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FIELDS IN THE FOCAL SPACE OF SYMMETRICAL HYPERBOLOIDAL FOCUSING LENS

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Abstract—In this paper, Maslov's method has been used to obtain the high frequency field refracted by a hyperboloidal focusing lens. High frequency problem which contains caustic region is transformed into caustic free problem by transforming the situation into mixed domain. The high-frequency solution that includes the caustic region is obtained from geometrical optics. The defect in high frequency solution due to geometrical optics is overcomed by Maslov's method. Numerical computations are made for the field pattern around the caustic. The results are found in good agreement with obtained using Debye-Wolf focusing integral.

1. INTRODUCTION

Ray-based techniques for waveform modeling are attractive in electromagnetics because they provide insight into how a wave front responds to a given structure [1, 2]. In these techniques, user has the luxury of being able to monitor a given phase as it steps through the medium. Geometrical optics (GO) is concerned with only the relatively high frequency component of the waveform, provided the ray tube does not vanish. However, there exists situations where ray tube shrinks to zero at places called caustics.

Mathematically in physical space, there exists singularity in the caustic, but the singularity is not a real one. In fact, the solutions of electromagnetic wave equation is not singular therefore the GO approximation is not suitable for the caustic region where they predict singularity. A systematic procedure which remedies these defects has

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been proposed by Maslov [3]. This method puts the GO field into the phase space M which has the double dimensions of the physical space, $M = X \times K$, where X is the physical space, K is the wave vector space which has the same dimensions with the physical space. The rays in the phase space do not have singularity. At the singular point in the physical space, we can project the rays from the phase space to the hybrid space. We can use the high-frequency approximation in the hybrid space, and then transforming the solution to the physical space through Fourier transform. It is well known that we always can find such a hybrid space which has no singularity at the singular point of the physical space. According to Maslov's method, the field expression near the caustic can be constructed by using the GO information, though we must perform the integration in the spectrum domain in order to predict the field in the space domain.

For a review and application of the Maslov's method on different problems, the reader is referred to Kravtsov [4–7], Gorman [8, 9], Ziolkowski and Dechamps [10]. Hongo and Ji [11–16], Naqvi and co-workers [17–33]. Many investigations on the fields in focal space of focusing system have been carried out using different methods [33–42].

In present work, field refracted by a focusing geometry which contains a hyperboloidal focusing lens has been studied by using the Maslov's method. For a special case of axis symmetrical hyperboloidal focusing lens the integral with respect to one of the two angular coordinate variables can be performed analytically, so field behavior may be computed through an integral with single variable. The field distribution along the axis is nearly symmetrically with respect to a focal point and this is one of the characteristics of point focus system.

In 1909, Debye reformulated the scalar focusing problem using plane waves rather than spherical waves. Debye formulation was further treated by Wolf as vector generalization of Debye's representation and known as Debye-Wolf vector integral [36, 37]. Sherif and Török [38–42] reported an eigenfunction representation of the integrals of Richards and Wolf. It is found that Debye-Wolf focusing integral and Maslov's method are of comparable accuracy. Solutions fit very well.

2. DERIVATION OF GEOMETRICAL OPTICS FIELD EXPRESSION

Consider the geometry which contains a hyperboloidal focusing lens as shown in Figure 1. Electromagnetic plane wave polarized in x-direction and propagating in z-direction, is incident on a hyperboloidal focusing lens. After passing through the hyperboloidal focusing lens,

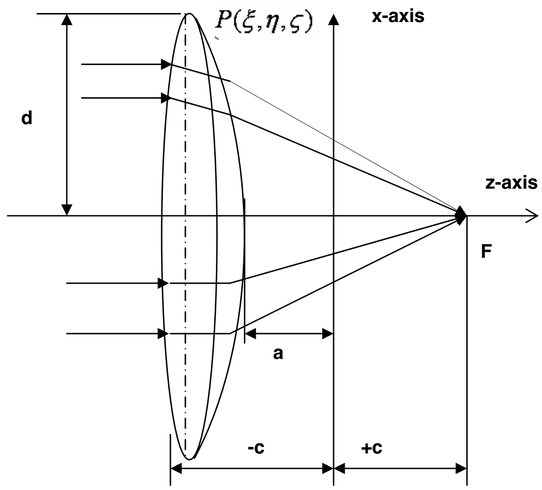

Figure 1. Hyperboloidal lens antenna.

ray is focused. We assume that the profile of the hyperboloidal lens is denoted by $g(\rho)$ and the rectangular coordinates of the point on curved surface of the lens are denoted as (ξ, η, ζ). Profile of the hyperboloidal lens is defined as

$$g(\rho) = \zeta = a\left[\frac{\xi^2+\eta^2}{b^2}+1\right]^{\frac{1}{2}} = \frac{a}{b}\left[\rho^2+b^2\right]^{\frac{1}{2}} \tag{1}$$

and

$$\rho^2 = \xi^2+\eta^2, \quad c^2 = a^2+b^2$$

An incident plane wave is given by

$$E_x^i = \exp(-jkz) \tag{2}$$

Our interest is to determine the field transmitted through the lens. Unit normal $\mathbf{N}$ of the curved surface is given by

$$\mathbf{N} = \sin\alpha\cos\beta\mathbf{i}_x + \sin\alpha\sin\beta\mathbf{i}_y + \cos\alpha\mathbf{i}_z \tag{3}$$

where (α, β) are angular polar coordinates of the point on the surface of lens and are related to the coordinates (ξ, η, ζ) by the following expressions

$$\xi = \frac{b^2\sin\alpha\cos\beta}{\sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}}$$

$$\begin{aligned}
\eta &= \frac{b^2 \sin\alpha \sin\beta}{\sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}}\\
\zeta &= \frac{a^2 \cos\alpha}{\sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}}\\
\sin\alpha &= -\frac{g'(\rho)}{\sqrt{1+(g'(\rho))^2}}\\
\cos\alpha &= \frac{1}{\sqrt{1+(g'(\rho))^2}}\\
\tan\beta &= \frac{\eta}{\xi} \qquad (4)
\end{aligned}$$

The ray vector of the refracted ray by hyperboloidal lens may be obtained using the relation $\mathbf{q} = n\mathbf{p^i} + \sqrt{1 - n^2 + n^2(\mathbf{p^i}\cdot\mathbf{N})^2}\mathbf{N} - n(\mathbf{p^i}\cdot\mathbf{N})\mathbf{N}$, which is derived from Snell's law. This can be rewritten in rectangular coordinates by using Equation (3) and is given below

$$\begin{aligned}
\mathbf{q} &= Q_t(\alpha)\cos\beta\mathbf{i}_x + Q_t(\alpha)\sin\beta\mathbf{i}_y + (n + Q_z(\alpha))\mathbf{i}_z\\
&= q_x\mathbf{i}_x + q_y\mathbf{i}_y + q_z\mathbf{i}_z \qquad (5)
\end{aligned}$$

where

$$\begin{aligned}
Q_t(\alpha) &= K(\alpha)\sin\alpha\\
Q_z(\alpha) &= K(\alpha)\cos\alpha\\
K(\alpha) &= \sqrt{1 - n^2\sin^2\alpha} - n\cos\alpha
\end{aligned}$$

where n is the refractive index of the dielectric medium. Once the ray vector of refracted field is obtained, the rectangular coordinates (x, y, z) of observation point on the refracted ray are given by

$$x = \xi + q_x t, \quad y = \eta + q_y t, \quad z = \zeta + q_z t \qquad (6)$$

where t is the parameter along the ray. Equation (6) is obtained from the solutions of Hamilton's equations and shows the relation between spatial coordinates (x, y, z) and ray vector components (q_x, q_y, q_z) of the ray. (ξ, η, ζ) are rectangular coordinates of initial point on the refracted ray or point on the curved surface of the lens. From Equations (5) and (6) and the using procedure employed in [11–13], the geometrical optics field is derived as

$$\mathbf{E}^r(x, y, z) = \mathbf{E}_T(\xi, \eta)\,[J(t)]^{-\frac{1}{2}} \exp\Big[-jk\Big(S_0(\xi, \eta) + t\Big)\Big] \qquad (7)$$

where $J(t)$ is the Jacobian of coordinate transformation from ray coordinates (ξ, η, t) to rectangular coordinates (x, y, z) and has been derived in the appendix

$$J(t) = \frac{D(t)}{D(0)} = \frac{1}{D(0)} \frac{\partial(x, y, z)}{\partial(\xi, \eta, t)} = \left(-P\frac{U_0}{E}t + 1\right)\left(\frac{Q_t(\alpha)}{\rho}t + 1\right)$$

where

$$\begin{aligned}
P &= -\frac{ab\cos^2\alpha}{(\rho^2 + b^2)^{\frac{3}{2}}} \\
U_0 &= n\frac{\partial Q_t(\alpha)}{\partial\alpha} + K^2(\alpha) \\
E &= \frac{n\cos\alpha + K(\alpha)}{\cos\alpha} \\
\rho &= \frac{b^2\sin\alpha}{\sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}} \\
D &= \frac{\partial Q_t(\alpha)}{\partial\alpha} = \frac{(1 - 2n^2\sin^2\alpha)\cos\alpha}{\sqrt{1 - n^2\sin^2\alpha}} - n\cos 2\alpha
\end{aligned}$$

$\mathbf{E}_T$ is the amplitude of the refracted ray at the refraction point. Initial phase S_0 on the surface of the lens and t are defined as

$$S_0 = n(-c + \zeta), \quad t = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2} \tag{8}$$

It is readily seen that the geometrical optics solution (7) of the refracted ray becomes infinity at the point F, where $J = 0$, in Figure 1.

3. DERIVATION OF THE EXPRESSION VALID AROUND CAUSTIC

According to Maslov's method, the three-dimensional expression for the field that is valid near the caustic is given by [11]

$$\begin{aligned}
\mathbf{E}^r(\underline{\mathrm{r}}) \;=\; &\frac{k}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}_T(\xi, \eta)\left[\frac{D(t)}{D(0)}\frac{\partial(q_x, q_y)}{\partial(x, y)}\right]^{-\frac{1}{2}} \\
&\exp\Big\{-jk\Big[S_0 + t - x(q_x, q_y, z)q_x \\
&-y(q_x, q_y, z)q_y + q_x x + q_y y\Big]\Big\}\, dq_x dq_x
\end{aligned} \tag{9}$$

Equation (9) is derived by applying the stationary phase method to the conventional Fourier-transform representation for $\mathbf{E}^r(\mathbf{r})$ and comparing it with the geometrical optics field given in Equation (7). Thus the integrand of the inverse Fourier transform of the wave function is derived through the information of the GO solution. It may be noted that $x(q_x, q_y, z)$ and $y(q_x, q_y, z)$ means that the coordinate x and y should be expressed in terms of mixed coordinates (q_x, q_y, z) by using solutions of Hamilton equations given in (6). The same is true for t and it is given by $t = \frac{z-\zeta}{q_z}$. The result is given by

$$\begin{aligned} J(t)\frac{\partial(q_x, q_y)}{\partial(x, y)} &= \frac{1}{D(0)}\frac{\partial(q_x, q_y, z)}{\partial(x, y, t)} \\ &= \frac{1}{D(0)}\left(\frac{\partial q_x}{\partial \xi}\frac{\partial q_y}{\partial \eta} - \frac{\partial q_y}{\partial \xi}\frac{\partial q_x}{\partial \eta}\right)\frac{\partial z}{\partial t} \\ &= -\frac{PQ_z(\alpha)DQ_t(\alpha)}{\rho E} \end{aligned} \tag{10}$$

The phase function is given by

$$\begin{aligned} S &= S_0 + t - x(q_x, q_y, z)q_x - y(q_x, q_y, z)q_y + q_x x + q_y y \\ &= S_0 + q_x(x-\xi) + q_y(y-\eta) + q_z(z-\zeta) \end{aligned} \tag{11}$$

We introduce polar coordinates as

$$\begin{aligned} x &= r\sin\theta_0\cos\phi_0 \\ y &= r\sin\theta_0\sin\phi_0 \\ z &= r\cos\theta_0 \end{aligned} \tag{12}$$

The phase function becomes as

$$\begin{aligned} S &= K(\alpha)r\sin\alpha\sin\theta_0\cos(\phi_0-\beta) + (n + K(\alpha)\cos\alpha)z \\ &\quad -K(\alpha)(\rho\sin\alpha + \zeta\cos\alpha) - nc \end{aligned} \tag{13}$$

Transforming the integration variables (q_x, q_y) into (α, β) that is,

$$dq_x dq_y = \left[\frac{(1-2n^2\sin^2\alpha)\cos\alpha}{\sqrt{1-n^2\sin^2\alpha}} - n\cos 2\alpha\right]Q_t(\alpha)d\alpha d\beta \tag{14}$$

Substituting (13) and (14) into (9), following is obtained

$$\begin{aligned} \mathbf{E}^r(x,z) = \frac{k}{2\pi}\int_0^T\int_0^{2\pi}&\mathbf{E}_T(\xi,\eta)\left[\frac{E\rho DQ_t(\alpha)}{PQ_z(\alpha)}\right]^{\frac{1}{2}} \\ &\exp\Big[-jk\Big(K(\alpha)r\sin\alpha\sin\theta_0\cos(\phi_0-\beta) \\ &+(n+K(\alpha)\cos\alpha)z \\ &-K(\alpha)(\rho\sin\alpha+\zeta\cos\alpha)-nc\Big)\Big]d\alpha d\beta \end{aligned} \tag{15}$$

The subtention angle T of lens is given by

$$T = \arctan\left(\frac{d}{2c}\right)$$

where d is height of hyperboloidal lens. The initial value $\mathbf{E}_T(\xi,\eta)$ in (15) may be obtained by using GO theory. The transmitted field by lens is given by

$$\begin{aligned}\mathbf{E}_T = &\left(T_\perp \sin^2\beta + \left[T_\perp + \left(n\sin^2\alpha + \cos\alpha\sqrt{1-n^2\sin^2\alpha}\right) T_\parallel \cos^2\beta\right)\mathbf{i}_x\right.\\ &+\left(-\sin\beta\cos\beta + \left(n\sin^2\alpha + \cos\alpha\sqrt{1-n^2\sin^2\alpha}\right)\right)T_\parallel\Big]\,\mathbf{i}_y\\ &+\left(T_\parallel\left(n\cos\alpha - \sqrt{1-n^2\sin^2\alpha}\right)\right)\sin\alpha\cos\beta]\mathbf{i}_z\end{aligned}$$

where

$$\begin{aligned}T_\parallel &= \frac{2n\cos\alpha}{\cos\alpha + n\sqrt{1-n^2\sin^2\alpha}}\\ T_\perp &= \frac{2n\cos\alpha}{n\cos\alpha + \sqrt{1-n^2\sin^2\alpha}}\end{aligned}$$

Finally the expression which is valid around the caustic is

$$\begin{aligned}\mathbf{E}^r(x,z) = &\frac{k}{2\pi}\int_0^T\int_0^{2\pi}\mathbf{E}_T\left[\frac{E\rho DQ_t(\alpha)}{PQ_z(\alpha)}\right]^{\frac{1}{2}}\\ &\exp\Big[-jk\Big(K(\alpha)r\sin\alpha\sin\theta_0\cos(\phi_0-\beta)\\ &+(n+K(\alpha)\cos\alpha)z\\ &-K(\alpha)(\rho\sin\alpha+\zeta\cos\alpha)-nc)\Big)\Big]d\alpha d\beta \qquad (16)\end{aligned}$$

The integration with respect to β can be performed by using the integral representation of Bessel function. The results are expressed as

$$E_x^r = \frac{k}{2}\Big[P(r,\theta_0) - R(r,\theta_0)\cos 2\phi_0\Big] \qquad (17)$$

$$E_y^r = \frac{k}{2}[Q(r,\theta_0)\sin 2\phi_0] \qquad (18)$$

$$E_z^r = jk[R(r,\theta_0)\sin\phi_0] \qquad (19)$$

where

$$
\begin{aligned}
P(r,\theta_0) \;=\; & \int_0^T \left[T_\perp + \left(n\sin^2\alpha + \cos\alpha\sqrt{1-n^2\sin^2\alpha}\right)T_\parallel\right] \\
& J_0(kK(\alpha)r\sin\theta_0\sin\alpha)\left[\frac{E\rho DQ_t(\alpha)}{PQ_z(\alpha)}\right]^{\frac{1}{2}} \\
& \exp\Big[-jk\Big((n+K(\alpha)\cos\alpha)\,z \\
& -K(\alpha)\,(\rho\sin\alpha+\zeta\cos\alpha)-nc\Big)\Big]d\alpha \\
Q(r,\theta_0) \;=\; & \int_0^T \left[\left(T_\parallel\left(n\cos\alpha-\sqrt{1-n^2\sin^2\alpha}\right)\right)\sin\alpha\right] \\
& J_1\left(kK(\alpha)r\sin\theta_0\sin\alpha\right)\left[\frac{E\rho DQ_t(\alpha)}{PQ_z(\alpha)}\right]^{\frac{1}{2}} \\
& \exp\Big[-jk\Big((n+K(\alpha)\cos\alpha)\,z \\
& -K(\alpha)\,(\rho\sin\alpha+\zeta\cos\alpha)-nc\Big)\Big]d\alpha \\
R(r,\theta_0) \;=\; & \int_0^T \left[\left(-T_\perp + \left(n\sin^2\alpha + \cos\alpha\sqrt{1-n^2\sin^2\alpha}\right)T_\parallel\right)\right. \\
& J_2(kK(\alpha)r\sin\theta_0\sin\alpha)\left[\frac{E\rho DQ_t(\alpha)}{PQ_z(\alpha)}\right]^{\frac{1}{2}} \\
& \exp\Big[-jk\Big((n+K(\alpha)\cos\alpha)\,z \\
& -K(\alpha)\,(\rho\sin\alpha+\zeta\cos\alpha)-nc\Big)\Big]d\alpha
\end{aligned}
$$

4. COMPARISON TO THE DEBYE-WOLF FOCUSING INTEGRAL

The electric field distribution around focal point F of hyperboloidal lens, which has been excited by an electromagnetic plane wave polarized in x-direction and propagating in z-direction as shown in Figure 2, may be obtained using the Debye-Wolf focusing integral [36–42] and is given below

$$\mathbf{E}(x,y,z) = -\frac{jk}{2\pi}\iint_T \frac{\mathbf{a}(\xi,\eta)}{\zeta}\exp(-jk\,(\xi x+\eta y+\zeta z))d\xi d\eta \tag{20}$$

where $k=\frac{2\pi}{\lambda}$ is wave number, $\mathbf{a}(\xi,\eta)$ is the strength vector of a ray at rear face of lens, ξ, η and ζ are the cartesian components of the

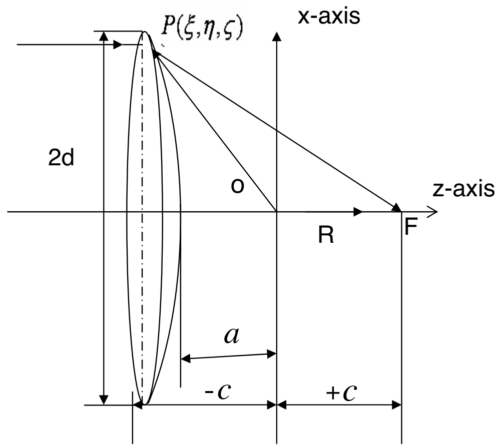

Figure 2. Geometry for Deby-Wolf integral.

position vector **OP** and T is the solid angle associated with all the ray which reaches the image space through exit pupil of the lens. The coordinates of the point $P(\xi, \eta, \zeta)$ on lens are defined in (4) and the co-ordinates (x, y, z) of a point F in the image region may be expressed in the form

$$F(x, y, z) = r\left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right) \tag{21}$$

so the term in the exponent of integral (20) becomes

$$\xi x + \eta y + \zeta z = r\rho\sin\theta\sin\alpha\cos(\phi - \beta) + r\zeta\cos\theta \tag{22}$$

The strength factor may be determined by [38] as

$$\begin{aligned}\mathbf{a}(\xi, \eta) \ &= \ f\sqrt{\cos\alpha}\Big[(\cos\alpha + \sin^2\beta(1 - \cos\alpha))\mathbf{i}_x \\ &\quad +(\cos\alpha - 1)\cos\alpha\sin\beta\mathbf{i}_y - \cos\beta\sin\alpha\mathbf{i}_z\Big]\end{aligned} \tag{23}$$

Expression for quantity $\frac{d\xi d\eta}{\zeta}$ in terms of α and β is required for our basic diffraction integral (20). This quantity represents the element dS of the solid angle and is given by

$$\frac{d\xi d\eta}{\zeta} = dS = \rho\sin\alpha d\alpha d\beta \tag{24}$$

Substituting (21) to (24) into (20), we obtain following field expression

$$\begin{aligned}\mathbf{E}^r(x,z) \;=\; & \frac{-jk}{2\pi}\int_0^T\int_0^{2\pi} f\Big[(\cos\alpha+\sin^2\beta(1-\cos\alpha))\mathbf{i}_x+(\cos\alpha-1)\\ & \cos\alpha\sin\beta\mathbf{i}_y-\cos\beta\sin\alpha\mathbf{i}_z\Big]\sqrt{\cos\alpha}\sin\alpha\\ & \exp\Big[-jk\Big(r\rho\sin\alpha\sin\theta\cos(\phi-\beta)+r\zeta\cos\theta\Big)\Big]d\alpha d\beta \qquad (25)\end{aligned}$$

The integration with respect to β can be performed by using the integral representation of Bessel function. Finally, we obtain following expressions for the components of the vector field

$$E_x^r = \frac{-jkf}{2}\Big[I_0+I_2\cos 2\phi\Big] \qquad (26)$$

$$E_y^r = \frac{-jkf}{2}[I_2\sin 2\phi] \qquad (27)$$

$$E_z^r = \frac{-2kf}{2}[I_1\cos\phi] \qquad (28)$$

where

$$I_0=\int_0^T\sqrt{\cos\alpha}\sin\alpha(1+\cos\alpha)J_0(kr\rho\sin\theta\sin\alpha)\exp\Big[-jk\Big(r\zeta\cos\theta\Big)\Big]d\alpha$$

$$I_1=\int_0^T\sqrt{\cos\alpha}\sin^2\alpha J_1(kr\rho\sin\theta\sin\alpha)\exp\Big[-jk\Big(r\cos\alpha\zeta\Big)\Big]d\alpha$$

$$I_2=\int_0^T\sqrt{\cos\alpha}\sin\alpha(1-\cos\alpha)J_2(kr\rho\sin\theta\sin\alpha)\exp\Big[-jk\Big(r\zeta\cos\theta\Big)\Big]d\alpha$$

5. NUMERICAL RESULTS AND DISCUSSION

Field pattern around the caustic of a hyperboloidal focusing lens are determined by performing the integration, in Equations (17) and (26), numerically by using Mathcad software. It is observed that normalized results of (18) and (19) are similar to (17). It is assumed that $ka = 20$, $kb = 5$, $kd = 15$ and $n = 1.8$ for results in Figure 3 to Figure 8 and $ka = 16$, $kb = 4$, $kd = 20$ and $n = 1.8$ for results in Figure 9 to Figure 10. Figure 3 and Figure 4 show comparison between Maslov's method and Debye-Wolf focusing integral along x-axis and z-axis respectively. The solid line shows the results obtained using Maslov's method while dashed line is for result obtained using Debye-Wolf focusing integral which are in good agreement. This comparison proves validity of Maslov's method. Figure 5 and Figure 6 show the field distribution around focal point along x-axis and z-axis respectively. Figure 7 provides contour plots of normalized field distribution around focus of a hyperboloidal focusing lens by Maslov's method and Figure 8 provides contour plots by Debye-Wolf focusing integral at $ka = 20$, $kb = 5$, $kd = 15$ and $n = 1.8$. The location of the caustic may be observed and verified easily. Figure 9 and Figure 10 also provides the

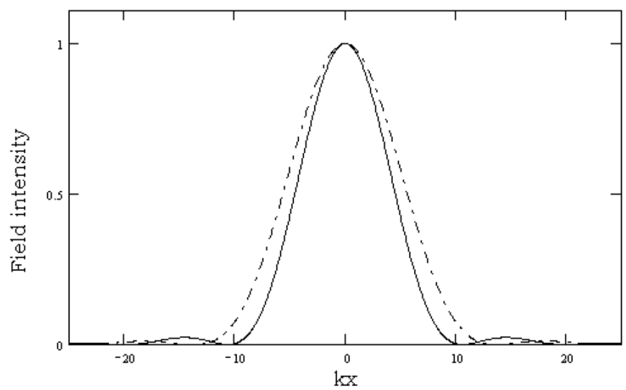

Figure 3. Comparison of normalized field intensity distribution anlog x-axis by Maslov's method (solid line) and Deby-Wolf integral (dashed line) at $ka = 20$, $kb = 5$, and $kd = 15$ and $n = 1.8$.

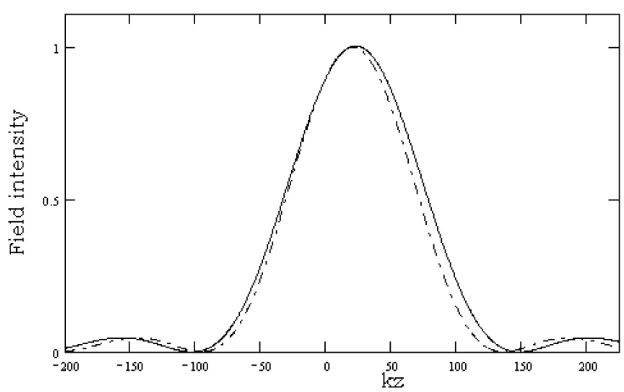

Figure 4. Comparison of normalized field intensity distribution anlog z-axis by Maslov's method (solid line) and Deby-Wolf integral (dashed line) at $ka = 20$, $kb = 5$, and $kd = 15$ and $n = 1.8$.

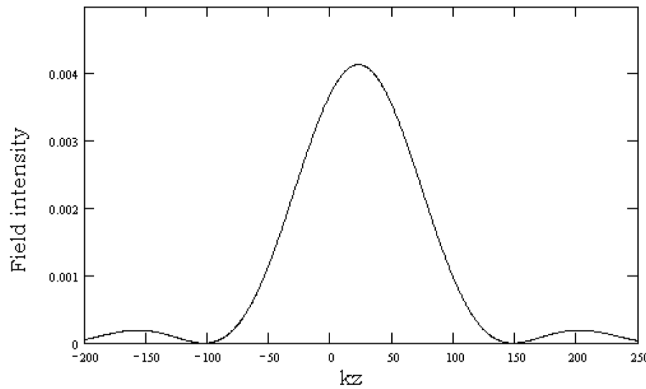

Figure 5. Field intensity distribution along z-axis at $ka = 20$, $kb = 5$, $kd = 15$ and $n = 1.8$ by Maslov's method.

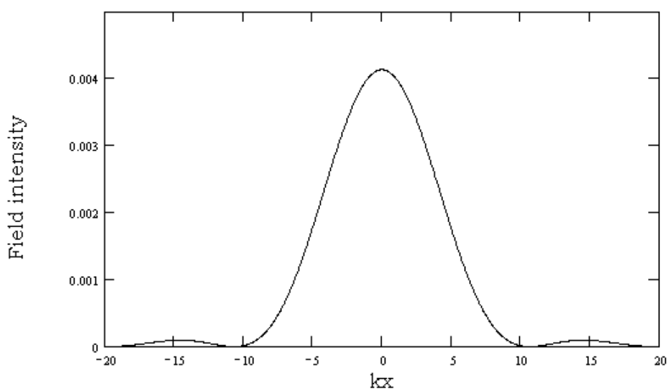

Figure 6. Field intensity distribution along z-axis at $ka = 20$, $kb = 5$, $kd = 15$ and $n = 1.8$ by Maslov's method.

contour plots by Maslov's method and Debye-Wolf focusing integral respectively at $ka = 16$, $kb = 4$, $kd = 20$ and $n = 1.8$.

All contour plots show equi-amplitude contours of the field distribution on the meridional plane line plots show the field behavior around caustic of lens. It is observed that the peak point of the field distribution are located at the caustic. It is also observed from Figure 7 to Figure 10 that the peak points of the field move towards lens if we decrease length of major axis and increase the height of hyperboloidal lens.

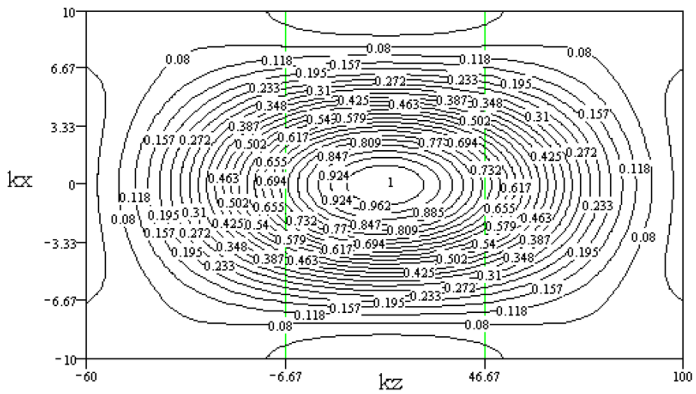

Figure 7. Equi-amplitude normalized contour plots of field intensity distribution at $ka = 20$, $kb = 5$, $kd = 15$ and $n = 1.8$ using Maslov's method.

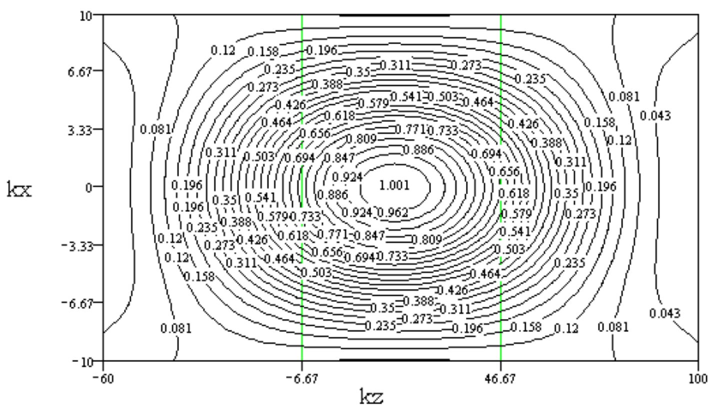

Figure 8. Equi-amplitude normalized contour plots of field intensity distribution at $ka = 20$, $kb = 5$, $kd = 15$ and $n = 1.8$ using Deby-Wolf inytegral.

APPENDIX A. EVALUATION OF THE $J(t)$

$$\begin{aligned} D(t) &= \frac{\partial(x,y,z)}{\partial(\xi,\eta,t)} = \begin{vmatrix} 1+\frac{\partial q_x}{\partial \xi}t & \frac{\partial q_y}{\partial \xi}t & \frac{\partial \zeta}{\partial \xi}+\frac{\partial q_z}{\partial \xi}t \\ \frac{\partial q_x}{\partial \eta}t & 1+\frac{\partial q_y}{\partial \eta}t & \frac{\partial \zeta}{\partial \eta}+\frac{\partial q_z}{\partial \eta}t \\ q_x & q_y & q_z \end{vmatrix} \\ &= Ut^2+Vt+W \end{aligned} \tag{A1}$$

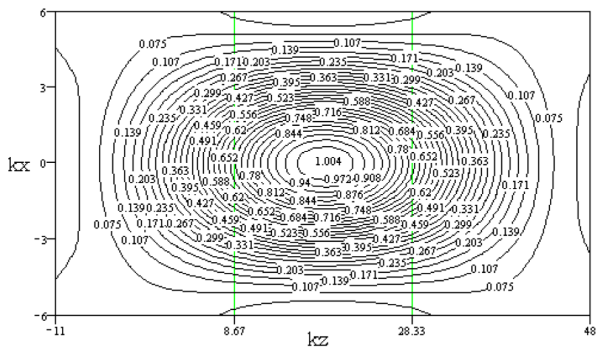

Figure 9. Equi-amplitude normalized contour plots of field intensity distribution at $ka = 16$, $kb = 4$, $kd = 20$ and $n = 1.8$ using Maslov's method.

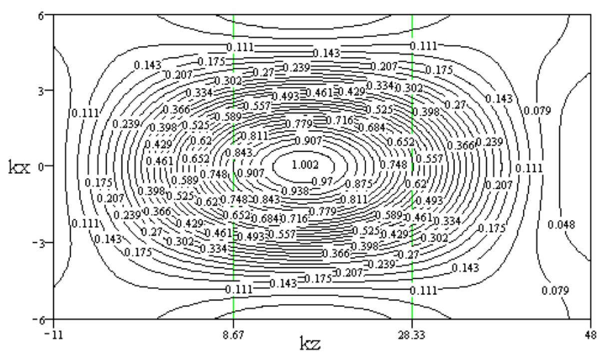

Figure 10. Equi-amplitude normalized contour plots of field intensity distribution at $ka = 16$, $kb = 4$, $kd = 20$ and $n = 1.8$ using Deby-Wolf inytegral.

where U, V, W are

$$\begin{aligned}
U &= \left(\frac{\partial q_y}{\partial \xi}\frac{\partial q_z}{\partial \eta} - \frac{\partial q_z}{\partial \xi}\frac{\partial q_y}{\partial \eta}\right) q_x + \left(\frac{\partial q_x}{\partial \eta}\frac{\partial q_z}{\partial \xi} - \frac{\partial q_z}{\partial \eta}\frac{\partial q_x}{\partial \xi}\right) q_y \\
&\quad + \left(\frac{\partial q_x}{\partial \xi}\frac{\partial q_y}{\partial \eta} - \frac{\partial q_y}{\partial \xi}\frac{\partial q_x}{\partial \eta}\right) q_z \\
V &= \left(\frac{\partial q_y}{\partial \xi}\frac{\partial \zeta}{\partial \eta} - \frac{\partial \zeta}{\partial \xi}\frac{\partial q_y}{\partial \eta} - \frac{\partial q_z}{\partial \xi}\right) q_x + \left(\frac{\partial q_x}{\partial \eta}\frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \eta}\frac{\partial q_x}{\partial \xi} - \frac{\partial q_z}{\partial \eta}\right) q_y \\
&\quad + \left(\frac{\partial q_y}{\partial \eta} + \frac{\partial q_x}{\partial \xi}\right) q_z \\
W &= \left(\frac{\partial \zeta}{\partial \xi} q_x + \frac{\partial \zeta}{\partial \eta} q_y\right) + q_z
\end{aligned} \tag{A2}$$

We may rewrite the values of U, V and W by using the following relations

$$\frac{\partial q_x}{\partial \xi} = \frac{\partial Q_t(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \xi}\cos\beta - Q_t(\alpha)\frac{\partial \beta}{\partial \xi}\sin\beta = -PD\cos^2\beta + \frac{Q_t}{\rho}\sin^2\beta$$
$$\frac{\partial q_y}{\partial \xi} = \frac{\partial Q_t(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \xi}\sin\beta - Q_t(\alpha)\frac{\partial \beta}{\partial \xi}\cos\beta = -\left(PD + \frac{Q_t(\alpha)}{\rho}\right)\cos\beta\sin\beta$$
$$\frac{\partial q_x}{\partial \eta} = \frac{\partial Q_t(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \eta}\cos\beta - Q_t(\alpha)\frac{\partial \beta}{\partial \eta}\sin\beta = -\left(PD + \frac{Q_t(\alpha)}{\rho}\right)\cos\beta\sin\beta$$
$$\frac{\partial q_y}{\partial \eta} = \frac{\partial Q_t(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \eta}\sin\beta - Q_t(\alpha)\frac{\partial \beta}{\partial \eta}\cos\beta = -PD\cos^2\beta + \frac{Q_t(\alpha)}{\rho}\sin^2\beta$$
$$\frac{\partial q_z}{\partial \xi} = \frac{\partial Q_z(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \xi} = -P\frac{\partial Q_z(\alpha)}{\partial \alpha}\cos\beta$$
$$\frac{\partial q_z}{\partial \eta} = \frac{\partial Q_z(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial \eta} = -P\frac{\partial Q_z(\alpha)}{\partial \alpha}\sin\beta$$

where

$$\tan\alpha = -g'(\rho), \qquad \tan\beta = \frac{\eta}{\xi}, \qquad \frac{\partial \alpha}{\partial \xi} = P\cos\beta, \qquad \frac{\partial \alpha}{\partial \xi} = P\sin\beta$$
$$\frac{\partial \beta}{\partial \xi} = -\frac{\sin\beta}{\rho}, \quad \frac{\partial \zeta}{\partial \xi} = \frac{\cos\beta}{\rho}, \quad \frac{\partial \beta}{\partial \xi} = g'(\rho)\cos\beta, \quad \frac{\partial \beta}{\partial \eta} = g'(\rho)\sin\beta$$

The new expressions for U, V and W are given by

$$\begin{aligned}
U &= P\frac{Q_t(\alpha)}{\rho}(Q_t(\alpha)D_1 - Q_z(\alpha)D)\\
V &= \frac{Q_t(\alpha)}{\rho}\left(\frac{n\cos\alpha + K(\alpha)}{\cos\alpha}\right) + P\left(Q_t(\alpha)D_1 - Q_z(\alpha)D\right)\\
W &= \frac{n\cos\alpha + K(\alpha)}{\cos\alpha}\\
D_1 &= \frac{\partial Q_z(\alpha)}{\partial \alpha} = n\sin 2\alpha - \frac{(1 + n^2\cos 2\alpha)\sin\alpha}{\sqrt{(1 - n^2\sin^2\alpha)}}
\end{aligned} \tag{A3}$$

where Hence we have

$$D(t) = Ut^2 + Vt + W = \left(P\frac{U_0}{E}t + 1\right)\left(\frac{Q_t(\alpha)}{\rho}t + 1\right)$$

where

$$P = -\frac{ab\cos^2\alpha}{(\rho^2 + b^2)^{\frac{3}{2}}}$$

$$U_0 = n\frac{\partial Q_t(\alpha)}{\partial \alpha} + K^2(\alpha)$$
$$E = \frac{n\cos\alpha + K(\alpha)}{\cos\alpha}$$

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