GREEN'S FUNCTION DERIVATION OF AN ANNULAR WAVEGUIDE FOR APPLICATION IN METHOD OF MOMENT ANALYSIS OF ANNULAR WAVEGUIDE SLOT ANTENNAS

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Abstract—Detailed procedure for Green's function derivation of an annular waveguide is presented in this paper for the first time. The proper components of the electromagnetic fields along with their corresponding Green's functions will be derived in a useful and applicable form. Based on the derived Green's functions, a proper set of integral equations are derived in a novel general form which MoM solution will lead to complete slot field distribution in the annular waveguide slot antenna (AWSA). The proposed theory is found having high value in circuit modelling and array design of such novel antenna structures in the future.

1. INTRODUCTION

Rectangular waveguide slot array antennas with circular boundary have been widely used in practical antennas. Low profile, ability for different current distribution implementation and ease of boundary formation are among the advantages of such antennas [1, 2]. On the other hand, the use of rectangular waveguides in circular boundary restricts optimum and efficient performance of such antennas. Such limitations are mainly due to the disharmony between the rectangular geometries of radiating waveguides and the circular boundary. This has led to alternative radiating structures having more geometrical balance with the boundary such as AWSA.

The concept of AWSA was introduced in [3] to be employed in circular boundary planar slot array antennas as an alternative to

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common rectangular waveguides. It was shown by initial simulation and measurement that such kind of annular slotted waveguides is feasible in the desired application. There was no theoretical knowledge of the annular waveguide in [3] and it was considered to have the same characteristics as the rectangular waveguide. It was predicted that by using the proposed structure it would be possible to achieve polarization agility, with less discritization error and higher gain from the slot array antennas. Resonant characteristic at the design frequency, linear polarization normal to the slot extension direction and the ability to be modelled as a shunt conductance were among the properties proposed for the single slot AWSA shown in that paper. The AWSA idea was derived from the azimuthal waveguide discussed by Ishimaru [4]. More recent reports on other kinds of waveguide bends following the same procedure for characterising the waveguide for complicated applications can be found in [5–8].

There were two main assumptions in [3] regarding the employed *annular waveguide* in the designed AWSA which were proved to be valid in more recent reports: sinusoidal field distribution along the waveguide cross section and single mode transmission through the waveguide extension. The initial discussion on the theory of annular waveguide was briefly provided in [9, 10] with the complete details recently provided in [11]. Exact field distributions of an annular waveguide were proposed in this last paper along with interesting approximations which highly simplify the design of complicated antenna arrays using such waveguide.

From the application point of view, the ability of AWSA array to produce linear horizontal or vertical polarization as well as right or left hand circular polarization were introduced in [12, 13]. It was shown that by choosing proper excitation phase at the four quarters of an AWSA array, the radiated polarization could be easily switched among all practical cases. This is among the main advantages of AWSA arrays, which is open to more theoretical investigations in the future. As the main standard requirement for such an improvement, Green's function formulation of the annular waveguide and MoM analysis of the slotted annular waveguide (AWSA) are the vital steps to be performed as in other similar design approaches [14–23].

In this paper, the details of Green's function formulation will be reviewed and the special considerations needed for the concerned geometry will be provided. Then, based on the fields derived in [11], the Green's function of an annular waveguide will be formulated for electric and magnetic forms. Based on the approach of Elliott for rectangular slots [2], in future reports, we will be interested in answering the question "*given that an annular slot is cut in one of the broad walls*

of an annular waveguide, what electric field distribution arises in the slot in response to an incident dominant mode?" It will be shown that the field distribution can be found by matching internal and external expressions for the normal magnetic and electric fields in the slot and solving the resulted equations by MoM. The internal expression consists of the sum of the incident dominant mode and the scattered fields in all modes. The scattered field can be formulated with the aid of Green's functions for an annular waveguide. These functions will be derived in a useful form in the next sections of this paper in details.

2. THE TRANSVERSE OR NORMAL FIELD COMPONENTS AS FUNCTIONS OF THE ANNULAR FIELD COMPONENTS

In this section, a set of relations between the transverse or normal field components and annular field components of the annular waveguide will be derived. Using this method, it will be finally shown that it is enough to only solve for the normal z-directed components and solve for the others using the results of this section.

(a)   (b)
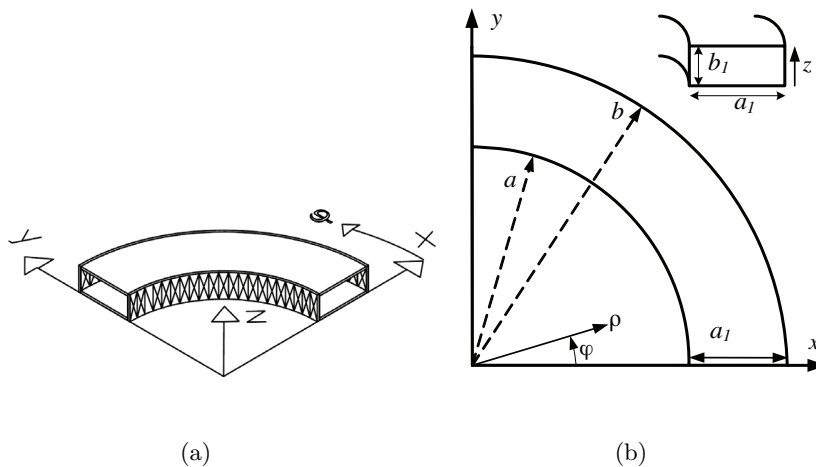
Figure 1. Demonstration of an annular waveguide: (a) 3-D and (b) top views. The wave will propagate in cylindrical φ direction in an AWSA.

The annular waveguide is shown in Fig. 1. It will be assumed that:

1- The waveguide is filled with a homogeneous, isotropic dielectric with constitutive parameters μ_0, ε and σ.

2- The walls are vanishingly thin and composed of perfect conductor.

3- The upper broad wall is imbedded in a perfectly conducting ground plane of infinite extent.

4- The outer half-space is a vacuum.

The last two items will be used in future MoM analysis of the AWSA. In the region R ($0 < z < b_1$, $a < \rho < b$, $0 < \varphi < \pi/2$), Maxwell's equations are

$$
\begin{aligned}
\nabla \times H = (\sigma + j\omega\varepsilon)E \qquad & \nabla \cdot H = 0 \\
\nabla \times E = -j\omega\mu_0 H \qquad & \nabla \cdot E = 0
\end{aligned} \tag{1}
$$

Since the waveguide is filled with homogeneous, isotropic dielectric, equation (1) is valid for the time harmonic field at our frequency of interest as long as the domain of study inside the annular waveguide is source free. In (1) $e^{j\omega t}$ time variations have been assumed. We shall define k by the formula

$$
k = \omega\sqrt{\mu_0\varepsilon}\sqrt{1 + \frac{\sigma}{j\omega\varepsilon}}, \qquad k^2 = \omega\mu(\omega\varepsilon - j\sigma) = -j\omega\mu(\sigma + j\omega\varepsilon) \tag{2}
$$

with k a complex number which lies in the fourth quadrant ($k = k_r + jk_i$, k_r positive, k_i negative). This will ensure waves which decay as they progress, consistent with the behaviour of a lossy dielectric. In cylindrical coordinate we can rewrite Maxwell equations as the following two sets of equations

$$
\begin{cases}
\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi}H_z - \dfrac{\partial}{\partial z}H_\varphi = (\sigma + j\omega\varepsilon)E_\rho \\[2mm]
\dfrac{\partial}{\partial z}H_\rho - \dfrac{\partial}{\partial\rho}H_z = (\sigma + j\omega\varepsilon)E_\varphi \\[2mm]
\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho H_\varphi) - \dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi}H_\rho = (\sigma + j\omega\varepsilon)E_z
\end{cases} \tag{3}
$$

$$
\begin{cases}
\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi}E_z - \dfrac{\partial}{\partial z}E_\varphi = -j\omega\mu H_\rho \\[2mm]
\dfrac{\partial}{\partial z}E_\rho - \dfrac{\partial}{\partial\rho}E_z = -j\omega\mu H_\varphi \\[2mm]
\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho E_\varphi) - \dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi}E_\rho = -j\omega\mu H_z
\end{cases} \tag{4}
$$

If all components of the field in region R are assumed to propagate according to the factor $e^{(j\omega t \pm \gamma\varphi)}$, then a standard development by using proper pair of equations from the above Maxwell's equations and the equation for k reveals that the transverse field components can be

expressed in terms of the annular field components via the following equations. Thus if the annular components E_φ and H_φ can be found in R, the entire fields will be known in R. This is the *conventional* form used in rectangular slot array antenna design and analysis to present transverse components in term of the field components along the direction of propagation,

$$
\begin{aligned}
H_z &= \frac{\rho}{\gamma^2 + \rho^2 k^2} \left[\pm\gamma\frac{1}{\rho}\frac{\partial}{\partial z}(\rho H_\varphi) + (\sigma + j\omega\varepsilon)\frac{\partial}{\partial \rho}(\rho E_\varphi) \right] \\
H_\rho &= \frac{\rho}{\gamma^2 + \rho^2 k^2} \left[\pm\gamma\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\varphi) - (\sigma + j\omega\varepsilon)\frac{\partial}{\partial z}(\rho E_\varphi) \right] \\
E_z &= \frac{\rho}{\gamma^2 + \rho^2 k^2} \left[\pm\gamma\frac{1}{\rho}\frac{\partial}{\partial z}(\rho E_\varphi) - j\omega\mu\frac{\partial}{\partial \rho}(\rho H_\varphi) \right] \\
E_\rho &= \frac{\rho}{\gamma^2 + \rho^2 k^2} \left[\pm\gamma\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\varphi) + j\omega\mu\frac{\partial}{\partial z}(\rho H_\varphi) \right]
\end{aligned}
\tag{5a}
$$

In our case of the novel annular waveguide, since the only components of the electromagnetic field that can be solved directly in cylindrical coordinate are the z components (E_z & H_z), we will use the following set of equations to express other field components in terms of those under study

$$
\begin{aligned}
E_\rho &= \frac{1}{k^2 - \beta_z^2} \left[\frac{\partial}{\partial z}\frac{\partial}{\partial \rho}(E_z) - \frac{1}{\rho}j\omega\mu\frac{\partial}{\partial \varphi}(H_z) \right] \\
E_\varphi &= \frac{1}{k^2 - \beta_z^2} \left[\frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial \varphi}(E_z) + j\omega\mu\frac{\partial}{\partial \rho}(H_z) \right] \\
H_\rho &= \frac{1}{k^2 - \beta_z^2} \left[\frac{1}{\rho}(\sigma + j\omega\varepsilon)\frac{\partial}{\partial \varphi}(E_z) + \frac{\partial}{\partial z}\frac{\partial}{\partial \rho}(H_z) \right] \\
H_\varphi &= \frac{1}{k^2 - \beta_z^2} \left[-(\sigma + j\omega\varepsilon)\frac{\partial}{\partial \rho}(E_z) + \frac{1}{\rho}\frac{\partial}{\partial z}\frac{\partial}{\partial \varphi}(H_z) \right]
\end{aligned}
\tag{5b}
$$

3. GREEN'S INTEGRAL THEOREMS

To analyse the AWSA, we will need to develop formulas for E_z and H_z in terms of the electric field distribution in a slot. To do this, we need to establish several integral theorems from Green's functions. Consider the first two scalar functions, $v(\rho', \varphi', z')$ and $w(\rho', \varphi', z')$, which satisfy the homogenous wave equation in cylindrical coordinate in the region R.

$$
\left(\nabla_S^2 + k^2\right) v\left(P'\right) = 0 \qquad \left(\nabla_S^2 + k^2\right) w\left(P'\right) = 0 \tag{6}
$$

In (6), $P'(\rho', \varphi', z')$ is any point in R and

$$\nabla_S = l_\rho \frac{\partial}{\partial \rho'} + l_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi'} + l_z \frac{\partial}{\partial z'}$$

$$\nabla_S^2 w = \frac{1}{\rho'} \frac{\partial}{\partial \rho'} \left(\rho' \frac{\partial}{\partial \rho'} w \right) + \frac{1}{\rho'^2} \frac{\partial^2}{\partial \varphi'^2} w + \frac{\partial^2}{\partial z'^2} w \qquad (7)$$

The use of a standard vector identity gives

$$\nabla_S \cdot (v \nabla_S w) = v \nabla_S^2 w + \nabla_S v \cdot \nabla_S w$$

$$\nabla_S \cdot (w \nabla_S v) = w \nabla_S^2 v + \nabla_S w \cdot \nabla_S v \qquad (8)$$

The difference of these two relations yields

$$\nabla_S \cdot (v \nabla_S w - w \nabla_S v) = v \nabla_S^2 w - w \nabla_S^2 v = v \left(-k^2 w \right) - w \left(-k^2 v \right) = 0 \quad (9)$$

When (7) is integrated throughout a volume V within R, and the divergence theorem is employed, one obtains Green's identity

$$\oint_S (v \nabla_S w - w \nabla_S v) \cdot l_{n'} dS' = 0 \qquad (10)$$

In which $l_{n'}$ is the outward-drawn normal at the surface element dS'. In our analysis n' will be equal to only z' or ρ', therefore

$$\nabla_S w \cdot l_{n'} = \frac{\partial w}{\partial n'} \qquad \nabla_S v \cdot l_{n'} = \frac{\partial v}{\partial n'} \qquad (11)$$

where we interpret n' to be a spatial variable which increases in the direction of $l_{n'}$ at dS'. It follows that (10) can be written in the form

$$\oint_S \left(v \frac{\partial w}{\partial n'} - w \frac{\partial v}{\partial n'} \right) dS' = 0 \qquad (12)$$

Next we wish to expand this result to include the case that w is the Green's function $G(P, P')$, which satisfies the inhomogeneous wave equation

$$\left(\nabla_S^2 + k^2 \right) G \left(P, P' \right) = -\delta \left(P - P' \right) \qquad (13)$$

With $P(x, y, z)$, any particular point in R, and with $\delta \left(P - P' \right)$, the three-dimensional Dirac delta function, defined by

$$\delta \left(P - P' \right) = 0, \quad P' \neq P \qquad \int_{V_\infty} \delta \left(P - P' \right) dV = 1 \qquad (14)$$

The particular solution of (13) is

$$G_P\left(P, P'\right) = \int\limits_V \delta\left(P'' - P\right) \frac{e^{-jkr(P'',P')}}{4\pi r\left(P'', P'\right)} dV\left(P''\right) = \frac{e^{-jkr(P,P')}}{4\pi r\left(P, P'\right)} \quad (15)$$

Whereas the complementary solution of (13) is $G_C\left(P, P'\right)$ which satisfies

$$\left(\nabla_S^2 + k^2\right) G_C\left(P, P'\right) = 0 \quad (16)$$

Thus, the general solution to (13) is

$$G_P\left(P, P'\right) = G_C\left(P, P'\right) + \frac{e^{-jkr(P,P')}}{4\pi r\left(P, P'\right)} \quad (17)$$

In which $G_C\left(P, P'\right)$ is well-behaved at P since it satisfies (16) everywhere in R, including the point P. To apply (12) to the case that $w = G\left(P, P'\right)$ we must erect a small sphere σ, of radius ε, around P as center and exclude the region inside σ. Then (12) becomes

$$\oint_S \left(v\frac{\partial G}{\partial n'} - G\frac{\partial v}{\partial n'}\right) dS'$$

$$= -\oint_\sigma \left(v\frac{\partial G}{\partial n'} - G\frac{\partial v}{\partial n'}\right) \varepsilon^2 d\Omega = \int\limits_0^{4\pi} \left(v\frac{\partial G}{\partial r} - G\frac{\partial v}{\partial r}\right)_{r=\varepsilon} \varepsilon^2 d\Omega$$

$$= \int\limits_0^{4\pi} \left[v\left(\frac{\partial G_C}{\partial r} - jk\frac{e^{-jk\varepsilon}}{4\pi\varepsilon} - \frac{e^{-jk\varepsilon}}{4\pi\varepsilon^2}\right) - \frac{\partial v}{\partial r}\left(G_C + \frac{e^{-jk\varepsilon}}{4\pi\varepsilon}\right)\right] \varepsilon^2 d\Omega \quad (18)$$

As $\varepsilon \to 0$, the right side of (18) approaches the limiting value $-v(P)$. Thus, we get the important result

$$v(P) = \oint_S \left(G\frac{\partial v}{\partial n'} - v\frac{\partial G}{\partial n'}\right) dS' \quad (19)$$

which is known as Green's theorem. If we impose the Dirichlet condition that $G\left(P, P'\right) = 0$ on S, then

$$v(P) = -\oint_S v\left(P'\right) \frac{\partial G\left(P, P'\right)}{\partial n'} dS' \quad (20)$$

whereas if we impose the Neumann condition that $\frac{\partial G}{\partial n'} = 0$ on S, then

$$v(P) = \oint_S \frac{\partial v(P')}{\partial n'} G(P, P') dS' \tag{21}$$

Both of these formulas have been useful in conventional rectangular waveguide Green's function formulation. In our case of the annular waveguide, it will be shown that we will have the compound boundary conditions and will use the general form of (19) to develop our theory.

4. GREEN'S THEOREM APPLIED TO THE ANNULAR WAVEGUIDE

Now we can turn our attention to the waveguide problem. If $E_z(P')$ is a component of any electromagnetic field which can exist in R, then $E_z(P')$ satisfies

$$\left(\nabla_S^2 + k^2\right) E_z(P') = 0 \tag{22}$$

in R and thus is a candidate for $v(P')$. Thus

$$E_z(P) = \oint_S \left(G_1(P, P') \frac{\partial E_z(P')}{\partial n'} - E_z(P') \frac{\partial G_1(P, P')}{\partial n'} \right) dS' \tag{23}$$

In which the Green's function $G_1(P, P')$ has these properties:

a) $G_1(P, P') = G_{C1}(P, P') - \dfrac{e^{-jkr(P,P')}}{4\pi r(P, P')}$

b) $G_1(P, P') = 0$ on $S1$ and $S2$

c) $\dfrac{\partial G_1(P, P')}{\partial n'} = 0$ on $S3$ and $S4$

d) $G_{C1}(P, P')$ is regular in R and satisfies $(\nabla_S^2 + k^2) G_{C1}(P, P') = 0$ for all P' and any P

e) $G_1(P, P')$ satisfies $(\nabla_S^2 + k^2) G_1(P, P') = \delta(P - P')$

Since the dielectric is assumed to be slightly lossy, if the sources creating $E_z(P')$ are all in a finite region, then the contributions to (23) form the end caps at $\varphi = \pm\infty$ are nil and (23) becomes

$$E_z(P) = -\int_{S1} E_z(P') \frac{\partial G_1(P, P')}{\partial n'} dS' - \int_{S2} E_z(P') \frac{\partial G_1(P, P')}{\partial n'} dS'$$

$$+ \int_{S3} G_1(P, P') \frac{\partial E_z(P')}{\partial n'} dS' + \int_{S4} G_1(P, P') \frac{\partial E_z(P')}{\partial n'} dS' \tag{24}$$

Generally, by $S1$ to $S4$ we shall mean the skin-tight surface which lies up against the waveguide walls. To be precise, by $S1$ and $S2$ we mean the faces at $\rho = a$ and $\rho = b$ and by $S3$ and $S4$ we mean the faces at $z = 0$ and $z = b1 = h$ respectively. The details can be found in Fig. 2.

Using the rule that the tangential component and the derivative of normal components of the electric field are zero next to a conductor, it can be verified that the right side of (24) will be always zero except where there are slots. Similarly, we can write

$$H_z(P) = \int\limits_{S1} G_2(P, P') \frac{\partial H_z(P')}{\partial n'} dS' + \int\limits_{S2} G_2(P, P') \frac{\partial H_z(P')}{\partial n'} dS'$$

$$- \int\limits_{S3} H_z(P') \frac{\partial G_2(P, P')}{\partial n'} dS' - \int\limits_{S4} H_z(P') dS' \frac{\partial G_2(P, P')}{\partial n'} dS' \quad (25)$$

In which $G_2(P, P')$ has the following properties:

a) $G_2(P, P') = G_{C2}(P, P') - \dfrac{e^{-jkr(P,P')}}{4\pi r(P, P')}$

b) $G_2(P, P') = 0$ on $S3$ and $S4$

c) $\dfrac{\partial G_2(P, P')}{\partial n'} = 0$ on $S1$ and $S2$

d) $G_{C2}(P, P')$ is regular in R and satisfies $(\nabla_S^2 + k^2)G_{C2}(P, P') = 0$ for all P' and any P

e) $G_2(P, P')$ satisfies $(\nabla_S^2 + k^2)G_2(P, P') = \delta(P - P')$

Also here, using the fact that the normal components of the magnetic field and the normal derivatives of its tangential components vanish near a conductor, it can be interestingly observed that the right side of (25) will also vanish except where there are slots in the waveguide.

5. EXPLICIT EXPRESSIONS FOR THE GREEN'S FUNCTIONS G_1 AND G_2 FOR AN ANNULAR WAVEGUIDE

As the next step, we need to find specific expressions for $G_1(P, P')$ and $G_2(P, P')$. For G_1, we start by rewriting the following equation

$$\nabla^2 G_1 + \beta_0^2 G_1 = \delta(P - P') = \frac{1}{\rho}\delta(\rho - \rho')\,\delta(\varphi - \varphi')\,\delta(z - z') \quad (26)$$
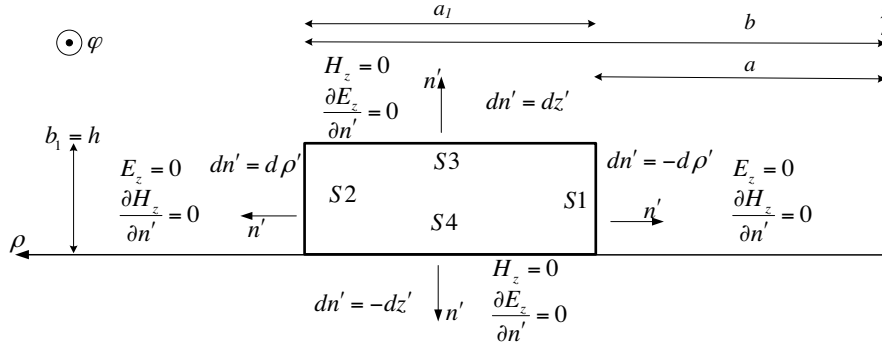
Figure 2. Annular waveguide cross section: visualization of the four different faces in a cross section of the annular waveguide along with the boundary conditions and the normal vector.

In which we can expand G_1 as follows for the waves travelling in positive φ direction and also for $\varphi > \varphi'$

$$G_1 = \sum_{n=1}^{\infty} \int_{m=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} g_1\left(m, \varphi'\right) e^{-jm\varphi} g_{mn}\left(\rho, \rho', z'\right) \cos\left(\frac{n\pi}{h} z\right) dm \quad (27)$$

which satisfies boundary condition in z direction. By replacing (27) in (26) we will have

$$\sum_{n=1}^{\infty} \int_{m=-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \left(\beta_0^2 - \frac{m^2}{\rho^2} - \left(\frac{n\pi}{h}\right)^2 \right) \right] \right.$$

$$\left. \times g_{mn}\left(\rho, \rho', z'\right) g_1\left(m, \varphi'\right) e^{-jm\varphi} \cos\left(\frac{n\pi}{h} z\right) \right) dm$$

$$= \frac{1}{\rho} \delta\left(\rho - \rho'\right) \delta\left(\varphi - \varphi'\right) \delta\left(z - z'\right) \quad (28)$$

By multiplying the resulted equation by $\cos(\frac{t_1\pi}{h} z)$ and using the following equality

$$\int_0^h \cos\left(\frac{n\pi}{h} z\right) \cos\left(\frac{t_1\pi}{h} z\right) dz = \begin{cases} \dfrac{h}{2} & n = t_1 \\ 0 & n \neq t_1 \end{cases}$$

Our equation will be simplified to

$$
\int\limits_{m=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} + \rho \left(\beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \right) - \frac{m^2}{\rho} \right]
$$

$$
\times g_{mn}\left(\rho, \rho'\right) g_1\left(m, \varphi'\right) e^{-jm\varphi} dm = \frac{2}{h} \delta\left(\rho - \rho'\right) \delta\left(\varphi - \varphi'\right) \cos\left(\frac{n\pi}{h} z'\right) \quad (29)
$$

As the next step, we can expand the delta function in terms of spectrum of plane waves as follows

$$
\delta\left(\varphi - \varphi'\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{m=-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{jm\varphi'} \right) e^{-jm\varphi} dm \qquad (30)
$$

Bu using (30) in (29) we will derive

$$
\int\limits_{m=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} + \rho \left(\beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \right) - \frac{m^2}{\rho} \right]
$$

$$
\times g_{mn}\left(\rho, \rho'\right) g_1(m, \varphi') e^{-jm\varphi} dm
$$

$$
= \frac{2}{h} \delta(\rho - \rho') \cos\left(\frac{n\pi}{h} z'\right) \frac{1}{\sqrt{2\pi}} \int\limits_{m=-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{jm\varphi'} \right) e^{-jm\varphi} dm \quad (31)
$$

If the above equation is simplified and the two integrals are combined we will have

$$
\int\limits_{m=-\infty}^{\infty} \left(\left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} + \rho \left(\beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \right) - \frac{m^2}{\rho} \right] g_{mn}\left(\rho, \rho'\right) g_1(m, \varphi') \right.
$$

$$
\left. - \frac{2}{h} \delta\left(\rho - \rho'\right) \cos\left(\frac{n\pi}{h} z'\right) \frac{1}{\sqrt{2\pi}} e^{jm\varphi'} \right) e^{-jm\varphi} dm = 0 \qquad (32)
$$

Which is always satisfied provided

$$
\left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} + \rho \left(\beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \right) - \frac{m^2}{\rho} \right] g_{mn}\left(\rho, \rho'\right) g_1\left(m, \varphi'\right)
$$

$$
- \frac{2}{h} \delta\left(\rho - \rho'\right) \cos\left(\frac{n\pi}{h} z'\right) \frac{1}{\sqrt{2\pi}} e^{jm\varphi'} = 0 \qquad (33)
$$

Reforming the derived equation, we will get the following differential equation

$$\left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} + \rho \left(\beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \right) - \frac{m^2}{\rho} \right] g_{mn} \left(\rho, \rho' \right)$$
$$= \delta \left(\rho - \rho' \right) \cos \left(\frac{n\pi}{h} z' \right) \frac{1}{h} \sqrt{\frac{2}{\pi}} \frac{e^{jm\varphi'}}{g_1 \left(m, \varphi' \right)} \qquad (34)$$

This is a one dimensional equation which homogeneous form can be written as

$$\left[\rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} + \rho^2 \left(\beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \right) - m^2 \right] g_{mn} \left(\rho, \rho' \right) = 0 \qquad (35)$$

or

$$\left[\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} + \rho \left(\beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \right) - \frac{m^2}{\rho} \right] g_{mn} \left(\rho, \rho' \right) = 0 \qquad (36)$$

This is actually a Sturm-Liouville with the following parameters

$$P(\rho) = \rho, \quad q(\rho) = \frac{m^2}{\rho}, \quad r(\rho) = \rho, \quad \lambda = \beta_\rho^2 = \beta_0^2 - \left(\frac{n\pi}{h} \right)^2 \qquad (37)$$

The two possible solutions can be written as follows

$$\begin{cases} g_{mn}^1 = A_{mn} J_m(\beta_\rho \rho) + B_{mn} Y_m(\beta_\rho \rho) & \text{for} \quad a < \rho < \rho' \\ g_{mn}^2 = C_{mn} J_m(\beta_\rho \rho) + D_{mn} Y_m(\beta_\rho \rho) & \text{for} \quad \rho' < \rho < b \end{cases} \qquad (38)$$

By applying proper boundary conditions to each of the equations we will have

$$\begin{cases} g_{mn}^1 = A_{mn} \left[J_m(\beta_\rho \rho) - \frac{J_m(\beta_\rho a)}{Y_m(\beta_\rho a)} Y_m(\beta_\rho \rho) \right] & \text{for } a < \rho < \rho' \\ g_{mn}^2 = C_{mn} \left[J_m(\beta_\rho \rho) - \frac{J_m(\beta_\rho b)}{Y_m(\beta_\rho b)} Y_m(\beta_\rho \rho) \right] & \text{for } \rho' < \rho < b \end{cases} \qquad (39)$$

To form the final solution we should use the following definition for Wronskian

$$W = y_1 \left(x' \right) y_2' \left(x' \right) - y_2 \left(x' \right) y_1' \left(x' \right) \qquad (40)$$

And use it in the following well known equation

$$G = \begin{cases} \dfrac{y_2 \left(x' \right)}{P \left(x' \right) W \left(x' \right)} y_1(x) \\ \dfrac{y_1 \left(x' \right)}{P \left(x' \right) W \left(x' \right)} y_2(x) \end{cases} \qquad (41)$$

By replacing

$$y_1 = g_m^1 \qquad y_2 = g_m^1 \tag{42}$$

We will derive the Wronskian as

$$W(\rho') = \beta_\rho A_{mn} C_{mn} \left[\frac{J_m(\beta_\rho b)}{Y_m(\beta_\rho b)} - \frac{J_m(\beta_\rho a)}{Y_m(\beta_\rho a)} \right]$$
$$\times \left\{ J_m'\left(\beta_\rho \rho'\right) Y_m\left(\beta_\rho \rho'\right) - J_m\left(\beta_\rho \rho'\right) Y_m'\left(\beta_\rho \rho'\right) \right\} \tag{43}$$

By using the following definition the above equation can be simplified

$$J(\alpha x) Y'(\alpha x) - Y(\alpha x) J'(\alpha x) = \frac{2}{\pi \alpha x} \tag{44}$$

The result can be simplified as

$$W(\rho') = \frac{-2}{\pi} A_{mn} C_{mn} \left[\frac{J_m(\beta_\rho b)}{Y_m(\beta_\rho b)} - \frac{J_m(\beta_\rho a)}{Y_m(\beta_\rho a)} \right] \frac{1}{\rho'} \tag{45}$$

Now, we are able to form the desired g_{mn} as

$$g_{mn} = -\frac{1}{h}\sqrt{\frac{\pi}{2}} \frac{e^{jm\varphi'}}{g_1(m,\varphi')} \frac{\cos\left(\dfrac{n\pi}{h}z'\right)}{\left[\dfrac{J_m(\beta_\rho b)}{Y_m(\beta_\rho b)} - \dfrac{J_m(\beta_\rho a)}{Y_m(\beta_\rho a)}\right]}$$

$$\times \left\{ \begin{array}{l} \left[J_m\left(\beta_\rho \rho'\right) - \dfrac{J_m(\beta_\rho b)}{Y_m(\beta_\rho b)} Y_m\left(\beta_\rho \rho'\right) \right] \\[4mm] \left[J_m\left(\beta_\rho \rho'\right) - \dfrac{J_m(\beta_\rho a)}{Y_m(\beta_\rho a)} Y_m\left(\beta_\rho \rho'\right) \right] \end{array} \right.$$

$$\times \left\{ \begin{array}{ll} \left[J_m(\beta_\rho \rho) - \dfrac{J_m(\beta_\rho a)}{Y_m(\beta_\rho a)} Y_m(\beta_\rho \rho) \right] & \text{for } a < \rho < \rho' \\[4mm] \left[J_m(\beta_\rho \rho) - \dfrac{J_m(\beta_\rho b)}{Y_m(\beta_\rho b)} Y_m(\beta_\rho \rho) \right] & \text{for } \rho' < \rho < b \end{array} \right. \tag{46}$$

Which can be written in a more simplified form as

$$g_{mn} = -\frac{1}{h}\sqrt{\frac{\pi}{2}} \frac{e^{jm\varphi'}}{g_1(m,\varphi')} \frac{\cos\left(\dfrac{n\pi}{h}z'\right)}{J_m(\beta_\rho b)Y_m(\beta_\rho a) - J_m(\beta_\rho a)Y_m(\beta_\rho b)}$$

$$\times \left\{ \begin{array}{l} [J_m\left(\beta_\rho \rho'\right) Y_m(\beta_\rho b) - J_m(\beta_\rho b) Y_m\left(\beta_\rho \rho'\right)] \\ [J_m\left(\beta_\rho \rho'\right) Y_m(\beta_\rho a) - J_m(\beta_\rho a) Y_m\left(\beta_\rho \rho'\right)] \end{array} \right.$$

$$\left\{ \begin{array}{l} [J_m(\beta_\rho \rho)Y_m(\beta_\rho a) - J_m(\beta_\rho a)Y_m(\beta_\rho \rho)], \ a < \rho < \rho' \\ [J_m(\beta_\rho \rho)Y_m(\beta_\rho b) - J_m(\beta_\rho b)Y_m(\beta_\rho \rho)], \ \rho' < \rho < b \end{array} \right. \tag{47}$$

To have more compact results, we define the following functions

$$\psi_m(x,y) = J_m(x)Y_m(y) - J_m(y)Y_m(x)$$
$$\psi_m^1(x,y) = J_m(x)Y_m'(y) - J_m'(y)Y_m(x) \tag{48}$$
$$\psi_m^2(x,y) = J_m'(x)Y_m'(y) - J_m'(y)Y_m'(x)$$

Using the above functions, the desire G_1 can be written as following for both positive and negative φ direction traveling waves

$$
\begin{aligned}
G_1 = \sum_{n=1}^{\infty} \int_{m=-\infty}^{\infty} &\left(-\frac{1}{2h} e^{jm|\varphi-\varphi'|} \frac{\cos\left(\frac{n\pi}{h}z'\right)\cos\left(\frac{n\pi}{h}z\right)}{\psi_{mn}(\beta_\rho b, \beta_\rho a)} \right.\\
&\left. \times \begin{cases} \psi_{mn}(\beta_\rho\rho', \beta_\rho b)\psi_{mn}(\beta_\rho\rho, \beta_\rho a) & \text{for } a < \rho < \rho' \\ \psi_{mn}(\beta_\rho\rho', \beta_\rho a)\psi_{mn}(\beta_\rho\rho, \beta_\rho b) & \text{for } \rho' < \rho < b \end{cases} \right) dm
\end{aligned} \tag{49}
$$

By following the same procedure, one can also derive G_2 as

$$
\begin{aligned}
G_2 = \sum_{n=1}^{\infty} \int_{m=-\infty}^{\infty} &\left(-\frac{1}{2h} e^{jm|\varphi-\varphi'|} \frac{\sin\left(\frac{n\pi}{h}z'\right)\sin\left(\frac{n\pi}{h}z\right)}{\psi_{mn}^2(\beta_\rho b, \beta_\rho a)} \right.\\
&\left. \times \begin{cases} \psi_{mn}^1(\beta_\rho\rho', \beta_\rho b)\psi_{mn}^1(\beta_\rho\rho, \beta_\rho a) & \text{for } a < \rho < \rho' \\ \psi_{mn}^1(\beta_\rho\rho', \beta_\rho a)\psi_{mn}^1(\beta_\rho\rho, \beta_\rho b) & \text{for } \rho' < \rho < b \end{cases} \right) dm
\end{aligned} \tag{50}
$$

6. FINAL FORMS FOR $E_Z(P)$ & $H_Z(P)$ IN THE ANNULAR WAVEGUIDE WITH A SLOT IN THE TOP BROAD FACE ($S3$)

As it was stated before, the main goal of this discussion is to find the proper forms for the components of the electromagnetic fields needed for method of moment analysis of an annular waveguide slot antenna. In other words, we are interested in finding the scattered fields due to the unknown field distribution in the slot. Considering that we will be involved in solving the problem of a slot cut in the broad face of the annular waveguide $S3$, in this part, the final forms for the main required components will be formed.

Combining (24) and (49) we will derive

$$
E_z(P) = \int_{\substack{slot \\ \rho',\varphi',z'}} \sum_{n=1}^{\infty} \int_{m=-\infty}^{\infty} \left(-\frac{1}{2h} e^{jm|\varphi-\varphi'|} \frac{\cos\left(\frac{n\pi}{h}z'\right)\cos\left(\frac{n\pi}{h}z\right)}{\psi_{mn}(\beta_\rho b, \beta_\rho a)} \right.
$$

$$\times \left\{ \begin{array}{l} \psi_{mn}(\beta_\rho \rho', \beta_\rho b)\psi_{mn}(\beta_\rho \rho, \beta_\rho a) \text{ for } a < \rho < \rho' \\ \psi_{mn}(\beta_\rho \rho', \beta_\rho a)\psi_{mn}(\beta_\rho \rho, \beta_\rho b) \text{ for } \rho' < \rho < b \end{array} \right) dm \frac{\partial E_z\left(P'\right)}{\partial z'} dS' \quad (51)$$

By the same way, combining (25) and (50) we will derive

$$H_z(P) = - \int\limits_{\substack{slot \\ \rho',\varphi',z'}} \sum_{n=1}^{\infty} \int\limits_{m=-\infty}^{\infty} H_z\left(P'\right) \left(\frac{n\pi}{h}z'\right)^2$$

$$\left(-\frac{1}{2h} e^{jm|\varphi-\varphi'|} \frac{\cos\left(\dfrac{n\pi}{h}z'\right)\cos\left(\dfrac{n\pi}{h}z\right)}{\psi_{mn}^2(\beta_\rho b, \beta_\rho a)} \right.$$

$$\left. \times \left\{ \begin{array}{l} \psi_{mn}^1(\beta_\rho \rho', \beta_\rho b)\psi_{mn}^1(\beta_\rho \rho, \beta_\rho a) \text{ for } a < \rho < \rho' \\ \psi_{mn}^1(\beta_\rho \rho', \beta_\rho a)\psi_{mn}^1(\beta_\rho \rho, \beta_\rho b) \text{ for } \rho' < \rho < b \end{array} \right) dm dS' \quad (52) \right.$$

Equations (51) and (52) will play a central role in our analysis of the behaviour of slots cut in the broad wall of an annular waveguide.

7. APPLICATION REVIEW: MOM ANALYSIS OF AWSA

Using the derived theory, we are now able to construct a set of integral equations which solution will lead to the desired unknown field distribution in the slot. Unlike the conventional rectangular case, and due to limitation in scalar wave equation for field components in cylindrical coordinate, we should form the integral equation in terms of both electric and magnetic normal field components at the slot. Luckily, each of the electric or magnetic field components in (51) and (52) are dependent on only one of the field components at the slot and we will have two independent integral equations as follows which can be directly solved by methods such as MoM.

$$H_z^{Scattered}\left(H_z\left(P'\right)\right) + H_z^{Incident} = H_z^{Radiated}\left(H_z\left(P'\right)\right)$$
$$E_z^{Scattered}\left(E_z\left(P'\right)\right) + E_z^{Incident} = E_z^{Radiated}\left(E_z\left(P'\right)\right) \quad (53)$$

having solved the above equations, we will be able to solve for all desired field components in the slot.

8. DISCUSSION ON THE ANNULAR DIRECTION OF PROPAGATION IN CYLINDRICAL COORDINATE

As it was seen in the paper, the periodic variable 'ϕ' in cylindrical coordinate is treated as the main axis that the wave is guided through

its extension which extends to plus and minus infinity. As a result, the two ends of the antenna are assumed to be completely isolated from each other. This is a good approximation for an AWSA with large inner radius. When the two ends are not so far from each other, this approximation may not be completely valid and there will be some errors in the results. This error may be mainly due to the higher order modes produced from the interaction of the annular waveguide ends with the transitions to rectangular or coaxial waveguides for excitation purposes.

Generally, the aimed application for such annular waveguide will be an annular waveguide slot array antenna with various annular waveguides. As in conventional slot array antennas, such array antennas will be mainly used in a Monopulse system with four quadrants. So, the annular extension of our annular waveguide is supposed to be fixed to '$\pi/2$' which is a 1/4 of the total circle. On the other hand, regarding the radiation characteristics, since we are going to control the polarization of the final array antenna, it was shown in our recent papers [12, 13] that we will need to have control on the phase excitation of each quadrant separately. The reason for this fact is if you simply excite all the slots in a single waveguide with '2π' extension, each set of four slots in the four quadrant will cancel each other at far field and there will be no final radiation. But, by having isolated '$\pi/2$' waveguides and controlling the corresponding phases, we will be able to have each desired component of the total radiated field and as a result we will be able to have linear horizontal or vertical or circular right or left hand polarization. This is the other results for having waveguides with '$\pi/2$' annular extension.

Although even we have this '$\pi/2$' extension, there will be some small deviation in the actual case from the proposed theory. Actually, in the initial simulation and measurement experiences [3], we have connected the '$\pi/2$' annular waveguide to rectangular/coaxial transitions. Since the main modes of the annular and rectangular waveguide have a very near field distribution [11], there will be a good transition of the energy between the modes. This concept is under more study and we more theoretical description will be provided in the near future.

9. CONCLUSION

In this paper, the required Green's function formulation of an annular waveguide for future method of moment analysis of annular waveguide slot antennas is presented. Using the derived Green's functions, proper integral equations in terms of unknown field components at slot are also

presented. As one of the main important differences with rectangular case derived from the results of this paper, the final integral equations will be in terms of normal components of the field at the slot instead of conventional transverse components. Deriving both $E_z(P)$ and $H_z(P)$ at the slot, one can consequently derive other transverse components as well.

Future reports will focus on the solutions of the proposed integral equations using the method of moment. The detailed theory and formulation proposed in this paper will be the basis for proper solutions resulted from MoM which will result in information about circuit modeling, scattering and radiation of the novel annular waveguide slot antennas.

The another important topic to be discussed in the future reports is the mode conversion of the incident wave. From the theoretical point of view, there is no such problem since we are assuming that there is an incident wave with our desired distribution which will be the result of an infinitely extending annular waveguide. But in actual case of measurement experience, which is already started with the initial results provided in [3], it is needed to connect the annular waveguide to a rectangular cross section adaptor. Therefore there will be the mode conversion, which of course will not be so much due to the high similarity of the dominant modes between rectangular and annular waveguide as was studied with the results provided in [11]. One solution to this problem will be to excite the structure with coupling slots from the behind (the broad face of the waveguide without radiating slots) or with coaxial waveguides again from the same described face. In this case we can exactly predict the measurement results. More research is going on regarding the excitation of AWSA and future reports will have more discussions on this fact.

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