i\tau} \frac{1}{(-i\theta + a)^q}, \tag{10}$$

have been considered — and then, referred to as splash pulses as well — in [11], where, as mentioned earlier, exact solutions of the scalar wave equation were reconsidered from the viewpoint of the bidirectional representation. A review in the same vein is offered in [23]. Also, a generalization of the splash modes as solutions of the scalar wave equation to solutions of the spinor wave equation was suggested in [24]. The u_q's for $q > -1$ have been recently reconsidered in [25], where the authors investigated the behavior of the so-called double-exponential (DEX) pulses, obtained through the same superposition as in (8) of progressive and regressive u_q solutions, corresponding to different values of the parameter a. A detailed analysis of the behavior of the splash and DEX pulses for values of q within the range $-0.9 \leq q \leq 0.9$ is presented in the quoted reference.

Finally we recall that the waveform (10) enters as a modulating factor of the Gaussian pulse in the modified-power-spectrum (MPS) pulse [6, 7],

$$u_{\mathrm{MPS}}(x,y,z,t) = \frac{1}{z_0 + i\tau} \frac{1}{\left(-i\dfrac{\theta}{\beta} + a\right)^q} e^{i\frac{b}{\beta}\theta}, \tag{11}$$

from which it can be derived in fact for $b = 0$. Again z_0, a, b, q and β are free parameters, for suitable values of which the MPS pulses have the desired physical properties of localized propagation and amplitude maintenance over very large distances [6, 7].

Here, we reconsider the u_q's for $q > 0$, that we rename as "fundamental" Laguerre-Lorentzian solutions of the wave equation (1) for reasons that will become clear later. We may note indeed in the u_q's, apart from the usual attenuation factor $\frac{1}{\tau - iz_0}$, the presence of Lorentzian-like complex functions[†].

We show that "higher-order" Laguerre-Lorentzian solutions of the wave equation can be constructed by applying the same operators, through which higher-order Laguerre-Gaussian pulses are generated from (5). Such a procedure will result in producing the relativistic

[†] Although it is quite an improper terminology, by Lorentz-like (in general, complex) functions we mean here functions of the type $(A + \frac{\xi^2}{B})^{-C}$, where ξ denotes the coordinate of concern, A and B are complex constants (i.e., independent on ξ) and $C > 0$. It is evident that, when referred to a waveform, as in the present context, it will not in general yield a similar Lorentzian-like behavior (with respect to ξ) for the wave amplitude.

Laguerre polynomials (RLP), which so come to modulate the Lorentzian-like factor in the same way as the ordinary Laguerre polynomials modulate the Gaussian factor.

The RLPs have been introduced in [26] as the "radial" counterpart of the relativistic Hermite polynomials (RHP), which in turn were proposed in [27] as the polynomial component of the wave function of the quantum relativistic 1D harmonic oscillator. The latter have also been recently re-discussed within the context of the paraxial wave propagation in [28], where their relation with the Lorentz beams has been evidenced.

It is worth noting that both the RHPs and the RLPs are not independent polynomials, being indeed related to the Jacobi polynomials of appropriate parameters and arguments [29]. However, since they allow for a straight formal analogy with the Hermite-Gaussian and Laguerre-Gaussian pulses, we prefer to refer to them in accord with the original terminology.

Also, as is well known, further solutions of the wave equation can be generated from a given solution through well definite "recipes" [12, 30–32].

In Sect. 2, the basic properties of the RLPs are listed. In Sect. 3, we relate the RLPs to the solutions of the wave equation (1), which are then used to construct the solutions to Maxwell's equations in Sect. 4. Generalizations to solutions of the spinor wave equation are suggested in Sect. 5. Concluding notes are finally given in Sect. 6.

2. THE RELATIVISTIC LAGUERRE POLYNOMIALS: BASIC PROPERTIES

The RLPs $L_n^{(\alpha,N)}(x)$ have been originally worked out in the form [26]

$$L_n^{(\alpha,N)}(x) = \Gamma\left(N+n+\frac{1}{2}\right)\sum_{j=0}^{n}(-)^j\binom{n+\alpha}{n-j}\frac{1}{j!\Gamma\left(N+n-j+\frac{1}{2}\right)}\left(\frac{x}{N}\right)^j, \tag{12}$$

for non negative integer values of n and real positive values of the parameter N, which indeed is at the basis of the terminology "relativistic" used to identify the above polynomials as well as the RHPs. In fact, N was defined in [27] as the ratio of the oscillator energy mc^2 to its quantum of energy $\hbar\omega_0$: $N = \frac{mc^2}{\hbar\omega_0}$, thus signaling the "relativistic" character of the RHPs there introduced.

We see that in the limit $N \to \infty$ the polynomials (12) turn into the ordinary Laguerre polynomials $L_n^{(\alpha)}(x)$ [33]. Furthermore,

$L_n^{(\alpha,N)}(0) = 1$ and $L_0^{(\alpha,N)}(x) = 1$.

As mentioned earlier, the RLPs relate to the Jacobi polynomials, being indeed [26, 29]

$$L_n^{(\alpha,N)}(x) = (-)^n \left(1 + \frac{x}{N}\right)^n P_n^{\left(N-\frac{1}{2},\alpha\right)} \left(\frac{x-N}{x+N}\right). \tag{13}$$

As a basic characterization of the RLPs, we write down

i) the orthogonality relation (with respect to the order n), holding, as for the non relativistic polynomials, through the non-negative real axis, namely

$$\begin{aligned} &\int_0^\infty x^\alpha \left(1 + \frac{x}{N}\right)^{-N-n-m-\alpha-\frac{3}{2}} L_n^{(\alpha,N)}(x) L_m^{(\alpha,N)}(x) dx \\ &= \frac{N^2 2^{2N}}{n!} \frac{\Gamma\left(N+n+\frac{1}{2}\right)}{\left(N+2n+\alpha+\frac{1}{2}\right)\Gamma\left(N+n+\alpha+\frac{1}{2}\right)} \delta_{n,m}, \end{aligned} \tag{14}$$

ii) the differentiation formula

$$\frac{d}{dx} L_n^{(\alpha,N)}(x) = -\frac{N+n-\frac{1}{2}}{N} \left(1 + \frac{x}{N}\right)^2 L_{n-1}^{(\alpha+1,N)}(x) \tag{15}$$

iii) the contiguous relations

$$\begin{aligned} nL_n^{(\alpha,N)}(x) &= (\alpha+n)\left(1 + \frac{x}{N}\right) L_{n-1}^{(\alpha,N)}(x) \\ &\quad - \left(N+\alpha+2n-\frac{1}{2}\right) \frac{x}{N} \left(1 + \frac{x}{N}\right)^2 L_{n-1}^{(\alpha+1,N)}(x), \end{aligned}$$

$$\begin{aligned} \left(N+\alpha+2n+\frac{1}{2}\right) L_n^{(\alpha,N)} &= \left(N+\alpha+n+\frac{1}{2}\right) L_n^{(\alpha+1,N)} \\ &\quad - \left(N+n-\frac{1}{2}\right)\left(1+\frac{x}{N}\right) L_{n-1}^{(\alpha+1,N)}, \end{aligned} \tag{16}$$

iv) the differential equation they obey

$$\begin{aligned} &\left\{ x\left(1+\frac{x}{N}\right)\frac{d^2}{dx^2} - \left[\frac{2N+4n-3}{2N}x - (1+\alpha)\right]\frac{d}{dx} \right. \\ &\left. + n\frac{2N+2n-1}{2N} \right\} L_n^{(\alpha,N)}(x) = 0, \end{aligned} \tag{17}$$

v) the Rodrigues representation

$$L_n^{(\alpha,N)}(x) = \frac{1}{n!}x^{-\alpha}\left(1+\frac{x}{N}\right)^{N+2n+\alpha+\frac{1}{2}} \frac{d^n}{dx^n}\left[x^{\alpha+n}\left(1+\frac{x}{N}\right)^{-N-n-\alpha-\frac{1}{2}}\right]. \tag{18}$$

Then, we introduce the *Laguerre-Lorentzian functions* (LLFs) as

$$\Phi_{n,N}(r) = L_n^{(0,N)}\left(r^2\right)\left(1+\frac{r^2}{N}\right)^{-N-2n-\frac{1}{2}}. \tag{19}$$

By use of the Rodrigues formula (18) and the obvious identity: $r^2 = (x+iy)(x-iy)$, it is easily proved that the $\Phi_{n,N}$'s are generated from the Lorentz-like function‡

$$\Phi_{0,N}(r) = \left(1+\frac{r^2}{N}\right)^{-N-\frac{1}{2}} \tag{20}$$

by repeated application of the transverse Laplacian operator

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial(x+iy)}\frac{\partial}{\partial(x-iy)}, \tag{21}$$

which when acting on an axisymmetric function, as in the case we are considering here, is equivalent to its radial part

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \tag{22}$$

We explicitly have in fact

$$\Phi_{n,N}(r) = (-)^n \frac{1}{2^{2n} n!} \frac{N^n}{\left(N+\frac{1}{2}\right)_n} \left[\nabla_r^2\right]^n \Phi_{0,N}(r); \tag{23}$$

the action of $[\nabla_r^2]^n$ on (20) produces the RLP of order n and correspondingly increases by $2n$ the exponent of the Lorentz-like factor. In the limit $N \to \infty$ the above reproduces the well known analogous

‡ Of course, addressing $\Phi_{0,N}$ as Lorentz function would be correct only for $N = \frac{1}{2}$. However, as previously noted, here we address as Lorentz–like function any function of the type of $\Phi_{0,N}$ for any real negative exponent.

relation for the Laguerre-Gaussian functions (LGF) of azimuthal order 0, namely

$$\Phi_{n,N\to\infty}(r) \equiv \Phi_n(r) = (-)^n \frac{1}{2^{2n}n!}\left[\nabla_r^2\right]^n e^{-r^2} = L_n\left(r^2\right)e^{-r^2}. \quad (24)$$

Evidently the above follows also from (19) for $N \to \infty$.

The general generation scheme involving as well the azimuthal parameter α will be considered in Sect. 5.

In order to see the difference between the LLFs and the

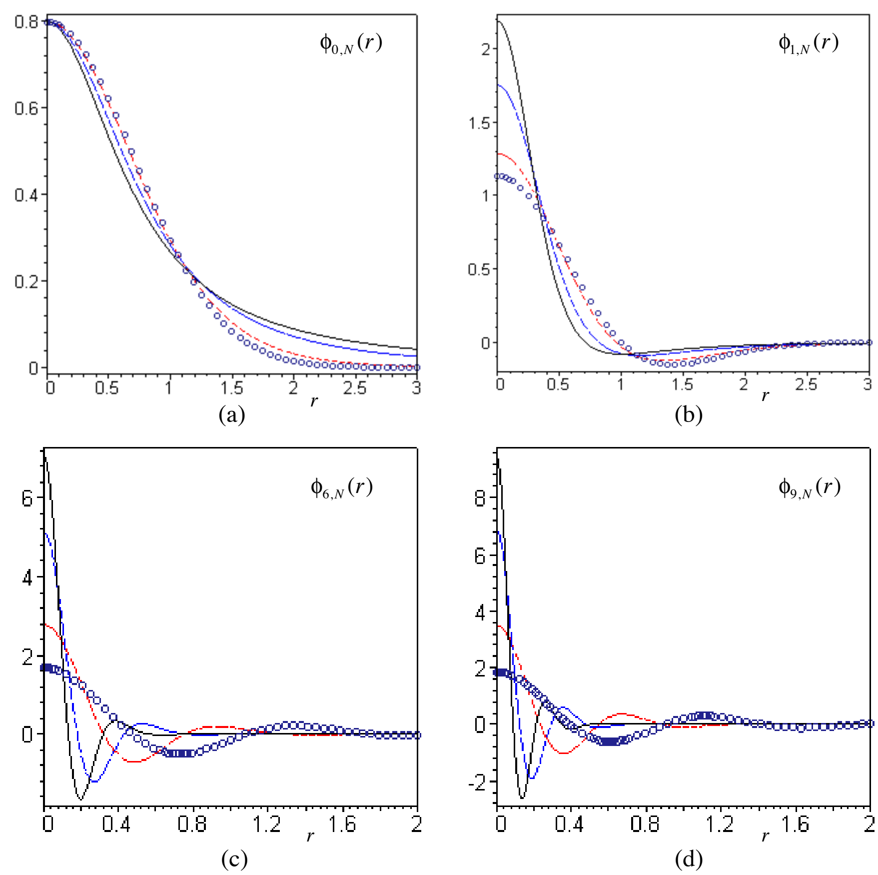

Figure 1. r-profiles of the LLFs $\varphi_{n,N}$'s for $n = 0, 1, 6, 9$ and $N = 0.5$ (solid line), $N = 1$ (dashed line), $N = 5$ (dash-dotted line). The profiles of the corresponding LGFs φ_n for each value of n are also plotted, marked by the o's.

corresponding LGFs, the correspondence being intended in the sense of the limit (24), we have plotted in Fig. 1 both functions for some values of the inherent parameters. More precisely, the plots reproduce the normalized functions, which write

$$\begin{aligned}\phi_{n,N}(r) &= 2^{n+\frac{1}{2}} n! \sqrt{\frac{\Gamma(N+2n+1)\,\Gamma\left(N+n+\frac{1}{2}\right)}{\pi N\,(2n)!\,\Gamma\left(N+2n+\frac{1}{2}\right)\Gamma(N+n)}}\,\Phi_{n,N}(r),\\ \phi_n(r) &= \frac{2^{n+\frac{1}{2}} n!}{\sqrt{\pi\,(2n)!}} L_n\left(r^2\right) e^{-r^2}.\end{aligned} \tag{25}$$

As to the former, we note that the relevant normalization factor can be evaluated resorting to the Parseval theorem and to the relation (23), which then allow us to write

$$\int_0^\infty r\,|\Phi_{n,N}(r)|^2\,dr = A_{n,N}^2 \int_0^\infty \kappa^{2n+1} \left|\widetilde{\Phi}_{0,N}(\kappa)\right|^2 d\kappa, \tag{26}$$

with $A_{n,N} = \frac{1}{2^{2n} n!} \frac{N^n}{(N+\frac{1}{2})_n}$. Here, $\widetilde{\Phi}_{0,N}(\kappa)$ denotes the Hankel transform of zero order of $\Phi_{0,N}(r)$, which on account of the integral (6.565.4) of [34] evaluates to

$$\begin{aligned}\widetilde{\Phi}_{0,N}(\kappa) &= \int_0^\infty r J_o(\kappa r)\Phi_{0,N}(r)dr\\ &= \frac{N^{N+\frac{1}{2}}}{\Gamma\left(N+\frac{1}{2}\right)}\left(\frac{\kappa}{2\sqrt{N}}\right)^{N-\frac{1}{2}} K_{\frac{1}{2}-N}\left(\sqrt{N}\kappa\right),\end{aligned} \tag{27}$$

$K_\nu(\cdot)$ denoting the modified Bessel function of the second kind [33].

Finally, the $\Phi_{n,N}$'s obey the differential equation

$$\left\{\left(1+\frac{r^2}{N}\right)\frac{d^2}{dr^2}+\left[1+\frac{2r^2}{N}(N+2n+2)\right]\frac{1}{r}\frac{d}{dr} + \frac{4(n+1)\left(N+n+\frac{1}{2}\right)}{N}\right\}\Phi_{n,N}(r)=0, \tag{28}$$

and on account of (23) and (27) can be given the integral representation

$$\Phi_{n,N}(r) = \frac{4}{n!\Gamma\left(N+n+\frac{1}{2}\right)}\left(\frac{\sqrt{N}}{2}\right)^{N+2n+\frac{3}{2}} \int_0^\infty \kappa^{N+2n+\frac{1}{2}} K_{\frac{1}{2}-N}\left(\sqrt{N}\kappa\right) J_o(\kappa r) d\kappa. \tag{29}$$

3. LAGUERRE-LORENTZIAN SOLUTIONS OF THE WAVE EQUATION

As mentioned earlier, it was proved in [5, 11] that Eq. (2) is solved by the axially symmetric functions u_q, that we recast as

$$u_{0,N}(r,z,t) = \frac{1}{\tau - iz_0}\left(\sigma + ia + \frac{r^2}{\tau - iz_0}\right)^{-N-\frac{1}{2}} \tag{30}$$

with r denoting the radial coordinate $r = \sqrt{x^2+y^2}$. The various parameters are chosen so that $N > 0$, $z_0 > 0$, and $a > 0$, the latter being aimed at avoiding singularities at $r = 0$ and similarly at ($z = 0$, $t = 0$). We refer to it as the "fundamental" Laguerre-Lorentzian solution of the wave equation. Needless to say, due to the symmetric appearance of σ and τ in the wave equation, the alternative solution formally similar to (30) with a simple interchange of σ and τ is also possible.

Although (30) has already been considered in the literature (some of the investigations presented in [6, 7] and [25], in fact, can be made to correspond to $0 < N \leq \frac{1}{2}$), for completeness' sake we briefly comment on some features of its. Thus, as an example, we show in Fig. 2 the surface plots and the corresponding contour plots of the amplitude of the Lorentzian-like solution (30) for $N = \frac{1}{10}, \frac{1}{2}, \frac{5}{2}, 5$ and $z_0 = 10^{-2}$ cm and $a = 2 \cdot 10^4$ cm. The graphs refer to the pulse center $z_c = ct = 0$, so that $z = \tau$ is just the distance along the direction of propagation away from the pulse center; in addition, the maximum in each plot is normalized to unity.

It is evident that the localization of $u_{0,N}$ along the longitudinal and radial directions is controlled by the length parameters z_0 and a, with the influence of the latter being increased by the possible greater than 1 power $N + \frac{1}{2}$ for $N > \frac{1}{2}$.

In particular, let us illustrate the behavior of $u_{0,N}$ at the pulse center $z = z_c = ct$. We see that the squared amplitude $|u_{0,N}|^2$ at

$z = z_c$ behaves with r and z as

$$|u_{0,N}(r, z = ct)|^2 = \frac{1}{4^{N+\frac{1}{2}} z_0^2} \frac{1}{\left[z^2 + \frac{a^2}{4}\left(1 + \frac{r^2}{a z_0}\right)^2\right]^{N+\frac{1}{2}}}, \qquad (31)$$

which conveys $\frac{a}{2}$ and $\sqrt{a z_0}$ as a sort of characteristic lengths for the variations of $u_{0,N}$ (at $z = z_c$) along the longitudinal and radial directions, respectively. Until $z \lesssim \frac{a}{2\sqrt{N+\frac{1}{2}}}$ the pulse shape does not vary much with z, being $|u_{0,N}(r, z = ct \lesssim \frac{a}{2\sqrt{N+\frac{1}{2}}})|^2 \sim$

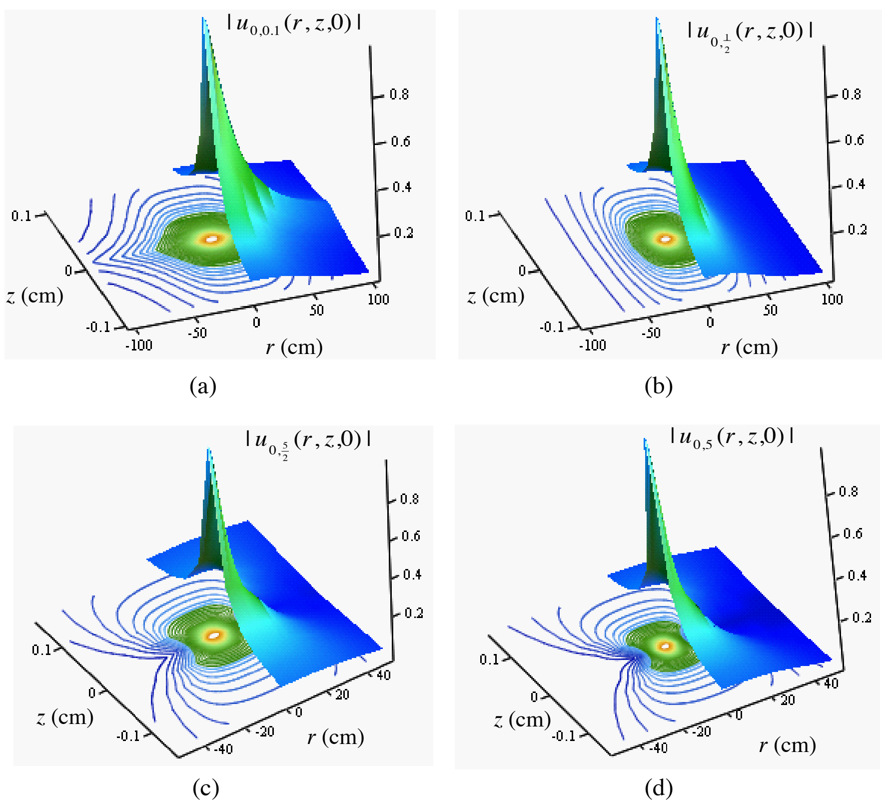

Figure 2. Amplitude $|u_{0,N}|$ vs. r and τ at the pulse center $z_c = ct = 0$ for $z_0 = 10^{-2}$ cm, $a = 2{\cdot}10^4$ cm, and (a) $N = \frac{1}{10}$, (b) $N = \frac{1}{2}$, (c) $N = \frac{5}{2}$, (d) $N = 5$.

$\frac{1}{z_0^2 a^{2N+1}} \frac{1}{(1+\frac{r^2}{az_0})^{2N+1}}$. And correspondingly until $r \lesssim \sqrt{\frac{az_0}{2N+1}}$ the relevant amplitude decreases with r roughly within a factor 2. Then, for $z > \frac{a}{2\sqrt{N+\frac{1}{2}}}$ the amplitude (at $z = z_c$) decays like $z^{-(N+\frac{1}{2})}$. In Fig. 3, we show the amplitudes $|u_{0,\frac{1}{2}}(r, z = ct)|$ and $|u_{0,5}(r, z = ct)|$ vs. r and z for the values $z_0 = 10^{-2}$ cm and $a = 2 \cdot 10^4$ cm.

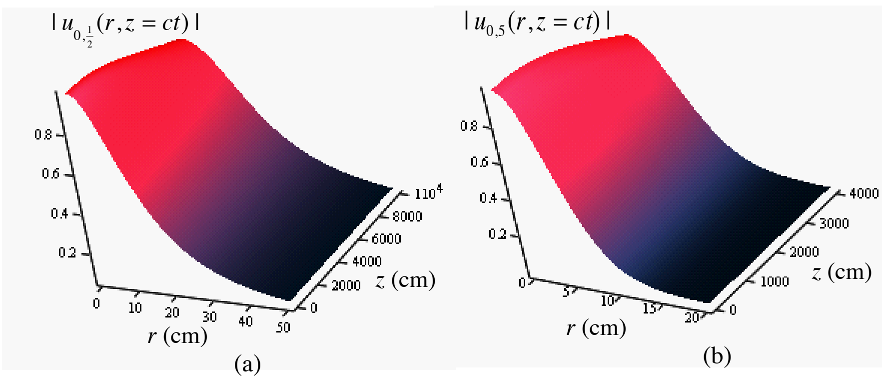

Figure 3. Amplitudes (a) $|u_{0,\frac{1}{2}}(r, z = ct)|$ and (b) $|u_{0,5}(r, z = ct)|$ vs. r and z for the values $z_0 = 10^{-2}$ cm and $a = 2 \cdot 10^4$ cm.

As illustrated in [11], the bidirectional traveling plane wave representation grounds on definite direct and inverse formulas, which in the case of (30) specifically write

$$u_{0,N}(r,\sigma,\tau) = \frac{1}{2\pi^2}\int_0^\infty \chi d\chi \int_0^\infty dv \frac{1}{v} G_{0,N} \left(\frac{\chi^2}{4v}, v, \chi\right) J_0(\chi r) e^{-i\frac{\chi^2\tau}{4v}} e^{iv\sigma}$$

$$G_{0,N}\left(\frac{\chi^2}{4v}, v, \chi\right) = \frac{\sqrt{\pi}}{2}\int_{-\infty}^\infty d\tau \int_{-\infty}^\infty d\sigma \int_0^\infty r dr u_{0,N} (r,\sigma,\tau)\, J_0\left(\chi r\right) e^{-\frac{\tau^2}{16v^2}} e^{i\frac{\chi^2\tau}{4v}} e^{-iv\sigma},$$

thus yielding

$$G_{0,N}\left(\frac{\chi^2}{4v}, v, \chi\right) = \frac{2\pi^2}{i^{N-\frac{1}{2}}\Gamma\left(N+\frac{1}{2}\right)} v^{N-\frac{1}{2}} e^{-av-\frac{\chi^2}{4v}z_o}. \tag{32}$$

It is well known that “higher-order” solutions of the wave equation can be generated from a given solution (the “fundamental” one) by applying to the latter the derivative operator — as well as any function of it —

$$D^{(m)} = \frac{\partial^p}{\partial x^p}\frac{\partial^h}{\partial y^h}\frac{\partial^j}{\partial \sigma^j}\frac{\partial^l}{\partial \tau^l}, \quad m = p + h + j + l$$

for any nonnegative integers (actually also nonnegative real), just because the differential operator in Eq. (2) commutes with $D^{(m)}$.

In particular, the differential operator in Eq. (2) commutes with the transverse Laplacian $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Therefore, if $v(r, \sigma, \tau)$ solves (2), so does also $v_\nu(r, \sigma, \tau) = [\nabla^2]^\nu v(r, \sigma, \tau) = [\nabla_r^2]^\nu v(r, \sigma, \tau)$ for any non-negative real value of ν, the latter identity holding in the case that v is axisymmetric.

Accordingly, from (30) we may generate further axisymmetric solutions of (2) as

$$u_{n,N}(r, z, t) = \frac{1}{(\tau - iz_0)^{n+1}(\sigma + ia)^{N+n+\frac{1}{2}}}\Phi_{n,N}\left(\sqrt{R_N(r, z, t)}\right) \quad (33)$$

where $\Phi_{n,N}(\cdot)$ denotes the LLFs discussed in the previous section, whose argument is

$$R_N(r, z, t) = \frac{Nr^2}{(\sigma + ia)(\tau - iz_0)}. \quad (34)$$

One may refer to the above as *Laguerre-Lorentzian* solutions of order n (and parameter N) of the homogeneous-wave equation (1).

It is worth noting that we consider here only non-negative integer values of n. However, any non-negative real value is allowed, resorting in that case to the expression for the relativistic Laguerre polynomials in terms of the proper Gauss hypergeometric function, deducible from (13). We might talk of *fractional order* Laguerre-Lorentzian solutions of the wave equation. Such a case is beyond the purposes of the present discussion.

On passing let us say that the 2D cartesian counterparts of (33) would involve the RHPs and so would solve the 2D wave equation $\phi_{qq} + \phi_{zz} - c^{-2}\phi_{tt} = 0$. Then, recalling that one can obtain solutions $u(x, y, z, t)$ of the 3D wave equation from those $\phi(q, z, t)$ of the 2D equation, for instance, as $u(x, y, z, t) = (x \pm iy)^{-\frac{1}{2}}\phi(q, z, t)$ [12, 30–32], the relevance of the RHPs for the 3D scalar wave equation as well becomes evident.

The behavior of $u_{n,N}(r, z, t)$ results from the synergistic contribution of the multi-Lorentzian-like factor $\frac{1}{\tau - iz_0}(\sigma + ia + \frac{r^2}{\tau - iz_0})^{-N-2n-\frac{1}{2}} = (\sigma + ia + \frac{r^2}{\tau - iz_0})^{-2n} u_{0,N}(r, z, t)$ and the r-depending polynomial component comprising also the σ- and τ-depending factor, i.e., $\frac{(\sigma+ia)^n}{(\tau-iz_0)^n} L_n^{(0,N)}(R_N(r))$. Clearly the former behaves as previously discussed, with the relative characteristic lengths being properly scaled to account for the further presence of the integer $2n$ in the exponent.

We may see again that $\frac{a}{2}$ and $\sqrt{az_0}$ play the role of characteristic lenghts for the variations of $u_{n,N}$ as well along the longitudinal and radial directions, respectively.

In fact, it is evident that the argument $R_N(r,z,t)$ of the RLP in (33) is in general complex, and hence the behavior of the polynomial component in (33) may significantly differ from that of the same polynomial with real argument. In particular, R_N becomes real at $z = ct = 0$, being $R_N(r, z = ct = 0) = \frac{Nr^2}{az_0}$. Also, at the pulse center $z = ct$ it turns out to be

$$R_N(r, z = ct) = \frac{iNr^2}{2z_0\left(z + i\dfrac{a}{2}\right)}. \tag{35}$$

Then, if $z \lesssim \frac{a}{2\sqrt{N+2n+\frac{1}{2}}}$ the above comes to be rather well approximated by the real z-independent expression

$$R_N\left(r, z = ct \lesssim \frac{a}{2\sqrt{N+2n+\dfrac{1}{2}}}\right) \simeq \frac{Nr^2}{z_0 a}. \tag{36}$$

Likewise, the multiplying factor remains almost constant to $\frac{i^n(2z+ia)^n}{{z_0}^n} \sim (-\frac{a}{z_0})^n$. Further, until $r \lesssim \sqrt{\frac{az_0}{N+2n+\frac{1}{2}}}$, $R_N < 1$, so that one may approximate the polynomial factor by the relevant zero-order power, thus yielding for the squared amplitude the expression

$$\left|u_{n,N}\left(r \lesssim \sqrt{\frac{az_0}{N+2n+\dfrac{1}{2}}}, z = ct \lesssim \frac{a}{2\sqrt{N+2n+\dfrac{1}{2}}}\right)\right|^2$$
$$\sim \frac{1}{a^{2N+2n+1} z_0^{2(n+1)}} \frac{1}{\left(1 + \dfrac{r^2}{az_0}\right)^{2N+4n+1}}. \tag{37}$$

In contrast, for $r > \sqrt{Naz_0}$ the polynomials might be approximated by the relevant highest powers thus yielding the expression

$$\left|u_{n,N}\left(r>\sqrt{Naz_0}, z=ct\lesssim\frac{a}{2\sqrt{N+2n+\frac{1}{2}}}\right)\right|^2$$
$$\sim\left[\frac{\left(N+\frac{1}{2}\right)_n}{n!}\right]^2\frac{1}{a^{2N+4n+1}z_0^{2(n+1)}}\frac{r^{4n}}{\left(1+\frac{r^2}{az_0}\right)^{2N+4n+1}}, \quad (38)$$

where the power r^{4n} mitigates the descending trend of the multi-Lorentzian factor $(1+\frac{r^2}{az_0})^{-2N-4n-1}$.

The 3D plots in Fig. 4 visually convey the above considerations, showing the amplitudes $|u_{n,N}|$ vs. r and τ at the pulse center $z_c=0$ and vs. r and $z=z_c$ for $n=2$, $N=5$ and $n=5$, $N=2.5$. In both cases, $z_0=10^{-2}$ cm and $a=2\cdot 10^4$ cm. Again, the maximum in each plot is normalized to unity.

Also, a certain insight into the decay rate of the peak amplitudes of the $u_{n,N}$'s vs. z as a result of the interplay of N, n and the parameters z_0 and a can be gained from Fig. 5. For comparison's purposes, the values in the graphs to the right have been properly scaled in order to have the same vertical ranges as those in the corresponding graphs to the left. We see that the peak amplitude starts to decay roughly at $z\gtrsim\frac{a}{2\sqrt{N+2n+\frac{1}{2}}}$.

Finally, since $\nabla_r^2 J_0(\chi r) = -\chi^2 J_0(\chi r)$, the bidirectional representation of $u_{n,N}$ simply writes

$$G_{n,N}\left(\frac{\chi^2}{4v}, v, \chi\right)=(-)^n\chi^{2n}G_{0,N}\left(\frac{\chi^2}{4v}, v, \chi\right). \quad (39)$$

We conclude noting that one could also refer to the $u_{n,N}$'s as higher-order splash pulses, taking into account however that for the relevant generation scheme (33) to be applicable the exponent q in the pertinent expression (10) must be $q>0.5$.

4. LAGUERRE-LORENTZIAN SOLUTIONS OF MAXWELL'S EQUATIONS

As is well known, solutions to the scalar wave equation convey solutions to Maxwell's equations. A well-established procedure resorts to the

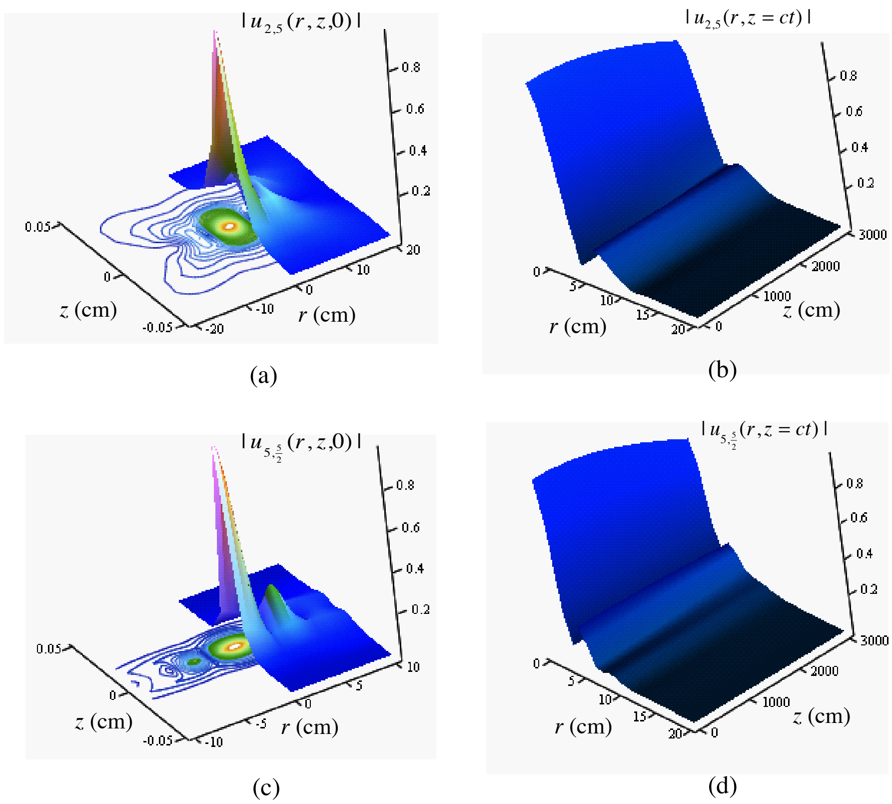

Figure 4. Amplitudes $|u_{n,N}|$ vs. r and τ at the pulse center $z_c = 0$ and vs. r and $z = z_c$ respectively in (a) and (b) for $n = 2$, $N = 5$ and in (c) and (d) for $n = 5$, $N = 2.5$. In both cases, $z_0 = 10^{-2}$ cm and $a = 2 \cdot 10^4$ cm.

Hertz electric and magnetic vector potentials $\overrightarrow{\Pi}_e$ and $\overrightarrow{\Pi}_m$ [6, 7, 35, 36], which over source-free regions obey the scalar wave equation (1), and so

$$\left[\nabla^2 - c^{-2}\frac{\partial^2}{\partial t^2}\right] \overrightarrow{\Pi}_{e,m}(x, y, z, t) = 0. \tag{40}$$

Definite relations hold between such potentials and the electromagnetic fields $\overrightarrow{E}(x, y, z, t)$ and $\overrightarrow{H}(x, y, z, t)$ [6, 7, 35, 36]. In particular, TE and TM modes are identified by the z-component respectively of $\overrightarrow{\Pi}_m$ and $\overrightarrow{\Pi}_e$, being indeed $\overrightarrow{\Pi}_e = 0$ and $\overrightarrow{\Pi}_m = \widehat{z}\, \Pi_m$ for the former and correspondingly $\overrightarrow{\Pi}_m = 0$ and $\overrightarrow{\Pi}_e = \widehat{z}\Pi_e$ for the latter. As already

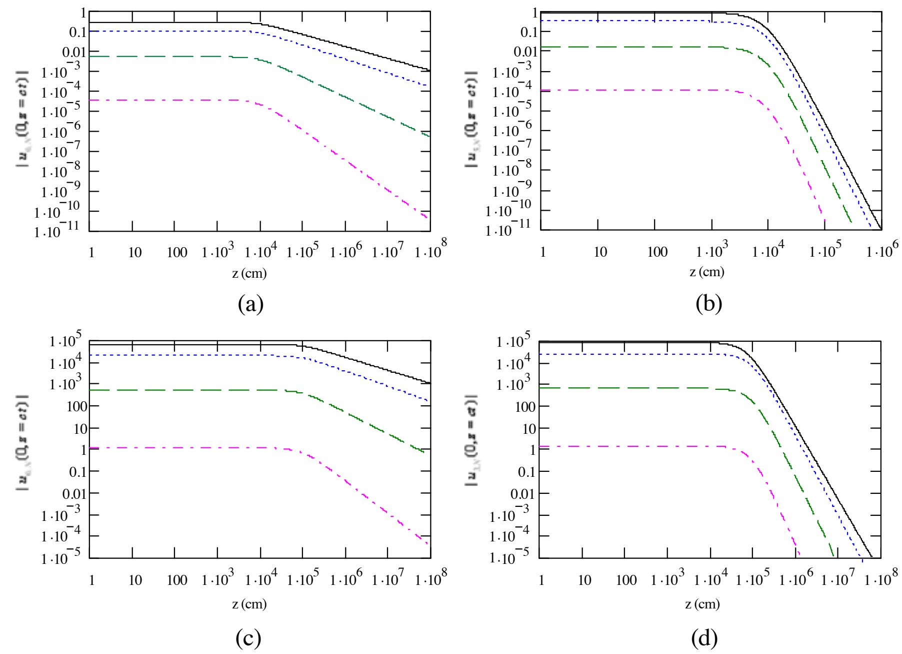

Figure 5. Peak amplitudes $|u_{n,N}(0, z = ct)|$ vs. z for $N = 0.1$ (solid line), 0.2 (dotted line), 0.5 (dashed line), 1 (dash-dotted line) and (a) $n = 0$, (b) $n = 5$ with $z_0 = 10^{-2}$ cm, $a = 2 \cdot 10^4$ cm and (c) $n = 0$, (d) $n = 3$ with $z_0 = 10^{-2}$ cm, $a = 2 \cdot 10^5$ cm.

noted, the z-axis is assumed as direction of propagation.

In the specific case of axial symmetry we are dealing with, according to which $\overrightarrow{\Pi}_{e,m}(x, y, z, t) = \overrightarrow{\Pi}_{e,m}(r, z, t)$, it turns out that the TE modes have only the field components E_φ, H_r and H_z (and, hence are azimuthally polarized, the polarization being defined in terms of the $\overrightarrow{E}$ field), whereas the TM modes have only the components E_r, E_z and H_φ (and, hence are radially polarized).

In symbols, denoting by $\widehat{r} = (\cos\varphi, \sin\varphi)$ and $\widehat{\varphi} = (-\sin\varphi, \cos\varphi)$ the unit vectors for polar coordinates, one finds that

$$\begin{aligned}\overrightarrow{E}_{TE}(x, y, z, t) &= \widehat{\varphi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{\partial}{\partial\sigma} - \frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial r}\Pi_m(r, z, t)\\ \overrightarrow{H}_{TE}(x, y, z, t) &= \widehat{r}\left(\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial r}\Pi_m(r, z, t) + \widehat{z}\,4\frac{\partial^2}{\partial\sigma\partial\tau}\Pi_m(r, z, t),\end{aligned} \tag{41}$$

where ε_0 and μ_0 respectively denote the free-space permittivity and permeability, and in conformity with the use here of the characteristic variables σ and τ the derivatives with respect to z and t have been expressed in terms of the derivatives with respect to σ and τ. Furthermore, on account of (40), it is evident that in the second of (41) $4\frac{\partial^2}{\partial\sigma\partial\tau}\Pi_m(r,z,t) = \nabla_r^2\Pi_m(r,z,t)$.

Dual expressions hold for the field components of the TM modes, for which in fact we have

$$\begin{aligned}\overrightarrow{E}_{TM}(x,y,z,t) &= \widehat{r}\left(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial r}\Pi_e(r,z,t) + \widehat{z}\,4\frac{\partial^2}{\partial\sigma\partial\tau}\Pi_e(r,z,t)\\ \overrightarrow{H}_{TM}(x,y,z,t) &= -\widehat{\varphi}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\frac{\partial}{\partial\sigma}-\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial r}\Pi_e(r,z,t).\end{aligned} \tag{42}$$

According to the above considerations, the Laguerre-Lorentzian solutions of the scalar wave equation deduced in the previous section, just convey the Hertz potential functions $\Pi_{e,m}(r,z,t)$, from which then the electromagnetic field components straightforwardly follow.

Thus, with $\Pi(r,z,t) = u_{n,N}(r,z,t)$, after some algebra we obtain for the TE mode field components the explicit expressions

$$\begin{aligned}E_\varphi(r,z,t) &= -2\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{(\sigma+ia)^{n+1}}{(\tau-iz_0)^{n+2}}\left(\sigma+ia+\frac{r^2}{\tau-iz_0}\right)^{-N-2n-\frac{5}{2}} r\\ &\quad\left\{\left(N+n+\frac{1}{2}\right)\left(1+\frac{R_N}{N}\right)^3\left[\Sigma_+-\Upsilon_+\right]L_{n-1}^{(1,N)}(R_N)\right.\\ &\quad\left.+\left[2S_-+n\Sigma_+-\left(N+n-\frac{1}{2}\right)\Upsilon_+\right]L_n^{(0,N)}(R_N)\right\},\\ H_r(r,z,t) &= 2r\frac{(\sigma+ia)^{n+1}}{(\tau-iz_0)^{n+2}}\left(\sigma+ia+\frac{r^2}{\tau-iz_0}\right)^{-N-2n-\frac{5}{2}}\\ &\quad\left\{\left(N+n+\frac{1}{2}\right)\left(1+\frac{R_N}{N}\right)^3[\Sigma_-+\Upsilon_-]L_{n-1}^{(1,N)}(R_N)\right.\\ &\quad\left.+\left[2S_++n\Sigma_-+\left(N+n-\frac{1}{2}\right)\Upsilon_-\right]L_n^{(0,N)}(R_N)\right\},\\ H_z(r,z,t) &= -4(n+1)\left(N+n+\frac{1}{2}\right)u_{n+1,N}(r,z,t),\end{aligned}$$

where R_N is defined in (34) and

$$\begin{aligned} S_{\pm}(\sigma,\tau) &= (n+1)\left(N+n+\frac{1}{2}\right)\left(\frac{1}{\tau-iz_0}\pm\frac{1}{\sigma+ia}\right), \\ \Sigma_{\pm}(r,\sigma,\tau) &= \frac{(n+1)}{\tau-iz_0}\left(1\pm\frac{r^2}{(\sigma+ia)^2}\right), \\ \Upsilon_{\pm}(r,\sigma,\tau) &= \frac{\left(N+n+\frac{1}{2}\right)}{\sigma+ia}\left(1\pm\frac{r^2}{(\tau-iz_0)^2}\right). \end{aligned}$$

Similar expressions are obtainable for the TM mode field components.

We see that, as expected on account of (40), (41) and (42) , the $u_{n,N}$'s directly convey one of the field components for both TE and TM modes, H_z and E_z respectively.

5. LAGUERRE-LORENTZIAN SOLUTIONS OF THE SPINOR WAVE EQUATION

Frequently, solutions of the wave equation are generalized to yield solutions of the spinor wave equation [24, 37, 38]. In [24, 37], for instance, spinor focus wave modes are presented in full analogy with the focus wave mode solutions of the wave equation, in the latter being also presented an extension of the Ziolkowski method of weighted superposition of Gaussian pulses to the realm of spinors.

Likewise, we may generalize the Laguerre-Lorentzian solutions of the wave equation, deduced in the previous section, to Laguerre-Lorentzian solutions of the spinor wave equation.

Let us write down the spinor wave equation in the coordinates r, φ, σ and τ:

$$2\frac{\partial}{\partial\sigma}\psi_1+\widehat{\mathcal{L}}_-(r,\varphi)\psi_2=0,\quad \widehat{\mathcal{L}}_+(r,\varphi)\psi_1-2\frac{\partial}{\partial\tau}\psi_2=0. \tag{43}$$

Here, r, φ denote polar coordinates in the x-y plane: $r=\sqrt{x^2+y^2}$, $\varphi=\arctan(\frac{y}{x})$, and ψ_1, ψ_2 are the two components of the spinor field. Furthermore, the operators $\widehat{\mathcal{L}}_{\pm}(r,\varphi)$ explicitly write

$$\widehat{\mathcal{L}}_{\pm}(r,\varphi)=\left(\frac{\partial}{\partial x}\pm i\frac{\partial}{\partial y}\right)=e^{\pm i\varphi}\left(\frac{\partial}{\partial r}\pm\frac{i}{r}\frac{\partial}{\partial\varphi}\right)=2\frac{\partial}{\partial(x\mp iy)}. \tag{44}$$

Equation (43) admit the solutions

$$\begin{aligned}\psi_1^{(N)} &= -e^{-i\varphi}\frac{r}{\tau - iz_0}\psi_2^{(N)}\\ \psi_2^{(N)} &= \frac{1}{\tau - iz_0}\left(\sigma + ia + \frac{r^2}{\tau - iz_0}\right)^{-N-\frac{1}{2}}\end{aligned} \tag{45}$$

where, as before, the various parameters are chosen so that $N > 0$, $z_0 > 0$, and $a > 0$. It is evident that $\psi_2^{(N)}$ just equals the "fundamental" Laguerre-Lorentzian solution (30) of the wave equation.

Then, from (45) we can generate higher-order solutions by applying to (45) the operators $\widehat{\mathcal{L}}_\pm$ in an arbitrary fashion, since $\left[\widehat{\mathcal{L}}_+, \widehat{\mathcal{L}}_-\right] = 0$ and $\left[\frac{\partial}{\partial\sigma,\tau}, \widehat{\mathcal{L}}_\pm\right] = 0$.

Accordingly, besides

$$\underline{\Psi}_{0,N} = \begin{pmatrix}\psi_1^{(N)}\\ \psi_2^{(N)}\end{pmatrix},$$

also

$$\underline{\Psi}_{n,l,N} \propto \widehat{\mathcal{L}}_+^n \widehat{\mathcal{L}}_-^{n+l}\underline{\Psi}_{0,N} = \begin{pmatrix}\widehat{\mathcal{L}}_+^n \widehat{\mathcal{L}}_-^{n+l}\psi_1^{(N)}\\ \widehat{\mathcal{L}}_+^n \widehat{\mathcal{L}}_-^{n+l}\psi_2^{(N)}\end{pmatrix} \equiv \begin{pmatrix}\psi_1^{(n,l,N)}\\ \psi_2^{(n,l,N)}\end{pmatrix}$$

turns out to be a solution of the spinor wave equation, with $\psi_1^{(N)}$ and $\psi_2^{(N)}$ given in (45) and n, l nonnegative integers such that $n \geq 0$ and $l \geq -n$.

In this connection, we note that the relation (23) can be generalized to include the angular index α of the RLPs there involved. In fact, let us introduce the Laguerre-Lorentzian functions of radial index n and angular index l,

$$\Phi_{n,N}^{(l)}(r,\varphi) = e^{-il\varphi} r^l L_n^{(l,N)}\left(r^2\right)\left(1 + \frac{r^2}{N}\right)^{-N-2n-l-\frac{1}{2}}, \tag{46}$$

which just parallels the definition of the Laguerre-Gaussian modes of order (n, l). Then, by use again of the Rodrigues representation (18), we may easily verify that

$$\Phi_{n,N}^{(l)}(r,\varphi) = (-)^{n+l}\frac{1}{2^{2n+l}n!}\frac{N^{n+l}}{\left(N + \frac{1}{2}\right)_{n+l}}\widehat{\mathcal{L}}_+^n \widehat{\mathcal{L}}_-^{n+l}\Phi_{0,N}(r). \tag{47}$$

In particular, when $l = 0$ we recover relation (23) since $\widehat{\mathcal{L}}_+^n \widehat{\mathcal{L}}_-^n \Phi_{0,N}(r) = [\nabla_\perp^2]^n \Phi_{0,N}(r) = [\nabla_r^2]^n \Phi_{0,N}(r)$.

Therefore, on account of

$$\begin{aligned} \widehat{\mathcal{L}}_-^j r e^{-i\varphi} f(r,\varphi) &= r e^{-i\varphi} \widehat{\mathcal{L}}_-^j f(r,\varphi), \\ \widehat{\mathcal{L}}_+^j r e^{-i\varphi} f(r,\varphi) &= 2j \widehat{\mathcal{L}}_+^{j-1} f(r,\varphi) + r e^{-i\varphi} \widehat{\mathcal{L}}_+^j f(r,\varphi), \end{aligned} \tag{48}$$

after some algebra we end up with

$$\begin{aligned} \psi_1^{(n,l,N)} &= -\frac{(\sigma + ia)^n}{(\tau - iz_0)^{n+l+2}} e^{-i(l+1)\varphi} r^{l+1} \\ &\quad \left(\sigma + ia + \frac{r^2}{\tau - iz_0}\right)^{-N-2n-l-\frac{1}{2}} \\ &\quad \times \left\{ \left[1 + \frac{R_N}{N}\right] L_{n-1}^{(l+1,N)}(R_N) + L_n^{(l,N)}(R_N) \right\}, \\ \psi_2^{(n,l,N)} &= \frac{(\sigma + ia)^n}{(\tau - iz_0)^{n+l+1}} e^{-il\varphi} r^l L_n^{(l,N)}(R_N) \\ &\quad \left(\sigma + ia + \frac{r^2}{\tau - iz_0}\right)^{-N-2n-l-\frac{1}{2}}, \end{aligned} \tag{49}$$

the argument R_N of the RLPs being given in (34).

The 1D counterpart of (49) would involve the 1D Lorentzian-like factor and the RHPs. In virtue of the aforementioned rule according to which solutions of the 3D scalar or spinor wave equation can be constructed from those of the corresponding 2D equations [12, 30–32, 36] such 1D forms of (49) come to be of relevance for the 3D spinor wave equation as the 1D forms of (33) are of relevance for the 3D scalar wave equation.

6. CONCLUSIONS

We have suggested solutions of the free-space 3D scalar wave equation, which resort to the splash pulses and have suitable polynomials as modulating factors. To the author's knowledge, such solutions are still undiscussed in the literature. Interestingly, a formal generation scheme of the higher-order solutions from the fundamental one has been highlighted, which basically parallels that holding for the Laguerre-Gaussian pulses. It resorts indeed to the same rising operators, and involves the relativistic Laguerre polynomials [26], introduced about one decade ago within a purely mathematical context as the "radial" counterpart of the relativistic Hermite polynomials [27]. Although such polynomials have later recognized to be related to the Jacobi polynomials, the original terminology has been retained here since it favours the direct correspondence with the Laguerre-Gaussian pulses-related formalism.

In fact, the results here presented further enlarge the correspondence between the Gaussian and the splash pulses to comprise also the respective higher-order pulses. As is well known, the Gaussian and the splash pulses represent two specific types of localized wave solutions of the homogeneous wave equation, and hence, as such, they may have potential applications in various research areas, like, for instance, impulse radar, high-resolution imaging, medical radiology, plasma physics, directed energy transfer and secure communications.

In addition, as mentioned earlier, the splash pulses have the evident advantage of being finite energy solutions of the wave equation [5], with the further interesting feature of exhibiting a missilelike behavior (namely, a decay at a slow rate before undergoing the usual z^{-1} decay along the z direction) [39, 40] for specific values of the exponent (i.e., $N < \frac{1}{2}$; see, indeed, Fig. 5(a)). Interestingly, as proved in [25], such a missilelike behavior is observed also in an aperture-generated spalsh pulse over an extended intermediate range between the near- and far-field regions.

Evidently, the higher-order pulses do not exhibit a quasi-missile decay since the exponent in the pertinent Lorentz-like factor increases by $2n$. However, the practical interest in such pulses may be dictated by their space-time structure, which can yield interestingly shaped pulses, when, for instance, progressive and regressive pulses are superimposed (see Fig. 6).

The solutions here obtained for the wave equation have been straightforwardly generalized to yield solutions of the spinor wave equation, which solutions have then the same basic algebraic dependencies.

Finally, we have also given the explicit expressions for the electromagnetic fields $\vec{E}$ and $\vec{H}$ in particular for the TE modes, on the basis of the Hertz potentials formalism, according to which such potentials over source-free regions just obey the scalar wave equation (1).

Of course, many issues are still to be addressed. It would be interesting, in fact, to establish whether the solutions of the scalar wave equation here presented are only a mathematical curiosity or may be amenable for a practical launching procedure.

In this regard, we recall the original hint, illustrated in detail in [6, 7] and further refined in several later publications (see, for

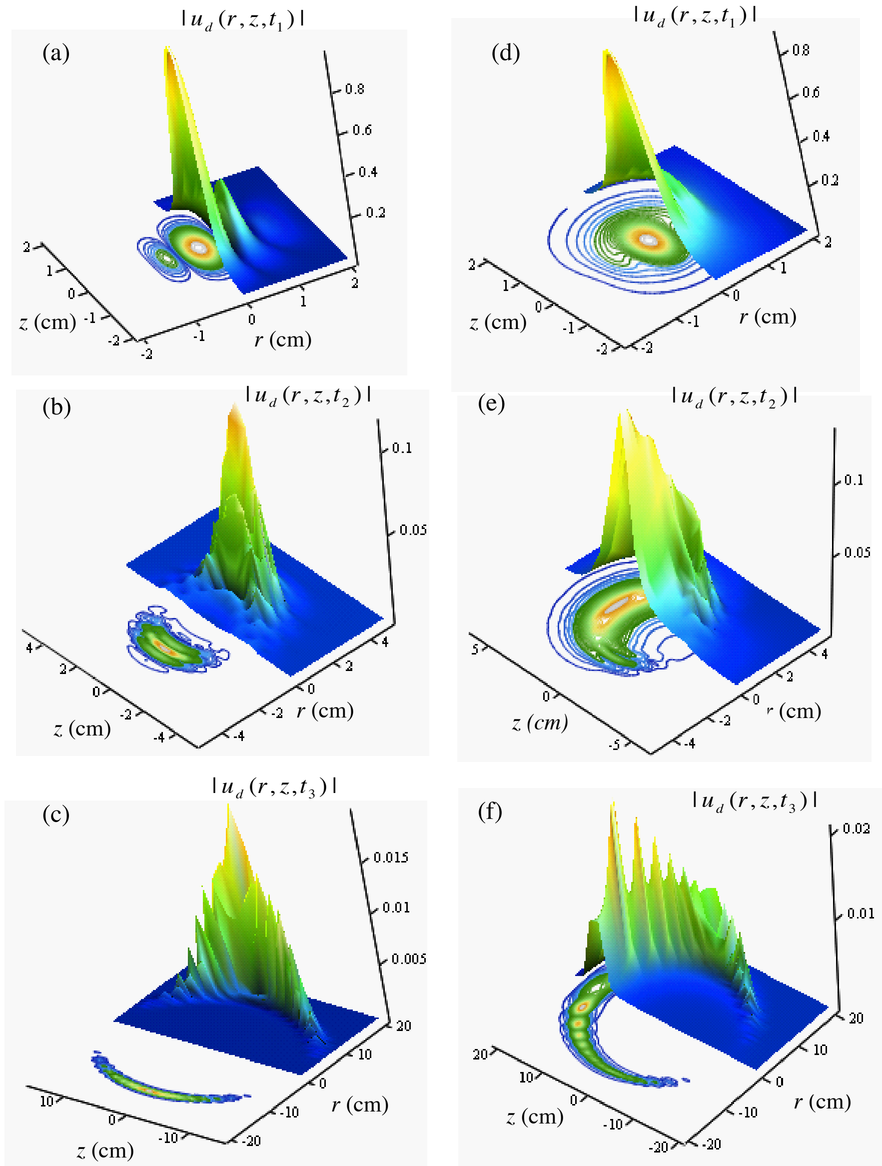

Figure 6. Amplitude $|u_d|$ vs. r and z for (a), (b), (c) $n = n' = 3$ and $N = N' = 0.5$ and (d), (e), (f) $n = n' = 3$, $N = 0.5$ and $N' = 2.5$ at $t_1 = 10^{-11}$ s, $t_2 = 8 \cdot 10^{-11}$ s and $t_3 = 5 \cdot 10^{-10}$ s.

instance, [41]), demonstrating the possibility of launching a MPS pulse from finite sources consisting of discrete circular array elements, driven by the exact field itself. The launching process, as described in [6, 7], is based on the Huygens construction yielding the scalar field generated into $z > 0$ half-plane by the source aperture from the relevant initial excitation. A comparative study of three basic reconstruction schemes is presented in [42].

The Huygens representation-based reconstruction scheme has been considered also in [25], where in fact an accurate analysis of the correspondence between source-free and aperture-generated MPS, splash and DEX pulses has been carried out. From such an analysis it emerges that (at least for the values of the exponent q considered in the quoted paper) the reconstruction of splash pulses is possible with a maintenance of the features of the source-free field over a rather extended z-range between the near- and the far-field regions, as, for instance, the aforementioned quasi-missile decay.

As to the Laguerre-Lorentzian solutions of the wave equation, discussed in the previous sections, that as noted earlier can be regarded as a sort of higher-order splash pulses with basic exponent $q > 0.5$, one might investigate their launchability through the same reconstruction process. On the other hand, according to the generation scheme (33) such higher-order pulses result formally from the repeated application of the Laplacian operator to the basic splash pulse, and so due to the axial symmetry of the latter, of the radial Laplacian. Therefore, on account of the basic properties of the Hankel transform, we may say that in principle the Laguerre-Lorentzian pulses could be produced by a sequence of direct and inverse Hankel transform of zero order with an intermediate propagation through a radial transmittance $T \propto (-)^n r^{2n}$. In formal terms, we have in fact

$$\left[\nabla_r^2\right]^n f(r) = \left[\widehat{\mathcal{H}}_0 (-)^n r^{2n} \widehat{\mathcal{H}}_0 f\right](r) \tag{50}$$

$\widehat{\mathcal{H}}_0$ denoting the Hankel transform of zero order. We recall that the Hankel transform is self-reciprocal: $\widehat{\mathcal{H}}_0^{-1} = \widehat{\mathcal{H}}_0$.

We know that the Hankel transform results from a 2D Fourier transform of an axially symmetric function operated by axially symmetric tools. The optical Hankel transform, for instance, may be realized by the so-called '*2f*' system — a spherical lens of focal length f sandwiched between two free-space section of length f — or the *Fourier tube* — a free-space section of length f sandwiched between two spherical lenses of focal length f.

One should therefore face the issues concerning the launchability of the u_q's for $q > 0.9$, which comprises the case of our interest (and should reasonably follow from the above considerations) and the practicability of the scheme (50) for the production of the higher order modes as well as the possible applications of such higher-order modes. Further investigations are evidently needed to master such issues.

As a conclusion, we show in Fig. 6 the surface plots and the relevant contour plots of the amplitude vs. r and z at different times of the wave solution obtained as difference of two Laguerre-Lorentzian solutions of the wave equation of given n's and N's:

$$u_d(r,z,t) \propto u_{n,N}(r,z,t) - u_{n',N'}(r,z,t)$$

with in general the two component wave fields being allowed to correspond to different values of the parameters z_0 and a. In particular, in the figure the values $z_0 = a = 1$ have been considered to favour the comparison with the plots of the original splash pulse reported in [5]. The maxima in the plots corresponding to the time $t_1 = 10^{-11}$ s have been normalized to unity.

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