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A DISCRETE TIME ELECTROMAGNETIC FORMULIZATION AND ITS APPLICATIONS

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Abstract—This paper presents a formulization of time domain (TD) discrete electromagnetic (EM) solution to provide an effective analysis over a variety of EM problems. This procedure first represents the time responses of EM fields in terms of a set of selected basis functions. Their coefficients are then employed to create matrix-form Maxwell's equations with forms analogous to frequency domain (FD) Maxwell's equations, which allows one to formulate TD solutions via utilizing their corresponding FD solutions that are existing and generally much mature. This work provides general characteristics with parts previously described in the discrete-time EM theory [20, 21] and, most importantly, provides rigorous theoretical derivations in formulating the TD solutions.

1. INTRODUCTION

The development of time domain (TD) solutions for the electromagnetic (EM) transient analysis retains interested because many problems are more nature to do the transient analysis in TD directly. However, in the past, the development of frequency domain (FD) solutions are found to be more mature than those in TD because of their capability to simplify the differential form of Maxwell's equations and reduce the analysis complexity.

Due to the needs of TD solutions, many efforts have been applied [1–19]. Typical TD approaches include using fast Fourier transform (FFT) or analytical time transform (ATT) [16, 17] to directly inverse FD solutions into TD, finite difference time domain (FDTD) [18], TD integral approaches [19] and TD uniform geometrical theory of diffractions (TD-UTD) [13–16]. Those approaches have been found to face some difficulties and challenges in their practical applications including computational inefficiency and not physical appealing in wave phenomena interpretations as having been widely discussed in the literatures. In particular, the most difficult challenge is that closed-form EM solutions are extremely difficult to be found in TD solutions.

Due to the maturity of FD solutions, it is motivated to develop techniques that allow one to take advantages of FD solutions in solving TD problems directly. This concept was first initiated by [20] (referred as a discrete-time (DT) EM theory), and later extended in [21] to develop a TD UTD formulation (referred as DT-UTD) for treating a transient analysis of scattering from curved wedges. The work in [20, 21] were found to perform well in some examples. Unfortunately, they were developed by a conjecture.

First, it was short of theoretic supports or rigorously derived procedures to formulate the DT EM solutions even though a formulation of plane wave propagation was symbolically derived in [21]. In particular, the duality relationship of the angular frequency in comparing FD and matrix-type Maxwell's equations was found from an observation without showing how the TD functions will be computed using this new matrix parameter. Second, the number of time segments using time discretization, which equals the dimension of the matrix parameter, will become extremely large if the problems under analysis are geometrically large, and subsequently cause an undurable cumbersome computation of matrix operations. Third, the matrices of angular frequency defined in [21] appears either nearly singular or complex eigen-values, which results in potential difficulties in computing special functions in the practical applications since most FD solutions were well developed using real angular frequencies.

This paper intends to resolve the above mentioned problems and generalizes the applicable scopes of this DT EM theory. It allows utilizing arbitrary, but continuous, basis functions to represent the TD fields' responses, and also obtains same matrix-type Maxwell's equations except the fields' matrix's elements are now formed by the expansion coefficients instead of fields' values at sampled times. This new definition of matrices' elements is advantageous in the practical applications because now the matrix size is determined by the number of basis functions instead of time segments, which may increase the computational efficiency in solving many practical problems. In particular, in this development the procedure to formulate the TD solutions are rigorously derived to theoretically support the duality relationship. Thus if the set of basis functions is complete, it can be shown the TD solutions are also exact.

The paper is organized as follows. Section 2 describes the fundamental derivations of this DT EM theory using a set of arbitrary basis functions to represent the TD response. In Section 3 the procedure to formulate TD solutions from their corresponding FD solutions are rigorously derived. In particular, the time functions' computations using the matrix angular frequency are shown with duality relationships between TD and FD. In Section 4 the matrix angular frequencies for two sets of time basis functions that are commonly used in the EM numerical analysis are derived, which are shown to have real eigen-values. Examples are presented in Section 5 to demonstrate the utilization of this generalized DT EM theory. Finally a short discussion is presented in Section 6 as a conclusion.

The FD EM solutions employed in this paper follow an $e^{j\omega t}$ time convention.

2. GENERAL TIME REPRESENTATION USING DIFFERENTIAL MAXWELL'S EQUATIONS

The time responses of EM fields at any space position, $\bar{r}$, are represented in terms of a set of basis functions by

$$\bar{A}(x,y,z,t) = \sum_{n=1}^{N} \bar{A}_n(x,y,z)B_n(t) \tag{1}$$

where $\bar{A}$ can be either $\bar{E}$, $\bar{H}$, $\bar{M}$ or $\bar{J}$, and $\Psi = \{B_n(t),\, n = 1 \sim N\}$ is a solution space with $B_n(t)$ being its nth time basis function. Assuming $\bar{A}$ satisfying the requirements of a regular Sturm-Liouville theory [22] with constant coefficients for the second order differential equations, $\bar{A}_n(x,y,z)$ can be solved by testing (1) with $B_n(t)$ ($n = 1 \sim N$) using a general formulation of dot product over time functions by

$$f(t)\cdot g(t) = \int_a^b f(t)g^*(t)dt \tag{2}$$

where $[a\ b]$ is the definition domain and "$*$" in (2) represents a complex conjugate. Thus $\bar{A}_n(x,y,z)$ can be obtained by solving the following matrix equation:

$$\begin{bmatrix} \bar{A}\cdot B_1 \\ \bar{A}\cdot B_2 \\ \vdots \\ \bar{A}\cdot B_{N-1} \\ \bar{A}\cdot B_N \end{bmatrix} = \begin{bmatrix} B_1\cdot B_1 & B_2\cdot B_1 & & B_{N-1}\cdot B_1 & B_N\cdot B_1 \\ B_1\cdot B_2 & B_2\cdot B_2 & & B_{N-1}\cdot B_2 & B_N\cdot B_2 \\ \vdots & & \ddots & & \vdots \\ B_1\cdot B_{N-1} & & & B_{N-1}\cdot B_{N-1} & \\ B_1\cdot B_N & \cdots & & \cdots & B_N\cdot B_N \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \vdots \\ \bar{A}_{N-1} \\ \bar{A}_N \end{bmatrix} \tag{3}$$

Note that if Ψ is complete, then (3) results in exact solutions. Otherwise, Ψ is a subspace and (3) represents a projection of the exact solution in this subspace.

Equations (1)–(3) may be employed to solve TD Maxwell's equations. Considering one of the Maxwell's two curl equations, given by

$$\nabla\times\bar{E}(x,y,z,t) = -\frac{\partial}{\partial t}\mu\bar{H}(x,y,z,t) - \bar{M}(x,y,z,t), \tag{4}$$

and substituting (1) into this equation give

$$\sum_{n=1}^{N}\nabla\times\bar{E}_n(x,y,z)B_n(t) = -\sum_{n=1}^{N}\mu\bar{H}_n(x,y,z)C_n(t) - \sum_{n=1}^{N}\bar{M}(x,y,z)B_n(t) \tag{5}$$

where

$$C_n(t) = \frac{d}{dt}B_n(t). \tag{6}$$

Using $B_m(t)$ $(m = 1 \sim N)$ as testing functions and employing (2) to solve (5) lead to the following equations that are functions of time expansion coefficients:

$$\sum_{n=1}^{N}\nabla\times\bar{E}_n(x,y,z)P_{nm} = -\sum_{n=1}^{N}\mu\bar{H}_n(x,y,z)Q_{nm} - \sum_{n=1}^{N}\bar{M}_n(x,y,z)P_{nm} \tag{7}$$

where

$$\begin{cases} P_{nm} = B_n(t)\cdot B_m(t) \\ Q_{nm} = C_n(t)\cdot B_m(t) \end{cases}. \tag{8}$$

In (8) the dot products are performed via (2). Note that in obtaining (7), it was assumed that the derivatives of electrical and magnetic fields are also sufficiently represented by $B_n(t)$ $(n = 1 \sim N)$. Otherwise, $C_n(t)$ in (6) should be also included in Ψ as basis functions, and used to test (5). Equation (7) can be symbolically expressed in a matrix form by

$$[P_{mn}]_{N\times N}\left(\nabla\times[\bar{E}_n]_{N\times 1}\right)=-[Q_{mn}]_{N\times N}\left[\mu\bar{H}_n\right]_{N\times 1}-[P_{mn}]_{N\times N}\left[\bar{M}_n\right]_{N\times 1} \tag{9}$$

or

$$\nabla\times[\bar{E}_n]_{N\times 1}=-j\,[W_{mn}]_{N\times N}\left[\mu\bar{H}_n\right]_{N\times 1}-\left[\bar{M}_n\right]_{N\times 1} \tag{10}$$

with

$$[W_{nm}]_{N\times N}=-j\left([P_{mn}]_{N\times N}\right)^{-1}[Q_{mn}]_{N\times N}\,. \tag{11}$$

It is observed that $[P_{mn}]_{N\times N}$ in (11) is identical to the $N\times N$ square matrix on the right-hand side of (3), which may be recognized as a normalization factor in the space expansion. Thus if $B_n(t)$ is an orthogonal set, then $[P_{mn}]_{N\times N}$ will be a diagonal matrix. Nevertheless, if $B_n(t)$ is an orthonormal set, then $[P_{mn}]_{N\times N}$ will be an unit identity matrix. Thus $[W_{mn}]_{N\times N}$ in (11) is primarily determined by $[Q_{mn}]_{N\times N}$ or the slope of $\bar{H}(x,y,z,t)$ with respect to its differentiation by t. Several phenomena can be observed:

(i) Equations (9)–(11) are in forms identical to (8)–(10) in [21], where integral form Maxwell's equations were used, except now the elements in $[W_{mn}]_{N\times N}$ are alternatively defined by (8) instead of (7) in [21] (also shown in (B1) of Appendix B). As to be shown in Section 4, both definitions result in a same matrix if time-harmonic basis functions are used. Otherwise, different values may occur and cause computational errors in practical applications. However, the current definition appears a better interpretation in the procedure of constructing DT EM theory.

(ii) As mentioned in the last part of first paragraph, if Ψ is a subspace, the solutions obtained by (3) represent a projection of the complete solution in this subspace. Thus (9) tends to link the fields' relationship at any position within this subspace. In another words, the solutions obtained by (9) provide exact solutions for every component of fields in this subspace. When the subspace becomes complete (or become the solution space), then the solutions also become exact.

(iii) Equation (9) allows one to find the coefficients of a time basis expansion defined in (1) that are now location-dependent in space coordinates. The correct time responses should be obtained by a superposition of the coefficients multiplied with the time basis functions, $B_n(t)$.

Apparently (9) has a form identical to FD Maxwell curl equations if one considers a dual relationship by replacing the angular frequency, ω, with $[W_{mn}]_{N\times N}$, which is now a matrix. This dual relationship appears a possibility to use existing FD formulations of EM solutions to solve transient analysis of the same problems with a proper translation and interpretation of the associated notations.

3. FD SOLUTION ESTABLISHMENT VIA DUAL RELATIONSHIPS WITH FOURIER TRANSFORM

The properties of this DT EM theory are characterized in this section, which allow one to establish TD EM solutions from their corresponding FD solutions in a rigorous way. The possibility is explored based on a fact that TD solutions can be generally obtained by applying an inverse FT over their solutions in FD. Thus if the dual relationship between ω with $[W_{mn}]_{N\times N}$ leads to perform FT's properties (summarized in Table 1) governing FT's operations over time and frequency functions, then the DT EM solutions' establishment from FD formulations becomes evident. Those properties, corresponding to each property in Table 1, are summarized in the following discussions with some derivations shown in the appendix.

Table 1. Relationships of FT.

No.	$f(t)$	FT, $F(\omega)$
1	$f'(t)$	$(j\omega)F(\omega)$
2	$f^{(n)}(t)$	$(j\omega)^n F(\omega)$
3	$f(t-t_0)$	$e^{-j\omega t_0}F(\omega)$

(i) If $f(t)$ and its derivative, $f'(t)$, are well represented via (1) with coefficients, $[a_n]_{N\times 1}$ and $[c_n]_{N\times 1}$, respectively, it is shown in Appendix A1 that

$$[c_n]_{N\times 1} = j[W_{nm}]_{N\times N}[a_m]_{N\times 1} \tag{12}$$

(ii) Similar to (i), if $f^{(q)}(t)$ has expansion coefficients, $[d_n]_{N\times 1}$, then it is shown in Appendix A2 that

$$[d_n]_{N\times 1} = (j[W_{nm}]_{N\times N})^q[a_m]_{N\times 1} \tag{13}$$

(iii) Let $[e_n]_{N\times 1}$ represent the expansion coefficients of $f(t-t_0)$. It is shown in Appendix A3 that

$$[e_n]_{N\times 1} = e^{-j[W_{nm}]_{N\times N}t_0}[a_m]_{N\times 1} \tag{14}$$

(iv) $h(t) = g(t) \otimes f(t)$ (Convolution theorem): Let $[h_n]_{N\times 1}$ represent the expansion coefficients of $h(t)$, then it is shown in Appendix A4 that

$$[h_n]_{N\times 1} = G([W_{nm}]_{N\times N})[a_n]_{N\times 1} \tag{15}$$

where $G(\omega)$ is the Fourier transform of $g(t)$ and $G([W_{nm}]_{N\times N})$ is a $N \times N$ matrix obtained by replacing ω with $[W_{mn}]_{N\times N}$. Thus as illustrated in Figure 1, if $g(t)$ is an impulse response and $G(\omega)$ is the solution of FD Maxwell's equations, its convolution in TD with a realistic time pulse, $f(t)$, to give general pulse-excited fields can be performed by its matrix form multiplied by the expansion coefficients of excitation pulse.

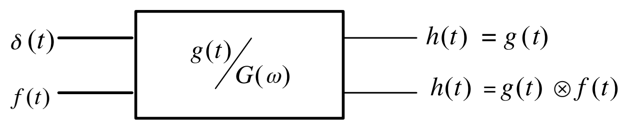


Figure 1. System diagram to show its response to a general input time pulse when the impulse response is available.

(v) Moreover, if $g(t)$ is a response to a time step input function, the response to a general pulse excitation can be expressed as $h(t) = g'(t) \otimes f(t)$. In this case, (15) becomes $[h_n]_{N\times 1} = j[W_{nm}]_{N\times N}G([W_{nm}]_{N\times N})[a_n]_{N\times 1}$.

The above summarized properties have concluded that the FD formulations expressed in a matrix form of expansion coefficients perform field computations in TD. These are particularly important in the implementation for TD-UTD solutions, which are primarily dependent on parameters determined by the ray propagation paths, because they are most used in the computations of UTD formulations.

4. $[W_{mn}]_{N\times N}$ OF COMMONLY USED BASIS FUNCTIONS

Apparently $[W_{mn}]_{N\times N}$ in (11) is primarily determined by $B_n(t)$ and their derivatives as shown in (8). This section presents the values of $[W_{mn}]_{N\times N}$ for two well-known types of time basis functions commonly used in the numerical EM analysis, which allows one to further visualize its characteristics.

(i) $B_n(t) = e^{j\omega_n t}(\omega_n \neq \omega_m \text{ for } n \neq m)$:
In this case, $B_n(t)$ are global functions that form an orthogonal set. Each basis is a time harmonic function with ω_n being referred as an angular frequency when it is used alone in the time harmonic EM analysis to come out FD solutions. Here it is found that

$$[W_{mn}]_{N\times N} = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \omega_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & \omega_N \end{bmatrix} \tag{16}$$

regardless the definition domain $t \in [a\ b]$, and is a real diagonal matrix formed by its eigen-values. Note that using (B1) to compute $[W_{mn}]_{N\times N}$ will result in a same formulation.

(ii) $B_n(t) = \Lambda_n(t)$ where $\Lambda_n(t)$ are triangular basis functions shown in Figure 2.

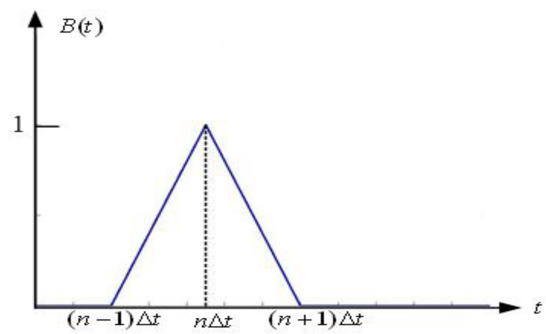


Figure 2. A triangular pulse functions.

The advantages of using these basis functions are that $f(t = n\Delta t)$ equal to the coefficients, $[a_n]_{N\times 1}$, which enables the definition of "Discrete-time EM theory" in the previous works [20, 21]. In another words, the solutions of (10) directly give the TD fields' values at the sampled times, $t = n\Delta t$, where Δt is a time interval.

Here it is found that

$$[W_{mn}]_{N\times N} = \frac{-j}{\Delta t}\begin{bmatrix} 2/3 & 1/6 & 0\cdots & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & \vdots \\ 0 & 1/6 & \ddots & & 0 \\ \vdots & 0 & \ddots & 2/3 & 1/6 \\ 0 & 0 & \cdots & 1/6 & 2/3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1/2 & \cdots & 0 & 0 \\ -1/2 & 0 & & 0 & 0 \\ 0 & -1/2 & \ddots & 1/2 & \vdots \\ \vdots & 0 & \ddots & 0 & 1/2 \\ 0 & 0 & \cdots & -1/2 & 0 \end{bmatrix} \quad (17)$$

Numerical experiments show that this matrix has real eigenvalues and complex eigenvectors, which is found more advantageous than (B2) obtained by using (7) in [21] because it results in either complex eigen-values. Note that other matrices in [21] appear nearly singular. Several cases of eigen-values with different matrix dimensions are shown in Tables 1 and 2, where $\Delta t = 1$ is assumed. Real eigen-values are important in the practical computations of TD solutions because their corresponding FD solutions were well developed using real angular frequencies and have mature computational characteristics. Complex eigen-values will increase the computational complexity and efforts in practical applications.

Table 2. Eigen-values of $[W_{mn}]_{N\times N}$ in (17) for $N = 6 \sim 9$ ($\Delta t = 1$).

N	1	2	3	4	5
6	±15.138	±9.8429	±3.3587		
7	±15.625	±11.339	±5.8483	0	
8	±15.968	±12.439	±7.746	±2.6146	
9	±16.217	±13.269	±9.2241	±4.6916	0

In addition to above described basis functions, basis functions of shifted sinc functions also give real eigen-values. The eigen-value and vector decomposition with real eigen-values increases the visualization to validate the DT-UTD solutions in [21] because its high frequency criterion ($\omega \to \infty$) to produce FD-UTD solutions can be visualized by the eigen-values. In (16) it requires $\omega_n \to \infty$ while in (17) it requires sufficiently small Δt ($[W_{mn}]_{N\times N}$ is proportional to $\frac{1}{\Delta t}$ in (17)).

Table 3. Eigen-values of $[W_{mn}]_{N\times N}$ in (B2) for $N = 6 \sim 9$ ($\Delta t = 1$).

N	1	2	3	4	5
6	(± 15.756,-9.9753)	(± 9.979,-10.748)	(± 3.3386,-11.215)		
7	(± 16.324,-9.8579)	(± 11.602,-10.509)	(± 5.8722,-10.989)	(0,-11.164)	
8	(± 16.722,-9.7781)	(± 12.812,-10.329)	(± 7.8451,-10.789)	(± 2.619,-11.041)	
9	(± 17.012,-9.7217)	(± 13.733,-10.191)	(± 9.4072,-10.622)	(± 4.7298,-10.903)	(0,-11.003)

5. DEMONSTRATING EXAMPLES

The formulation of a plane wave propagation in an unbounded isotropic homogeneous medium using (10) has been rigorously derived in [21] and is employed in this section to demonstrate the utilization of the concepts developed in this paper. According to [21], the plane wave propagating from $\bar{r}_1$ to $\bar{r}_2$ can be expressed as

$$\left[\bar{E}_m(\bar{r}_2)\right]_{N\times 1} = e^{-j[K_{nm}]_{N\times N}\left(\hat{k}\cdot(\bar{r}_2-\bar{r}_1)\right)}\left[\bar{E}_n(\bar{r}_1)\right]_{N\times 1} \tag{18}$$

where $[K_{nm}]_{N\times N} = \sqrt{\mu\varepsilon}\,[W_{nm}]_{N\times N}$ and $\hat{k}$ indicates the direction of propagation. It is assumed that the TD response at $\bar{r}_1$ is given by

$$\bar{E}(\bar{r}_1, t) = \hat{e}\cos(\omega t). \tag{19}$$

where $t \in [-\infty\ \infty]$. It is well-known that the TD response at $\bar{r}_2$ is identical to (18) with an additional time delay, and can be described as

$$\bar{E}(\bar{r}_2, t) = \hat{e}\cos(\omega t - k\ell) \tag{20}$$

where $k = \omega\sqrt{\mu\varepsilon}$ and ℓ is the distance between $\bar{r}_1$ and $\bar{r}_2$. Cases with different types of basis functions are discussed as follows.

(i) $\Psi = \left\{e^{-j\omega t}, e^{j\omega t}\right\}$ with $N = 2$:
In this case, $\left[\bar{E}_n(\bar{r}_1)\right]_{2\times 1} = [0.5\ 0.5]^T$ by using (3), and the $[W_{nm}]_{2\times 2}$ is given by

$$[W_{nm}]_{2\times 2} = \begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix}. \tag{21}$$

Utilizing (18) gives

$$\left[\bar{E}_m(\bar{r}_2)\right]_{2\times 1} = e^{-j[K_{nm}]_{2\times 2}\ell}\left[\bar{E}_n(\bar{r}_1)\right]_{2\times 1} = \frac{1}{2}\begin{bmatrix} e^{jk\ell} \\ e^{-jk\ell} \end{bmatrix} \tag{22}$$

because $[W_{nm}]_{2\times 2}$ in (21) is diagonal. Substituting (22) into (1) to find the fields of the plane wave at $\bar{r}_2$ gives

$$\bar{E}(\bar{r}_2, t) = \hat{e}\frac{1}{2}\left\{e^{-j(\omega t - k\ell)} + e^{j(\omega t - k\ell)}\right\} = \hat{e}\cos(\omega t - k\ell) \tag{23}$$

which is identical to (20). Thus in this case $\Psi = \left\{e^{-j\omega t}, e^{j\omega t}\right\}$ is sufficient to represent the TD response at both $\bar{r}_1$ and $\bar{r}_2$, and therefore result in a complete solution.

(ii) $\Psi = \{\cos\omega t\}$ with $N = 1$:
In this case, the derivative of electrical field in (19) is $\frac{\partial}{\partial t}\bar{E}(\bar{r}_1, t) = \hat{e}\left[-\omega\sin(\omega t)\right]$, which can not be represented by $\Psi = \{\cos\omega t\}$. Thus it is not sufficient to be used to solve the problem.

(iii) $\Psi = \{\cos\omega t, \sin\omega t\}$ with $N = 2$:
In this case, $\left[\bar{E}_n(\bar{r}_1)\right]_{1\times 1} = [1\ 0]^T$ by using (3), and the $[W_{nm}]_{2\times 2}$ is given by

$$[W_{nm}]_{2\times 2} = \begin{bmatrix} 0 & -j\omega \\ j\omega & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -j & -j \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\omega & 0 \\ 0 & \omega \end{bmatrix}\begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix}, \tag{24}$$

where the eigen-value and eigen-vector decomposition was performed. Utilizing (18) gives

$$\begin{aligned}\left[\bar{E}_m(\bar{r}_2)\right]_{2\times 1} &= e^{-j[K_{nm}]_{2\times 2}\ell}\left[\bar{E}_n(\bar{r}_1)\right]_{2\times 1} \\ &= \frac{1}{2}\begin{bmatrix} -j & -j \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{jk\ell} & 0 \\ 0 & e^{-jk\ell} \end{bmatrix}\begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{25}$$

Substituting (25) into (1) to find the field of the plane wave at $\bar{r}_2$ gives

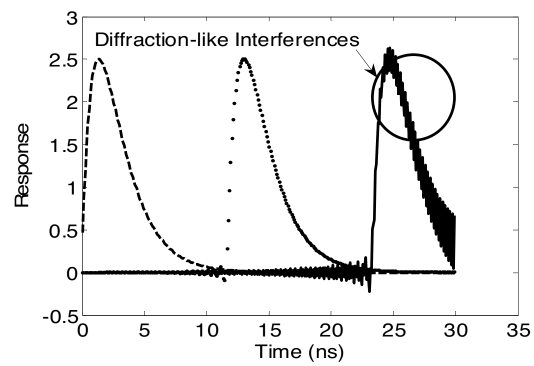
$$\bar{E}(\bar{r}_2, t) = \hat{e}\left\{\cos(k\ell)\cos(\omega t) - \sin(k\ell)\sin(\omega t)\right\}, \tag{26}$$

which is also identical to (20).

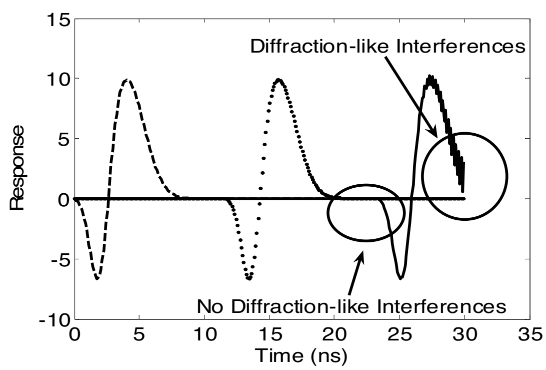
(iv) $B_n(t) = \Lambda_n(t)$ where $\Lambda_n(t)$ are triangular basis functions:
Since $\Lambda_n(t)$ is a local basis function, thus the transient waveform of the propagating plane wave at $\bar{r}_1 = (0, 0, 0)$ is assumed to be

$$E_0\left(t\right) = e^{-\frac{t}{2T}} - e^{-\frac{t}{T}}. \tag{27}$$

where $T = 10\Delta t$. Note that if $\Psi = \left\{e^{-\frac{t}{2T}}, e^{-\frac{t}{T}}\right\}$ is selected, then following the procedure in part (i) gives the exact solution by $E(\bar{r}_2, t) = e^{-\frac{t-\ell/c}{2T}} - e^{-\frac{t-\ell/c}{T}}$ where c is speed of light. In the current case, the three curves in Figure 3(a) show the TD waveforms at $\ell = 0$, 3.5 m and 7 m, respectively. In this case, $\Delta t = 0.1$ ns and $N = 300$ are used ($t \in [0\ 30\,\text{ns}]$). In principle, the TD waveforms should remain identical at different locations. The case at $\ell = 7$ m in Figure 3(a) shows diffraction-like interferences, which occurs due to the presence of the discontinuities of the fields' derivatives as one may observe from the comparison between cases at $\ell = 3.5$ m and 7 m. To further verify it, one considers a short pulse with a smooth variation to zero, where



(a)



(b)

Figure 3. Transient waveform of a plane wave propagation at distances of $\ell = 0$, 3.5 m and 7 m. ($[W_{nm}]_{2\times 2}$ uses (17)). (a) Equation (27), (b) Equation (28).

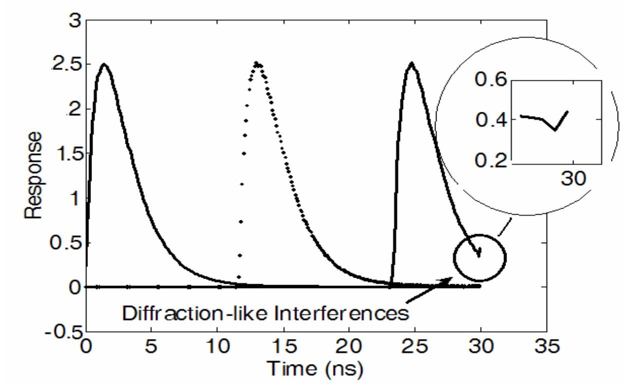


Figure 4. Transient waveform of a plane wave propagation at distances of $\ell = 0$, 3.5 m and 7 m. ($[W_{nm}]_{2\times2}$ uses (B2)).

it is assumed that

$$E_0(t) = 10\left(e^{-\left(\frac{t}{2T}-20\Delta t\right)^2} - e^{-\left(\frac{t}{T}-20\Delta t\right)^2}\right) \tag{28}$$

The results are shown in Figure 3(b). In this case the response starts smoothly from zero, and thus will not cause the diffraction-like interferences at the rising time of each pulse. Also the curve at $\ell = 7$ m shows similar diffraction-like interferences, which are however less serve than that in Figure 3(a) because field's values near the time truncation are smaller than those in Figure 3(a). Similar studies were also performed using $[W_{nm}]_{2\times2}$ in (B2), and are shown in Figure 4. It can be observed that in this case Figure 4 shows better results in comparison with Figure 3, which can be interpreted by a fact that an integration of a function will not appear severe discontinuities. Numerical experiments show that if the input pulse has severe discontinuities, such as a rectangular pulse, both expressions of $[W_{nm}]_{2\times2}$ give strong diffraction-like interferences. In this case, the formulation of correction was shown in [21] and needs to be used.

6. CONCLUSIONS

Previous works on DT EM theory appear to have shortcomings because of its need to discretize the entire time domain into time segments, and shortage of theoretical background to support its solution developments. Apparently it causes cumbersome matrix computations if the problem domain is sufficient large, which commonly occurs in the practical problems. The works presented in this paper generalize its application scopes and allow using general basis expansions, instead of simply time segment discretization, form the matrix solutions. With a proper selection of time basis functions, the size of the matrix computation can be significantly reduced regardless the size of the problem domain, where the dimension of matrix becomes dependent on the number of basis functions. Most importantly theoretical procedures to establish the TD solutions from their corresponding FD solutions are rigorously derived. This discrete TD EM theory allows the FD solutions to solve directly in TD a variety of transient EM analysis problems.

APPENDIX A.

Appendix A1–A4 derives the formulations in (12)–(16), respectively presented in Section 3.

A1 $f(t)$ and $f'(t)$ are first expressed via (1) by

$$f(t) = \sum_{n=1}^{N} a_n B_n(t) \tag{A1}$$

and

$$f'(t) = \sum_{n=1}^{N} c_n B_n(t). \tag{A2}$$

$f'(t)$ can also be obtained by differentiating (A1) and given by

$$f'(t) = \sum_{n=1}^{N} a_n B'_n(t). \tag{A3}$$

Solving $[c_n]_{N\times 1}$ and $[a_n]_{N\times 1}$ by performing (2) and (3) over (A2) and (A3) simultaneously and comparing them with (6) and (8) give

$$[P_{nm}]_{N\times N}[c_n]_{N\times 1} = [Q_{nm}]_{N\times N}[a_m]_{N\times 1} \tag{A4}$$

or

$$[c_n]_{N\times 1} = ([P_{nm}]_{N\times N})^{-1}[Q_{nm}]_{N\times N}[a_m]_{N\times 1} \tag{A5}$$

where it is noted that $[W_{nm}]_{N\times N} = -j\left([P_{mn}]_{N\times N}\right)^{-1}[Q_{mn}]_{N\times N}$, which results in (12).

A2 In Appendix A2, it is be identified that $j[W_{nm}]_{N\times N}[a_m]_{N\times 1}$ performs a differentiation over $f(t)$, and thus $(j[W_{nm}]_{N\times N})^q[a_m]_{N\times 1}$ will perform q times repeated differentiations over $f(t)$ (i.e., $f^{(q)}(t)$).

A3 Using a Taylor's series expansion over $f(t-t_0)$ gives

$$f(t-t_0)=\sum_{m=0}^{\infty}\frac{1}{m!}f^{(m)}(t)(-t_0)^m \tag{A6}$$

The relation between the expansion coefficients of $f(t-t_0)$ and $f^{(m)}(t)$ can be obtained by performing (2) and (3) over (A6). Using (13) to represent the expansion coefficients of $f^{(m)}(t)$ gives

$$[e_n]_{N\times 1}=\sum_{m=0}^{\infty}\frac{1}{m!}(j[W_{nm}]_{N\times N})^m[a_m]_{N\times 1}(-t_0)^m, \tag{A7}$$

which can be simplified to (14) because

$$e^{-j[W_{nm}]_{N\times N}t_0}=\sum_{m=0}^{\infty}\frac{1}{m!}(-j[W_{nm}]_{N\times N}t_0)^m \tag{A8}$$

Note that this Taylor's expansion is valid only when (A8) converges.

A4 The convolution of $g(t)$ and $f(t)$ can be performed by

$$h(t)=\int_a^b g(t')f(t-t')dt \tag{A9}$$

Performing (2) and (3) over both sides of (A9) gives

$$[h_n]_{N\times 1}=\int_a^b g(t')[e_n]_{N\times 1}dt \tag{A10}$$

Substituting (14) into (A10) gives

$$\begin{aligned}[h_n]_{N\times 1} &= \int_a^b g(t')e^{-j[W_{nm}]_{N\times N}t'}[a_m]_{N\times 1}dt' \\ &= \left\{\int_a^b g(t')e^{-j[W_{nm}]_{N\times N}t'}dt'\right\}[a_m]_{N\times 1}\end{aligned} \tag{A11}$$

Thus if one defines the Fourier Transform of $g(t)$ by $G(\omega)$ given by

$$G(\omega) = \int_a^b g(t')e^{-j\omega t'}dt' \tag{A12}$$

then the term inside the bracket on the right-hand side of (A11) can be identified by $G([W]_{N\times N})$ which is obtained by replacing ω with $[W_{mn}]_{N\times N}$ in (A12). Thus (A11) directly gives (15).

APPENDIX B.

The definitions of $[W_{mn}]_{N\times N}$ in [21] are summarized in this appendix:

B1 The elements of $[P_{mn}]_{N\times N}$ and $[Q_{mn}]_{N\times N}$ to define $[W_{mn}]_{N\times N}$ are given by

$$\begin{cases} P_{nm} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{t} B_n(t)d\tau\right] B_m^*(t)dt \\ Q_{nm} &= \int_{-\infty}^{\infty} B_n(t)B_m^*(t)dt \end{cases} \tag{B1}$$

B2 For a triangular basis function, $[W_{mn}]_{N\times N}$ is given by

$$[W_{mn}]_{N\times N} = \frac{-j}{\Delta t}\begin{bmatrix} 1/2 & 1/24 & 0 & \cdots & 0 \\ 23/24 & 1/2 & 1/24 & \ddots & \vdots \\ 1 & 23/24 & \ddots & \ddots & 0 \\ \vdots & 1 & \ddots & 1/2 & 1/24 \\ 1 & 1 & \cdots & 23/24 & 1/2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 2/3 & 1/6 & 0\cdots & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & \vdots \\ 0 & 1/6 & \ddots & & 0 \\ \vdots & 0 & \ddots & 2/3 & 1/6 \\ 0 & 0 & \cdots & 1/6 & 2/3 \end{bmatrix} \tag{B2}$$

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