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DIFFRACTION OF PLANE WAVES BY A SLIT IN AN INFINITE SOFT-HARD PLANE

M. Ayub, A. B. Mann, M. Ramzan †, and M. H. Tiwana

Department of Mathematics
Quaid-i-Azam University
45320, Islamabad 44000, Pakistan

Abstract—We have studied the problem of diffraction of plane waves by a finite slit in an infinitely long soft-hard plane. Analysis is based on the Fourier transform, the Wiener-Hopf technique and the method of steepest descent. The boundary value problem is reduced to a matrix Wiener-Hopf equation which is solved by using the factorization of the kernel matrix. The diffracted field, calculated in the far-field approximation, is shown to be the sum of the fields (separated and interaction fields) produced by the two edges of the slit. Some graphs showing the effects of various parameters on the diffracted field produced by two edges of the slit are also plotted.

1. INTRODUCTION

The problem of plane wave diffraction by a half plane which is soft at the top and hard at the bottom was first solved by Rawlins [1] who adopted an ad-hoc method for the solution of this problem. Later on Büyükaksoy [2] reconsidered the problem solved by [1] and proposed an appropriate method for the factorization of the kernel matrix appearing in it. The continued interest in the problem is due to the fact that it constitutes the simplest half plane problem which can be formulated as a system of coupled Wiener-Hopf (WH) equations that cannot be decoupled trivially [2].

In this paper we have studied the problem of diffraction of plane waves by a slit in an infinite soft-hard plane. From the existing literature it is evident that numerous past investigations have been devoted to the study of diffraction of acoustic/electromagnetic waves

† The third author is also with Department of Computer and Engineering Sciences, Bahria University, Islamabad 44000, Pakistan

by slits in various geometries and several authors adopted different analytical and numerical approaches to study the phenomenon of diffraction of waves by slits. To name a few only, e.g., the problem of diffraction of electromagnetic waves by slits in thick/thin screens have been treated by the authors [3–7]. Morse and Rubenstein [8], Asghar et al. [9] and Hayat et al. [10] studied the problem of diffraction of acoustic waves by slits by using the method of separation of variables and the WH technique, respectively. It is pertinent to mention here that scattering from strips, slits, half-planes, impedance surfaces and study of high frequency diffraction are the topics of current interest [11–24].

In the present analysis, the three-part boundary value problem connected with diffraction of plane acoustic waves by a slit in an infinite soft-hard plane is reduced to a matrix Wiener-Hopf equation. It is well-known that the solution of a matrix Wiener-Hopf problem requires the factorization of the kernel matrix as the product of two non-singular matrices such that these component matrices and their inverses have regular entries and are of algebraic growth at infinity in certain overlapping halves of the complex plane. To find these explicit factors of kernel matrix is vital and important at the same time. The non-commutativity of factor matrices and the requirements of satisfaction of radiation conditions present further problems. There is, as yet, no general procedure of factorization of such matrices, although the factorization for a restricted class of matrices has been achieved. For example the Wiener-Hopf-Hilbert method, introduced by Hurd [25], Rawlins [26] and Rawlins and Williams [27], is a powerful tool in the case when the kernel matrix contains branch-point singularities, while the Daniele-Kharapkov method, proposed independently by Daniele [28] and Kharapkov [29], is effective for the class of matrices having only pole-singularties and branch-point singularities. Another detailed survey for the matrix factorization methods with reference to applications of these methods to different diffraction problems may also be found in a paper by Büyükaksoy et al. [30]. Diffraction from a slit is a well-studied phenomenon in the diffraction theory and relevant for many applications. For the problem under consideration the kernel matrix remains the same and has been factorized by [2] with the help of Daniele-Kharapkov method [28, 29]. Using the factorization of the kernel matrix we then follow the Noble's approach [31] to calculate the diffracted field produced by the slit. Some graphs showing the effect of various parameters on the separated field are also plotted.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let (x, y, z) define the Cartesian coordinate system with respect to the origin O. We consider the diffraction of a plane acoustic wave by a slit occupying the position $\{p \leq x \leq q,\ y = 0,\ z \in (-\infty, \infty)\}$. The positions of the soft-hard planes located on both sides of the slit are given by $\{-\infty < x \leq p,\ y = 0,\ z \in (-\infty, \infty)\}$ and $\{q \leq x < \infty,\ y = 0,\ z \in (-\infty, \infty)\}$, respectively and these are assumed to have vanishing thicknesses. A time factor of the type $e^{-i\omega t}$ is assumed and suppressed throughout the calculations. The geometry of the problem is depicted in Fig. 1. For harmonic acoustic vibrations of time dependence $e^{-i\omega t}$,

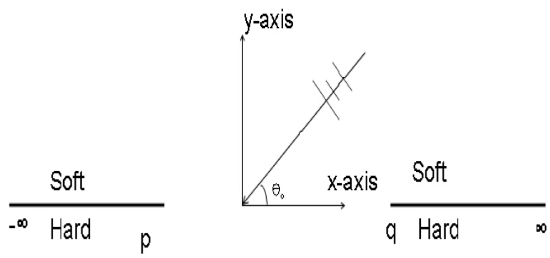

Figure 1. Geometry of the problem.

we require the solution of the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) \psi_t\,(x, y) = 0, \tag{1}$$

where ψ_t is the total velocity potential and the boundary and continuity conditions are given by

$$\psi_t\left(x, 0^+\right) = 0, \qquad \text{on} \qquad \left\{ \begin{array}{c} -\infty < x \leq p \\ q \leq x < \infty \end{array} \right\}, \tag{2a}$$

$$\frac{\partial \psi_t\,(x, 0^-)}{\partial y} = 0, \qquad \text{on} \qquad \left\{ \begin{array}{c} -\infty < x \leq p \\ q \leq x < \infty \end{array} \right\}, \tag{2b}$$

and

$$\psi_t\left(x, 0^+\right) = \psi_t\left(x, 0^-\right), \qquad \text{on} \qquad p < x < q, \tag{3a}$$

$$\frac{\partial \psi_t\,(x, 0^+)}{\partial y} = \frac{\partial \psi_t\,(x, 0^-)}{\partial y}, \qquad \text{on} \qquad p < x < q. \tag{3b}$$

In Eqs. (2), (3), $0^{\pm}$ refers to the situation that $y \to 0$ through positive or negative $y - axis$. Let a plane acoustic wave

$$\psi_i = e^{-ik(x\cos\theta_0 + y\sin\theta_0)}, \tag{4}$$

be incident upon the slit occupying the position $p \leq x \leq q$, $y = 0$. In Eq. (4), θ_0 is the angle of incidence and for the analytic convenience it is assumed that the wave number k has positive imaginary part. For the analysis purpose it is convenient to express the total field ψ_t as

$$\psi_t = \begin{cases} \psi_i + \psi_r + \psi & y > 0 \\ \psi & y < 0 \end{cases}, \tag{5}$$

where ψ is the diffracted field and ψ_r is the reflected field given by

$$\psi_r = -e^{-ik(x\cos\theta_0 + y\sin\theta_0)}.$$

For the unique solution of the problem, the edge conditions require that ψ_t and its normal derivative must be bounded and satisfy [2].

$$\psi_t\,(x,0) = \begin{cases} -1 + O(x-p)^{\frac{1}{4}} & \text{as} \quad x \to p^-, \\ -1 + O(x-q)^{\frac{1}{4}} & \text{as} \quad x \to q^+, \end{cases} \tag{6}$$

$$\frac{\partial \psi_t(x,0)}{\partial y} = \begin{cases} O(x-p)^{-\frac{3}{4}} & \text{as} \quad x \to p^-, \\ O(x-q)^{-\frac{3}{4}} & \text{as} \quad x \to q^+, \end{cases} \tag{7}$$

where a negative sign indicates a limit taken from left and a positive sign indicates that a limit taken from right. Thus, the scattered field satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\psi(x,y) = 0, \tag{8}$$

subject to the boundary conditions

$$\psi\left(x, 0^+\right) = 0 \qquad \text{on} \begin{cases} -\infty < x < p \\ q < x < \infty \end{cases}, \tag{9a}$$

and

$$\frac{\partial \psi\left(x, 0^-\right)}{\partial y} = 0 \qquad \text{on} \begin{cases} -\infty < x < p \\ q < x < \infty \end{cases}, \tag{9b}$$

and the continuity conditions

$$\psi\left(x, 0^+\right) - \psi\left(x, 0^-\right) = 0 \qquad \text{on } p \leq x \leq q, \tag{10a}$$

and

$$\frac{\partial\psi\left(x,0^{+}\right)}{\partial y}-\frac{\partial\psi\left(x,0^{-}\right)}{\partial y}=2ik\sin\theta_0 e^{-ikx\cos\theta_0}\qquad \text{on } p\leq x\leq q. \tag{10b}$$

The Fourier transform pair is defined as follows

$$\begin{aligned}\overline{\psi}(\alpha,y)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x,y)e^{i\alpha x}dx,\\&=e^{i\alpha p}\overline{\psi}_{-}(\alpha,y)+Q(\alpha,y)+e^{i\alpha q}\overline{\psi}_{+}(\alpha,y),\end{aligned}\tag{11}$$

and its inverse as

$$\psi(x,y)=\int_{-\infty}^{\infty}\overline{\psi}(\alpha,y)e^{-i\alpha x}d\alpha,\tag{12}$$

where

$$\begin{aligned}\overline{\psi}_{-}(\alpha,y)&=\frac{1}{2\pi}\int_{-\infty}^{p}\psi(x,y)e^{i\alpha(x-p)}dx,\\Q(\alpha,y)&=\frac{1}{2\pi}\int_{p}^{q}\psi(x,y)e^{i\alpha x}dx,\\\overline{\psi}_{+}(\alpha,y)&=\frac{1}{2\pi}\int_{q}^{\infty}\psi(x,y)e^{i\alpha(x-q)}dx.\end{aligned}\tag{13}$$

The function $\overline{\psi}_{-}(\alpha,y)$ is regular in the lower half plane $\operatorname{Im}\alpha<\operatorname{Im}k$, $\overline{\psi}_{+}(\alpha,y)$ is regular in the upper half plane $\operatorname{Im}\alpha>\operatorname{Im}k\cos\theta_0$ and $Q(\alpha,y)$ is an analytic function and therefore regular in the common region $\operatorname{Im}k\cos\theta_0<\operatorname{Im}\alpha<\operatorname{Im}k$.

On taking the Fourier transform of the Eq. (8) we arrive at

$$\frac{d^2\overline{\psi}(\alpha,y)}{dy^2}+K^2\overline{\psi}(\alpha,y)=0,\tag{14}$$

where $K(\alpha)=\sqrt{k^2-\alpha^2}$.

Defining $K(\alpha)$, the square root function, to be that branch which reduces to $+k$ when $\alpha=0$ and when the complex α-plane is cut either from $\alpha = k$ to $\alpha = k\infty$ or from $\alpha = -k$ to $\alpha = -k\infty$. The solution of Eq. (14), representing the outgoing waves at infinity, can formally be written as

$$\overline{\psi}(\alpha, y) = \begin{cases} A(\alpha)e^{iK(\alpha)y} & y > 0, \\ B(\alpha)e^{-iK(\alpha)y} & y < 0, \end{cases} \tag{15}$$

where $A(\alpha)$ and $B(\alpha)$ are the unknown coefficients which are to be determined. The Fourier transform of the boundary conditions (9) and (10) yields

$$\overline{\psi}_{-1}\left(\alpha, 0^{+}\right) = 0, \tag{16a}$$
$$\overline{\psi}_{+1}\left(\alpha, 0^{+}\right) = 0, \tag{16b}$$
$$\overline{\psi}_{-2}\left(\alpha, 0^{-}\right) = 0, \tag{16c}$$
$$\overline{\psi}_{+2}\left(\alpha, 0^{-}\right) = 0, \tag{16d}$$

$$Q_1\left(\alpha, 0^{+}\right) - Q_1\left(\alpha, 0^{-}\right) = 0, \tag{17a}$$
$$Q_2\left(\alpha, 0^{+}\right) - Q_2\left(\alpha, 0^{-}\right) = k \sin\theta_0 G(\alpha), \tag{17b}$$

where

$$\overline{\psi}_{-1}\left(\alpha, 0^{-}\right) = \frac{1}{2\pi}\int\limits_{-\infty}^{p} \psi\left(x, 0^{-}\right) e^{i\alpha(x-p)} dx, \tag{18a}$$

$$\overline{\psi}_{+1}\left(\alpha, 0^{-}\right) = \frac{1}{2\pi}\int\limits_{q}^{\infty} \psi\left(x, 0^{-}\right) e^{i\alpha(x-q)} dx, \tag{18b}$$

$$\overline{\psi}_{-2}\left(\alpha, 0^{+}\right) = \frac{1}{2\pi i}\int\limits_{-\infty}^{p} \frac{\partial\psi\left(x, 0^{+}\right)}{\partial y} e^{i\alpha(x-p)} dx, \tag{18c}$$

$$\overline{\psi}_{+2}\left(\alpha, 0^{+}\right) = \frac{1}{2\pi i}\int\limits_{q}^{\infty} \frac{\partial\psi\left(x, 0^{+}\right)}{\partial y} e^{i\alpha(x-q)} dx, \tag{18d}$$

$$Q_1\left(\alpha, 0^{+}\right) = \frac{1}{2\pi}\int\limits_{p}^{q} \psi\left(x, 0^{+}\right) e^{i\alpha x} dx, \tag{18e}$$

$$Q_2\left(\alpha, 0^{-}\right) = \frac{1}{2\pi i}\int\limits_{p}^{q} \frac{\partial\psi\left(x, 0^{-}\right)}{\partial y} e^{i\alpha x} dx, \tag{18f}$$

and

$$G(\alpha) = \frac{e^{i(\alpha - k\cos\theta_0)q} - e^{i(\alpha - k\cos\theta_0)p}}{\pi\left(\alpha - k\cos\theta_0\right)}. \tag{19}$$

Using Eqs. (16a)–(16d) and (17a)–(17b) in Eq. (15), we obtain

$$A(\alpha) = Q_1\left(\alpha, 0^+\right), \tag{20a}$$

$$B(\alpha) = -\frac{Q_2\left(\alpha, 0^-\right)}{K(\alpha)}, \tag{20b}$$

$$A(\alpha) - B(\alpha) = -e^{i\alpha p}\overline{\psi}_{-1}\left(\alpha, 0^-\right) - e^{i\alpha q}\overline{\psi}_{+1}\left(\alpha, 0^-\right), \tag{20c}$$

$$-K(\alpha)\left[A(\alpha) + B(\alpha)\right] = -e^{i\alpha p}\overline{\psi}_{-2}\left(\alpha, 0^+\right) - e^{i\alpha q}\overline{\psi}_{+2}\left(\alpha, 0^+\right) + ik\sin\theta_0 G(\alpha). \tag{20d}$$

The elimination of the coefficients $A(\alpha)$ and $B(\alpha)$ among the Eqs. (20a)–(20d) will lead to the following matrix Wiener-Hopf equation valid in the strip of analyticity $\operatorname{Im} k\cos\theta_0 < \operatorname{Im}\alpha < \operatorname{Im} k$,

$$e^{i\alpha q}\begin{bmatrix}\overline{\psi}_{+1}(\alpha)\\ \overline{\psi}_{+2}(\alpha)\end{bmatrix} + \begin{bmatrix}1 & \frac{1}{K(\alpha)}\\ -K(\alpha) & 1\end{bmatrix}\begin{bmatrix}Q_1(\alpha)\\ Q_2(\alpha)\end{bmatrix} + e^{i\alpha p}\begin{bmatrix}\overline{\psi}_{-1}(\alpha)\\ \overline{\psi}_{-2}(\alpha)\end{bmatrix} = G(\alpha)\begin{bmatrix}0\\ ik\sin\theta_0\end{bmatrix}. \tag{21}$$

In compact form, Eq. (21) can further be arranged as

$$e^{i\alpha q}\Psi_+(\alpha) + \mathbf{H}(\alpha)\mathbf{Q}(\alpha) + e^{i\alpha p}\Psi_-(\alpha) = G(\alpha)\mathbf{A}, \tag{22}$$

where bold letters are used to denote the matrices. Eq. (22) is an equation analogous to the Eq. (5.60) available in [31]. In Eq. (22), $\mathbf{H}(\alpha)$ is the kernel matrix and in order to solve it, we have to factorize the matrix $\mathbf{H}(\alpha)$ as the product of two non-singular factor matrices such that one factor matrix being regular in the lower half plane and the other factor matrix being regular in the upper half plane with the additional requirements that both the factor matrices as well as their inverses contains elements of algebraic growth at infinity and both of these factor matrices should commute with each other. The factorization of $\mathbf{H}(\alpha)$, satisfying these conditions, has been done in [2] by using the Daniele-Kharapkov method [28, 29] and the result is as follows:

$$\mathbf{H}_+(\alpha) = 2^{\frac{1}{4}}\begin{bmatrix}\cosh\chi(\alpha) & \sinh\chi(\alpha)/\gamma(\alpha)\\ \gamma(\alpha)\sinh\chi(\alpha) & \cosh\chi(\alpha)\end{bmatrix}, \tag{23a}$$

with

$$\mathbf{H}_-(\alpha) = \mathbf{H}_+(-\alpha), \tag{23b}$$

where

$$\chi(\alpha) = -\frac{i}{4}\arccos\frac{\alpha}{k}, \quad \chi(-\alpha) = -\frac{i}{4}\left[\pi - \arccos\frac{\alpha}{k}\right] \tag{23c}$$

and

$$\gamma(\alpha) = \sqrt{\alpha^2 - k^2}. \tag{23d}$$

Also as $|\alpha| \to \infty$, we note that

$$\mathbf{H}_{\pm}(\alpha) \sim (4k)^{-\frac{1}{4}} \begin{bmatrix} (\pm\alpha)^{\frac{1}{4}} & (\pm\alpha)^{-\frac{3}{4}} \\ (\pm\alpha)^{\frac{5}{4}} & (\pm\alpha)^{\frac{1}{4}} \end{bmatrix}. \tag{23e}$$

After accomplishing the factorization of the matrix $\mathbf{H}(\alpha)$, we can rearrange Eq. (22) as

$$e^{i\alpha q}\Psi_{+}(\alpha) + \mathbf{H}_{+}(\alpha)\mathbf{H}_{-}(\alpha)\mathbf{Q}(\alpha) + e^{i\alpha p}\Psi_{-}(\alpha) = G(\alpha)\mathbf{A}. \tag{24}$$

Pre-multiplying Eq. (24) by $e^{-i\alpha q}[\mathbf{H}_{+}(\alpha)]^{-1}$, substituting the value of $G(\alpha)$ from Eq. (19) and simplifying we arrive at

$$\begin{aligned} &[\mathbf{H}_{+}(\alpha)]^{-1}\Psi_{+}(\alpha) + e^{-i\alpha q}\mathbf{H}_{-}(\alpha)\mathbf{Q}(\alpha) + e^{i\alpha(p-q)}[\mathbf{H}_{+}(\alpha)]^{-1}\Psi_{-}(\alpha) \\ &= \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)}[\mathbf{H}_{+}(\alpha)]^{-1}\mathbf{A} - \frac{e^{i\alpha(p-q)-ik\cos\theta_0 p}}{\pi(\alpha - k\cos\theta_0)}[\mathbf{H}_{+}(\alpha)]^{-1}\mathbf{A}. \end{aligned} \tag{25}$$

According to the procedure defined in [31] different terms occurring Eq. (25) can be decomposed as follows,

$$e^{i\alpha(p-q)}[\mathbf{H}_{+}(\alpha)]^{-1}\Psi_{-}(\alpha) = \mathbf{U}_{+}(\alpha) + \mathbf{U}_{-}(\alpha), \tag{26}$$

$$\frac{e^{i\alpha(p-q)-ik\cos\theta_0 p}}{\pi(\alpha - k\cos\theta_0)}[\mathbf{H}_{+}(\alpha)]^{-1}\mathbf{A} = \mathbf{V}_{+}(\alpha) + \mathbf{V}_{-}(\alpha). \tag{27}$$

The pole contribution of the first term on right hand side of Eq. (25) can be expressed as

$$\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)}\left[\{\mathbf{H}_{+}(\alpha)\}^{-1} - \{\mathbf{H}_{+}(k\cos\theta_0)\}^{-1} + \{\mathbf{H}_{+}(k\cos\theta_0)\}^{-1}\right]\mathbf{A}. \tag{28}$$

Using Eqs. (26)–(28) in Eq. (25) and separating it into positive and negative terms, we obtain

$$\begin{aligned} &[\mathbf{H}_{+}(\alpha)]^{-1}\Psi_{+}(\alpha) + \mathbf{U}_{+}(\alpha) \\ &-\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)}\left[\{\mathbf{H}_{+}(\alpha)\}^{-1} - \{\mathbf{H}_{+}(k\cos\theta_0)\}^{-1}\right]\mathbf{A} + \mathbf{V}_{+}(\alpha) \\ &= -e^{-i\alpha q}\mathbf{H}_{-}(\alpha)\mathbf{Q}(\alpha) - \mathbf{U}_{-}(\alpha) - \mathbf{V}_{-}(\alpha) + \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)}\{\mathbf{H}_{+}(k\cos\theta_0)\}^{-1}\mathbf{A}, \end{aligned} \tag{29}$$

where

$$\mathbf{U}_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)}\left[\mathbf{H}_{+}(\xi)\right]^{-1}\Psi_{-}(\xi)}{\xi-\alpha}d\xi, \tag{30}$$

and

$$\mathbf{V}_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int\limits_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)-ik\cos\theta_0 p}\left[\mathbf{H}_{+}(\xi)\right]^{-1}\mathbf{A}}{\pi(\xi-\alpha)(\xi-k\cos\theta_0)}d\xi. \tag{31}$$

Now pre-multiplying Eq. (24) $e^{-i\alpha p}[\mathbf{H}_{-}(\alpha)]^{-1}$, substituting the value of $G(\alpha)$ from Eq. (19) and simplifying we arrive at

$$\begin{aligned}&e^{i\alpha(q-p)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{+}(\alpha)+e^{-i\alpha p}\mathbf{H}_{+}(\alpha)\mathbf{Q}(\alpha)+\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha)\\ &=\frac{e^{i\alpha(q-p)-ik\cos\theta_0 q}}{\pi\left(\alpha-k\cos\theta_0\right)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A}-\frac{e^{-ik\cos\theta_0 p}}{\pi\left(\alpha-k\cos\theta_0\right)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A}.\end{aligned} \tag{32}$$

Decomposing different terms in Eq. (32) by following [31], we obtain

$$e^{i\alpha(q-p)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{+}(\alpha) = \mathbf{R}_{+}(\alpha)+\mathbf{R}_{-}(\alpha), \tag{33}$$

$$\frac{e^{i\alpha(q-p)-ik\cos\theta_0 q}}{\pi\left(\alpha-k\cos\theta_0\right)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A} = \mathbf{S}_{+}(\alpha)+\mathbf{S}_{-}(\alpha), \tag{34}$$

so that

$$\mathbf{R}_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{e^{i\xi(q-p)}\left[\mathbf{H}_{-}(\xi)\right]^{-1}\Psi_{+}(\xi)}{\xi-\alpha}d\xi, \tag{35}$$

and

$$\mathbf{S}_{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int\limits_{-\infty+id}^{\infty+id}\frac{e^{i\xi(q-p)-ik\cos\theta_0 q}\left[\mathbf{H}_{-}(\xi)\right]^{-1}\mathbf{A}}{\pi(\xi-\alpha)\left(\xi-k\cos\theta_0\right)}d\xi, \tag{36}$$

where $-\mathrm{Im}\,\alpha < c < \mathrm{Im}\,k\cos\theta_0$ and $-\mathrm{Im}\,\alpha < d < \mathrm{Im}\,k\cos\theta_0$, also $\mathrm{Im}\,\alpha > c$ in Eqs. (30), (31) and $\mathrm{Im}\,\alpha < d$ in Eqs. (35), (36) as given in [31].

Using Eqs. (33), (34) in Eq. (32) and separating it into positive and negative portions we arrive at

$$\begin{aligned}&\mathbf{R}_{-}(\alpha)+\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\Psi_{-}(\alpha)-\mathbf{S}_{-}(\alpha)+\frac{e^{-ik\cos\theta_0 p}}{\pi\left(\alpha-k\cos\theta_0\right)}\left[\mathbf{H}_{-}(\alpha)\right]^{-1}\mathbf{A}\\ &=-e^{-i\alpha p}\mathbf{H}_{+}(\alpha)\mathbf{Q}(\alpha)-\mathbf{R}_{+}(\alpha)+\mathbf{S}_{+}(\alpha).\end{aligned} \tag{37}$$

The left hand side of Eq. (29) and right hand side of Eq. (37) are regular in $\operatorname{Im}\alpha > \operatorname{Im} k\cos\theta_0$ and right hand side of Eq. (29) and left hand side of Eq. (37) are regular in $\operatorname{Im}\alpha < \operatorname{Im} k$. Hence using the extended form of the Liouville's theorem each side of Eqs. (29) and (37) is equal to zero, i.e.,

$$\begin{aligned}&[\mathbf{H}_+(\alpha)]^{-1}\Psi_+(\alpha)+\mathbf{U}_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}\\&\left[\{\mathbf{H}_+(\alpha)\}^{-1}-\{\mathbf{H}_+(k\cos\theta_0)\}^{-1}\right]\mathbf{A}+\mathbf{V}_+(\alpha)=0,\end{aligned}\tag{38}$$

and

$$\mathbf{R}_-(\alpha)+[\mathbf{H}_-(\alpha)]^{-1}\Psi_-(\alpha)-\mathbf{S}_-(\alpha)+\frac{e^{-ik\cos\theta_0 p}}{\pi(\alpha-k\cos\theta_0)}[\mathbf{H}_-(\alpha)]^{-1}\mathbf{A}=0.\tag{39}$$

Using Eqs. (30), (31) in Eq. (38) and Eqs. (35), (36) in Eq. (39) and simplifying these equations we obtain

$$\begin{aligned}&[\mathbf{H}_+(\alpha)]^{-1}\Psi_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}[\mathbf{H}_+(k\cos\theta_0)]^{-1}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)}\\&+\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{i\xi(p-q)}[\mathbf{H}_+(\xi)]^{-1}\Psi_-(\xi)}{(\xi-\alpha)}d\xi=0\end{aligned}\tag{40}$$

and

$$[\mathbf{H}_-(\alpha)]^{-1}\Psi_-(\alpha)-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\frac{e^{i\xi(q-p)}[\mathbf{H}_-(\xi)]^{-1}\Psi_+^*(\alpha)}{(\xi-\alpha)}d\xi=0,\tag{41}$$

where

$$\Psi_+^*(\alpha)=\Psi_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)},\tag{42}$$

$$\Psi_-(\alpha)=\Psi_-(\alpha)+\frac{e^{-ik\cos\theta_0 p}\mathbf{A}}{\pi(\alpha-k\cos\theta_0)}.\tag{43}$$

From the assumption that $0<\theta_0<\frac{\pi}{2}$, we can choose a such that $-k_2\cos\theta_0<a<k_2\cos\theta_0$ and $d=-c=a$, [31]. In Eq. (40) replacing ξ by $-\xi$ and in Eq. (41) α by $-\alpha$ and also noting that $\mathbf{H}_-(-\alpha)=\mathbf{H}_+(\alpha)$

we arrive at

$$[\mathbf{H}_+(\alpha)]^{-1}\,\Psi_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_+\left(k\cos\theta_0\right)\right]^{-1}\mathbf{A}}{\pi\left(\alpha-k\cos\theta_0\right)}$$
$$-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\Psi_-(-\xi)}{(\xi+\alpha)}d\xi=0 \qquad (44)$$

and

$$[\mathbf{H}_+(\alpha)]^{-1}\,\Psi_-(-\alpha)-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\Psi_+^*(\alpha)}{(\xi+\alpha)}d\xi=0. \qquad (45)$$

Adding and subtracting Eqs. (44) and (45), we obtain

$$[\mathbf{H}_+(\alpha)]^{-1}\,\mathbf{S}_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_+\left(k\cos\theta_0\right)\right]^{-1}\mathbf{A}}{\pi\left(\alpha-k\cos\theta_0\right)}$$
$$-\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\mathbf{S}_+^*(\xi)}{(\xi+\alpha)}d\xi=0 \qquad (46)$$

and

$$[\mathbf{H}_+(\alpha)]^{-1}\,\mathbf{D}_+^*(\alpha)+\frac{e^{-ik\cos\theta_0 q}\left[\mathbf{H}_+\left(k\cos\theta_0\right)\right]^{-1}\mathbf{A}}{\pi\left(\alpha-k\cos\theta_0\right)}$$
$$+\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\left[\mathbf{H}_-(\xi)\right]^{-1}\mathbf{D}_+^*(\xi)}{(\xi+\alpha)}d\xi=0, \qquad (47)$$

where

$$\mathbf{S}_+^*(\alpha)\;=\;\Psi_+^*(\alpha)+\Psi_-(-\alpha), \qquad (48)$$
$$\mathbf{D}_+^*(\alpha)\;=\;\Psi_+^*(\alpha)-\Psi_-(-\alpha). \qquad (49)$$

The Eqs. (46), (47) are of the same type and we obtain an approximate solution by a method due to Jones [32]. Setting

$$\mathbf{S}_+^*(\alpha)=\mathbf{D}_+^*(\alpha)=\mathbf{F}_+^*(\alpha), \qquad (50)$$

the Eqs. (46), (47) will take the form

$$[\mathbf{H}_+(\alpha)]^{-1}\,\mathbf{F}^*_+(\alpha)+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}\,[\mathbf{H}_-(\xi)]^{-1}\,\mathbf{F}^*_+(\xi)}{(\xi+\alpha)}d\xi$$
$$=\ -\frac{e^{-ik\cos\theta_0 q}\,[\mathbf{H}_+\,(k\cos\theta_0)]^{-1}\,\mathbf{A}}{\pi\,(\alpha-k\cos\theta_0)},\tag{51}$$

where

$$\mathbf{F}^*_+(\alpha)\ =\ \mathbf{F}_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}\mathbf{A}}{\pi\,(\alpha-k\cos\theta_0)}+\frac{\lambda e^{-ik\cos\theta_0 p}\mathbf{A}}{\pi\,(\alpha+k\cos\theta_0)},\tag{52}$$
$$\mathbf{F}_+(\alpha)\ =\ \Psi_+(\alpha)-\lambda\Psi_-(-\alpha),\tag{53}$$

and $\lambda=\pm1$.

A more elaborative form of Eq. (51) is as follows:

$$\begin{bmatrix}\cosh\varkappa(\alpha)F^{1*}_+(\alpha)-\sinh\varkappa(\alpha)F^{2*}_+(\alpha)/\gamma(\alpha)\\-\gamma(\alpha)\sinh\varkappa(\alpha)F^{1*}_+(\alpha)+\cosh\varkappa(\alpha)F^{2*}_+(\alpha)\end{bmatrix}$$
$$+\frac{\lambda}{2\pi i}\int\limits_{-\infty+ia}^{\infty+ia}\frac{e^{i\xi(q-p)}}{(\xi+\alpha)}\begin{bmatrix}\cosh\varkappa(-\xi)F^{1*}_+(\xi)-\sinh\varkappa(-\xi)F^{2*}_+(\xi)/\gamma(-\xi)\\-\gamma(-\xi)\sinh\varkappa(-\xi)F^{1*}_+(\xi)+\cosh\varkappa(-\xi)F^{2*}_+(\xi)\end{bmatrix}d\xi$$
$$+\frac{e^{-ik\cos\theta_0 q}}{\pi\,(\alpha-k\cos\theta_0)}$$
$$\begin{bmatrix}A_1\cosh\varkappa\,(k\cos\theta_0)-A_2\sinh\varkappa\,(k\cos\theta_0)\,/\gamma\,(k\cos\theta_0)\\-A_1\gamma\,(k\cos\theta_0)\sinh\varkappa\,(k\cos\theta_0)+A_2\cosh\varkappa\,(k\cos\theta_0)\end{bmatrix}=0.\tag{54}$$

Eq. (52) in matrix form can be written as:

$$\begin{bmatrix}F^{1*}_+(\alpha)\\F^{2*}_+(\alpha)\end{bmatrix}=\begin{bmatrix}F^1_+(\alpha)\\F^1_+(\alpha)\end{bmatrix}-\frac{e^{-ik\cos\theta_0 q}}{\pi\,(\alpha-k\cos\theta_0)}\begin{bmatrix}A_1\\A_2\end{bmatrix}+\frac{\lambda e^{-ik\cos\theta_0 p}}{\pi\,(\alpha+k\cos\theta_0)}\begin{bmatrix}A_1\\A_2\end{bmatrix}.\tag{55}$$

Considering the first row of Eq. (54) and using the values of $F^{1*}_+(\alpha)$ and $F^{2*}_+(\alpha)$ in it, we obtain

$$\cosh\varkappa(\alpha)\left[F^1_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi\,(\alpha-k\cos\theta_0)}A_1+\frac{\lambda e^{-ik\cos\theta_0 p}}{\pi\,(\alpha+k\cos\theta_0)}A_1\right]$$
$$-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[F^2_+(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi\,(\alpha-k\cos\theta_0)}A_2+\frac{\lambda e^{-ik\cos\theta_0 p}}{\pi\,(\alpha+k\cos\theta_0)}A_2\right]+\frac{\lambda}{2\pi i}$$

$$\int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)}}{(\xi+\alpha)} \left[\cosh \varkappa(-\xi) \left\{F_+^1(\xi) - \frac{e^{-ik\cos\theta_0 q}}{\pi\,(\xi - k\cos\theta_0)} A_1 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi\,(\xi + k\cos\theta_0)} A_1\right\}\right.$$
$$\left. - \sinh \varkappa(-\xi)/\gamma(\xi) \left\{F_+^2(\xi) - \frac{e^{-ik\cos\theta_0 q}}{\pi(\xi - k\cos\theta_0)} A_2 + \frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\xi + k\cos\theta_0)} A_2\right\}\right] d\xi$$
$$+ \frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha - k\cos\theta_0)} [A_1 \cosh \varkappa(k\cos\theta_0) - A_2 \sinh \varkappa(k\cos\theta_0)/\gamma(k\cos\theta_0)]$$
$$= 0. \tag{56}$$

Writing $\gamma(\xi) = \gamma_+(\xi)\gamma_-(\xi) = \sqrt{\xi + k}\sqrt{\xi - k}$ and considering the integrals arising in Eq. (56), we have

$$I = I_1 - \frac{e^{-ik\cos\theta_0 q} A_1}{\pi} I_2 + \frac{\lambda e^{-ik\cos\theta_0 p} A_1}{\pi} I_3 - I_4 + \frac{e^{-ik\cos\theta_0 q} A_2}{\pi} I_5 + \frac{e^{-ik\cos\theta_0 p} A_2}{\pi} I_6, \tag{57}$$

where

$$I_1 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \cosh \varkappa(-\xi) F_+^1(\xi)}{(\xi + \alpha)} d\xi, \tag{58}$$

$$I_2 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \cosh \varkappa(-\xi)}{(\xi + \alpha)\,(\xi - k\cos\theta_0)} d\xi, \tag{59}$$

$$I_3 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \cosh \varkappa(-\xi)}{(\xi + \alpha)\,(\xi + k\cos\theta_0)} d\xi, \tag{60}$$

$$I_4 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} F_+^2(\xi) \sinh \varkappa(-\xi)/\sqrt{\xi + k}}{(\xi + \alpha)\,\sqrt{\xi - k}} d\xi, \tag{61}$$

$$I_5 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \sinh \varkappa(-\xi)/\sqrt{\xi + k}}{(\xi - k\cos\theta_0)\,(\xi + \alpha)\,\sqrt{\xi - k}} d\xi, \tag{62}$$

$$I_6 = \int_{-\infty+ia}^{\infty+ia} \frac{e^{i\xi(q-p)} \sinh \varkappa(-\xi)/\sqrt{\xi + k}}{(\xi + k\cos\theta_0)\,(\xi + \alpha)\,\sqrt{\xi - k}} d\xi. \tag{63}$$

Integrals (58)–(63) are solved by a method described in [31] and are substituted in Eq. (56) to get

$$\cosh\varkappa(\alpha)\left[F_+^1(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}A_1+\frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\alpha+k\cos\theta_0)}A_1\right]$$
$$-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[F_+^2(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}A_2+\frac{\lambda e^{-ik\cos\theta_0 p}}{\pi(\alpha+k\cos\theta_0)}A_2\right]=$$
$$-\lambda T(\alpha)F_+^1(k)+\lambda\frac{e^{-ik\cos\theta_0 q}}{\pi}$$
$$\times A_1\left\{\frac{e^{ikl\cos\theta_0}}{(\alpha+k\cos\theta_0)}\cosh\varkappa(-k\cos\theta_0)+R_2(\alpha)\right\}$$
$$-A_1\frac{e^{-ik\cos\theta_0 p}}{\pi}R_1(\alpha)+\lambda T_1(\alpha)F_+^2(k)-\lambda\frac{e^{-ik\cos\theta_0 q}}{\pi}A_2$$
$$\times\left\{\frac{e^{ikl\cos\theta_0}\sinh\varkappa(-k\cos\theta_0)/\gamma_+(k\cos\theta_0)}{(\alpha+k\cos\theta_0)\gamma_-(k\cos\theta_0)}+R_4(\alpha)\right\}$$
$$-A_2\frac{e^{-ik\cos\theta_0 p}}{\pi}R_3(\alpha)-\frac{e^{-ik\cos\theta_0 q}}{\pi(\alpha-k\cos\theta_0)}$$
$$\times\left[A_1\cosh\varkappa(k\cos\theta_0)-A_2\sinh\varkappa(k\cos\theta_0)/\gamma(k\cos\theta_0)\right],\tag{64}$$

where $l=q-p$ and

$$T(\alpha)=\frac{1}{2\pi i}E_{-\frac{1}{2}}W_{-\frac{1}{2}}\{-i(k+\alpha)l\},$$
$$T_1(\alpha)=\frac{1}{2\pi i}E_{-1}W_{-1}\{-i(k+\alpha)l\},$$

$$R_{1,2}(\alpha)=$$
$$\frac{\cosh\varkappa(-k)E_{-\frac{1}{2}}\left[W_{-\frac{1}{2}}\{-i(k\pm k\cos\theta_0)l\}-W_{-\frac{1}{2}}\{-i(k+\alpha)l\}\right]}{2\pi i(\alpha\mp k\cos\theta_0)},$$
$$R_{3,4}(\alpha)=$$
$$\frac{E_{-1}\left[W_{-1}\{-i(k\pm k\cos\theta_0)l\}-W_{-1}\{-i(k+\alpha)l\}\right]\sinh\varkappa(-k)/\sqrt{2k}}{2\pi i(\alpha\mp k\cos\theta_0)}.\tag{65}$$

Equation (64) can further be simplified according to the procedure described in [31] and the result is

$$\begin{aligned}
&\cosh\varkappa(\alpha)F_+^1(\alpha)-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}F_+^2(\alpha)\\
=&-\lambda T(\alpha)F_+^1(k)+\lambda T_1(\alpha)F_+^2(k)+A_1\frac{e^{-ik\cos\theta_0 q}}{\pi}P_1(\alpha)\\
&-\lambda A_1\frac{e^{-ik\cos\theta_0 p}}{\pi}P_2(\alpha)-A_2\frac{e^{-ik\cos\theta_0 q}}{\pi}P_3(\alpha)+\lambda A_2\frac{e^{-ik\cos\theta_0 p}}{\pi}P_4(\alpha)\\
&+\lambda A_1\frac{e^{-ik\cos\theta_0 q}}{\pi}R_2(\alpha)-A_1\frac{e^{-ik\cos\theta_0 p}}{\pi}R_1(\alpha)-\lambda A_2\frac{e^{-ik\cos\theta_0 q}}{\pi}R_4(\alpha)\\
&+A_2\frac{e^{-ik\cos\theta_0 p}}{\pi}R_3(\alpha),
\end{aligned}\tag{66}$$

where

$$\begin{aligned}
P_1(\alpha)&=\frac{1}{(\alpha-k\cos\theta_0)}\left[\cosh\varkappa(\alpha)-\cosh\varkappa\left(k\cos\theta_0\right)\right],\\
P_2(\alpha)&=\frac{1}{(\alpha+k\cos\theta_0)}\left[\cosh\varkappa(\alpha)-\cosh\varkappa\left(-k\cos\theta_0\right)\right],\\
P_3(\alpha)&=\frac{1}{(\alpha-k\cos\theta_0)}\left[\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}-\frac{\sinh\varkappa\left(k\cos\theta_0\right)}{\gamma\left(k\cos\theta_0\right)}\right],\\
P_4(\alpha)&=\frac{1}{\gamma_-\left(-k\cos\theta_0\right)(\alpha+k\cos\theta_0)}\left[\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}-\frac{\sinh\varkappa(-k\cos\theta_0)}{\gamma_+\left(k\cos\theta_0\right)}\right].
\end{aligned}\tag{67}$$

Further simplification of Eq. (66) will yield

$$\begin{aligned}
&\cosh\varkappa(\alpha)F_+^1(\alpha)-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}F_+^2(\alpha)\\
=&-\lambda T(\alpha)F_+^1(k)\cosh\varkappa(-k)+\lambda T_1(\alpha)F_+^2(k)\frac{\sinh\varkappa(-k)}{\sqrt{2k}}\\
&+\frac{A_1}{\pi}\{e^{-ik\cos\theta_0 q}P_1(\alpha)-e^{-ik\cos\theta_0 p}R_1(\alpha)\}\\
&-\frac{\lambda A_1}{\pi}\{e^{-ik\cos\theta_0 p}P_2(\alpha)-e^{-ik\cos\theta_0 q}R_2(\alpha)\}\\
&-\frac{A_2}{\pi}\{e^{-ik\cos\theta_0 q}P_3(\alpha)-e^{-ik\cos\theta_0 p}R_3(\alpha)\}\\
&+\frac{\lambda A_2}{\pi}\{e^{-ik\cos\theta_0 p}P_4(\alpha)-e^{-ik\cos\theta_0 q}R_4(\alpha)\}.
\end{aligned}\tag{68}$$

Letting

$$\begin{aligned}
G_1(\alpha) &= e^{-ik\cos\theta_0 q} P_1(\alpha) - e^{-ik\cos\theta_0 p} R_1(\alpha),\\
G_2(\alpha) &= e^{-ik\cos\theta_0 p} P_2(\alpha) - e^{-ik\cos\theta_0 q} R_2(\alpha),\\
G_3(\alpha) &= e^{-ik\cos\theta_0 q} P_3(\alpha) - e^{-ik\cos\theta_0 p} R_3(\alpha),\\
G_4(\alpha) &= e^{-ik\cos\theta_0 p} P_4(\alpha) - e^{-ik\cos\theta_0 q} R_4(\alpha),
\end{aligned}\tag{69}$$

in Eq. (68), the solution of the first WH equation, obtained by considering the first row of matrices in Eq. (54), is given as follows:

$$\begin{aligned}
&\cosh\varkappa(\alpha)F_+^1(\alpha) - \frac{\sinh\varkappa(\alpha)F_+^2(\alpha)}{\gamma(\alpha)} = -\lambda T(\alpha)\cosh\varkappa(-k)F_+^1(k)\\
&\quad+\lambda\frac{T_1(\alpha)\sinh\varkappa(-k)F_+^2(k)}{\sqrt{2k}} + \frac{A_1}{\pi}\left[G_1(\alpha)-\lambda G_2(\alpha)\right]\\
&\quad-\frac{A_2}{\pi}\left[G_3(\alpha)-\lambda G_4(\alpha)\right].
\end{aligned}\tag{70}$$

The second WH equation corresponds to the second row of the matrix Eq. (54) and its solution can be obtained in a similar manner as for the first row of Eq. (54). Omitting all the similar steps and quantities arose in the solution, we finally arrive at:

$$\begin{aligned}
&-\gamma(\alpha)\sinh\varkappa(\alpha)F_+^1(\alpha)+\cosh\varkappa(\alpha)F_+^2(\alpha)=\lambda T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)F_+^1(k)\\
&-\lambda T(\alpha)\cosh\varkappa(-k)F_+^2(k)+\frac{A_2}{\pi}\left[G_1(\alpha)-\lambda G_2(\alpha)\right]-\frac{A_1}{\pi}\left[G_5(\alpha)-\lambda G_6(\alpha)\right],
\end{aligned}\tag{71}$$

where

$$\begin{aligned}
T_2(\alpha) &= \frac{1}{2\pi i}E_0 W_0\{-i(k+\alpha)l\}\\
G_5(\alpha) &= e^{-ik\cos\theta_0 q} P_5(\alpha) - e^{-ik\cos\theta_0 p} R_5(\alpha),\\
G_6(\alpha) &= e^{-ik\cos\theta_0 p} P_6(\alpha) - e^{-ik\cos\theta_0 q} R_6(\alpha),\\
P_5(\alpha) &= \frac{\gamma(\alpha)\sinh\varkappa(\alpha)-\gamma(k\cos\theta_0)\sinh\varkappa(k\cos\theta_0)}{\alpha-k\cos\theta_0},\\
P_6(\alpha) &= \frac{\gamma(\alpha)\sinh\varkappa(\alpha)-\gamma(-k\cos\theta_0)\sinh\varkappa(-k\cos\theta_0)}{\alpha+k\cos\theta_0},\\
R_{5,6}(\alpha) &= \frac{D_0\left[W_0\{-i(k\pm k\cos\theta_0)l\}-W_0\{-i(k+\alpha)l\}\right]}{2\pi i(\alpha\mp k\cos\theta_0)},\\
D_0 &= E_0\sqrt{2k}\sinh\varkappa(-k).
\end{aligned}\tag{72}$$

In Eqs. (65) and (72), we have

$$\begin{aligned}W_{n-\frac{1}{2}}(z) &= \int_0^\infty \frac{u^n e^{-u}}{u+z} du \\ &= \Gamma(n+1)\, e^{\frac{1}{2}z} z^{\frac{1}{2}n-\frac{1}{2}} W_{-\frac{1}{2}(n+1),\frac{1}{2}n}(z), \qquad (73)\end{aligned}$$

where $z = -i(k+\alpha)l$ and $n = -\frac{1}{2}, 0, \frac{1}{2}$. $W_{m,n}$ is known as a Whittaker function [32]. The values of the functions $F_+^1(k)$ and $F_+^2(k)$ can be calculated by putting $\alpha = k$ in Eqs. (70) and (71) and solving these equations simultaneously. Now as

$$\mathbf{F}_+(\alpha) = \begin{bmatrix} F_+^1(\alpha) \\ F_+^2(\alpha) \end{bmatrix} = \begin{bmatrix} \overline{\psi}_{+1}(\alpha) \\ \overline{\psi}_{+2}(\alpha) \end{bmatrix} - \lambda \begin{bmatrix} \overline{\psi}_{-1}(\alpha) \\ \overline{\psi}_{-2}(\alpha) \end{bmatrix}, \qquad (74)$$

Eq. (74) is considered for the cases $\lambda = 1$ and $\lambda = -1$ and when the values of $F_+^1(\alpha)$ and $F_+^2(\alpha)$ are substituted in Eqs. (70) and (71) the results are as follows:

For $\lambda = 1$

$$\begin{aligned}&\cosh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) - \overline{\psi}_{-1}(\alpha)\right] - \frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[\overline{\psi}_{+2}(\alpha) - \overline{\psi}_{-2}(\alpha)\right] \\ &= -T(\alpha)\cosh\varkappa(-k)\, F_+^1(k)\big|_{\lambda=1} + \frac{T_1(\alpha)\sinh\varkappa(-k)}{\sqrt{2k}}\, F_+^2(k)\big|_{\lambda=1} \\ &\quad + \frac{A_1}{\pi}\left[G_1(\alpha) - G_2(\alpha)\right] - \frac{A_2}{\pi}\left[G_3(\alpha) - G_4(\alpha)\right], \qquad (75)\end{aligned}$$

and

$$\begin{aligned}&-\gamma(\alpha)\sinh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) - \overline{\psi}_{-1}(\alpha)\right] + \cosh\varkappa(\alpha)\left[\overline{\psi}_{+2}(\alpha) - \overline{\psi}_{-2}(\alpha)\right] \\ &= -T(\alpha)\cosh\varkappa(-k)\, F_+^2(k)\big|_{\lambda=1} + \sqrt{2k}T_2(\alpha)\sinh\varkappa(-k)\, F_+^1(k)\big|_{\lambda=1} \\ &+ \frac{A_2}{\pi}\left[G_1(\alpha) - G_2(\alpha)\right] - \frac{A_1}{\pi}\left[G_5(\alpha) - G_6(\alpha)\right], \qquad (76)\end{aligned}$$

and for $\lambda = -1$

$$\begin{aligned}&\cosh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha) + \overline{\psi}_{-1}(\alpha)\right] - \frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\left[\overline{\psi}_{+2}(\alpha) + \overline{\psi}_{-2}(\alpha)\right] \\ &= +T(\alpha)\cosh\varkappa(-k)\, F_+^1(k)\big|_{\lambda=-1} - \frac{T_1(\alpha)\sinh\varkappa(-k)}{\sqrt{2k}}\, F_+^2(k)\big|_{\lambda=-1} \\ &\quad + \frac{A_1}{\pi}\left[G_1(\alpha) + G_2(\alpha)\right] - \frac{A_2}{\pi}\left[G_3(\alpha) + G_4(\alpha)\right], \qquad (77)\end{aligned}$$

and

$$\begin{aligned}
&-\gamma(\alpha)\sinh\varkappa(\alpha)\left[\overline{\psi}_{+1}(\alpha)+\overline{\psi}_{-1}(\alpha)\right]+\cosh\varkappa(\alpha)\left[\overline{\psi}_{+2}(\alpha)+\overline{\psi}_{-2}(\alpha)\right]\\
&= T(\alpha)\cosh\varkappa(-k)\,F_{+}^{2}(k)\big|_{\lambda=-1}-T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\,F_{+}^{1}(k)\big|_{\lambda=-1}\\
&+\frac{A_2}{\pi}\left[G_1(\alpha)+G_2(\alpha)\right]-\frac{A_1}{\pi}\left[G_5(\alpha)+G_6(\alpha)\right]. \qquad (78)
\end{aligned}$$

Adding Eqs. (75) and (77), we obtain

$$\begin{aligned}
&\cosh\varkappa(\alpha)\overline{\psi}_{+1}(\alpha)-\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\overline{\psi}_{+2}(\alpha)\\
&=\frac{A_1}{\pi}G_1(\alpha)-\frac{A_2}{\pi}G_3(\alpha)-\frac{T(\alpha)\cosh\varkappa(-k)}{2}C_1+\frac{T_1(\alpha)\sinh\varkappa(-k)}{2\sqrt{2k}}C_2, \qquad (79)
\end{aligned}$$

and Eqs. (76) and (18) will yield

$$\begin{aligned}
&-\gamma(\alpha)\sinh\varkappa(\alpha)\overline{\psi}_{+1}(\alpha)+\cosh\varkappa(\alpha)\overline{\psi}_{+2}(\alpha)\\
&=\frac{A_2}{\pi}G_1(\alpha)-\frac{A_1}{\pi}G_5(\alpha)-\frac{T(\alpha)\cosh\varkappa(-k)}{2}C_2\\
&\quad+\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)}{2}C_1. \qquad (80)
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= F_{+}^{1}(k)\big|_{\lambda=1}-F_{+}^{1}(k)\big|_{\lambda=-1},\\
C_2 &= F_{+}^{2}(k)\big|_{\lambda=1}-F_{+}^{2}(k)\big|_{\lambda=-1}. \qquad (81)
\end{aligned}$$

Eliminating $\overline{\psi}_{+2}(\alpha)$ from Eqs. (79) and (80), we obtain

$$\begin{aligned}
\overline{\psi}_{+1}(\alpha)&=\left(\frac{A_1}{\pi}\cosh\varkappa(\alpha)+\frac{A_2}{\pi}\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)G_1(\alpha)-\frac{A_2}{\pi}G_3(\alpha)\cosh\varkappa(\alpha)\\
&-\frac{A_1}{\pi}G_5(\alpha)\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}-\frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\cosh\varkappa(\alpha)+C_2\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)\\
&+\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_2}{2\sqrt{2k}}\\
&+\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(\alpha)/\gamma(\alpha)C_1}{2}. \qquad (82)
\end{aligned}$$

and eliminating $\overline{\psi}_{+1}(\alpha)$ between Eqs. (79) and (80), will yield

$$\begin{aligned}\overline{\psi}_{+2}(\alpha) = &\left(\frac{A_1}{\pi}\gamma(\alpha)\sinh\varkappa(\alpha) + \frac{A_2}{\pi}\cosh\varkappa(\alpha)\right)G_1(\alpha)\\ &-\frac{A_2}{\pi}G_3(\alpha)\gamma(\alpha)\sinh\varkappa(\alpha) - \frac{A_1}{\pi}G_5(\alpha)\cosh\varkappa(\alpha)\\ &-\frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\gamma(\alpha)\sinh\varkappa(\alpha) + C_2\cosh\varkappa(\alpha)\right)\\ &+\frac{T_1(\alpha)\sinh\varkappa(-k)\gamma(\alpha)\sinh\varkappa(\alpha)C_2}{2\sqrt{2k}}\\ &+\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_1}{2}. \end{aligned} \tag{83}$$

Now in order to calculate the function $\overline{\psi}_{-1}(\alpha)$ and $\overline{\psi}_{-2}(\alpha)$ we replace $G_1(\alpha)$ by $G_2(\alpha)$ (and $G_2(\alpha)$ by $G_1(\alpha)$), $G_3(\alpha)$ by $G_4(\alpha)$ (and $G_4(\alpha)$ by $G_3(\alpha)$) and $G_5(\alpha)$ by $G_6(\alpha)$ (and $G_6(\alpha)$ by $G_5(\alpha)$) and also changing α to $-\alpha$ in the Eqs. (82)and (83), respectively, we arrive at:

$$\begin{aligned}\overline{\psi}_{-1}(\alpha) = &\left(\frac{A_1}{\pi}\cosh\varkappa(-\alpha) + \frac{A_2}{\pi}\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)G_2(-\alpha)\\ &-\frac{A_2}{\pi}G_4(-\alpha)\cosh\varkappa(-\alpha) - \frac{A_1}{\pi}G_6(-\alpha)\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\\ &-\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(\widetilde{C}_1\cosh\varkappa(-\alpha) + \widetilde{C}_2\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)\\ &+\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}}\\ &+\frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\sinh\varkappa(-\alpha)/\gamma(-\alpha)\widetilde{C}_1}{2}, \end{aligned} \tag{84}$$

and

$$\begin{aligned}\overline{\psi}_{-2}(\alpha) = &\left(\frac{A_1}{\pi}\gamma(-\alpha)\sinh\varkappa(-\alpha) + \frac{A_2}{\pi}\cosh\varkappa(-\alpha)\right)G_2(-\alpha)\\ &-\frac{A_2}{\pi}G_4(-\alpha)\gamma(-\alpha)\sinh\varkappa(-\alpha) - \frac{A_1}{\pi}G_6(-\alpha)\cosh\varkappa(-\alpha)\\ &-\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(\widetilde{C}_1\gamma(-\alpha)\sinh\varkappa(-\alpha)+\widetilde{C}_2\cosh\varkappa(-\alpha)\right)\\ &+\frac{T_1(-\alpha)\sinh\varkappa(-k)\gamma(-\alpha)\sinh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}}\\ &+\frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_1}{2}. \end{aligned} \tag{85}$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are given by

$$\begin{aligned}\widetilde{C}_1 &= \widetilde{F}_+^1(k)\Big|_{\lambda=1} - \widetilde{F}_+^1(k)\Big|_{\lambda=-1},\\ \widetilde{C}_2 &= \widetilde{F}_+^2(k)\Big|_{\lambda=1} - \widetilde{F}_+^2(k)\Big|_{\lambda=-1},\end{aligned} \tag{86}$$

and $\widetilde{F}_+^1(k)$ and $\widetilde{F}_+^2(k)$ denote the functions in which G_1 by G_2 and G_2 by G_1, G_3 by G_4 and G_4 by G_3 and G_5 by G_6 and G_6 by G_5 have also been interchanged and then evaluated for $\lambda = 1$ and $\lambda = -1$ respectively. Since the functions $\overline{\psi}_{\pm1}(\alpha)$ and $\overline{\psi}_{\pm2}(\alpha)$ have been calculated, therefore we now manipulate Eqs. (20c) and (20d) and the unknown coefficient $A(\alpha)$ is determined to be

$$\begin{aligned}A(\alpha) &= \frac{1}{2K(\alpha)}\left[e^{i\alpha p}\overline{\psi}_{-2}(\alpha) - ik\sin\theta_0 G(\alpha) + e^{i\alpha q}\overline{\psi}_{+2}(\alpha)\right]\\ &\quad -\frac{e^{i\alpha p}\overline{\psi}_{-1}(\alpha)}{2} - \frac{e^{i\alpha q}\overline{\psi}_{+1}(\alpha)}{2}.\end{aligned} \tag{87}$$

Substituting the values of $\overline{\psi}_{\pm1}(\alpha)$ and $\overline{\psi}_{\pm2}(\alpha)$ in Eq. (87) and simplifying we obtain

$$\begin{aligned}A(\alpha) = &\left[\frac{1}{2\pi K(\alpha)}\left\{\frac{ik\sin\theta_0\cosh\varkappa(-\alpha)\cosh\varkappa(-k\cos\theta_0)}{\alpha - k\cos\theta_0}e^{i(\alpha-k\cos\theta_0)p}\right.\right.\\ &-ik\sin\theta_0 R_2(-\alpha)\cosh\varkappa(-\alpha)e^{i\alpha p - k\cos\theta_0 q}\\ &-\frac{ik\sin\theta_0\sinh\varkappa(-\alpha)\gamma(-\alpha)\sinh\varkappa(-k\cos\theta_0)}{(\alpha - k\cos\theta_0)\gamma(-k\cos\theta_0)}e^{i(\alpha-k\cos\theta_0)p}\\ &+ik\sin\theta_0 R_4(-\alpha)\gamma(-\alpha)\sinh\varkappa(-\alpha)e^{i\alpha p - k\cos\theta_0 q}\\ &+\left(\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(-\gamma(-\alpha)\sinh\varkappa(-\alpha)\widetilde{C}_1 + \cosh\varkappa(-\alpha)\widetilde{C}_2\right)\right.\\ &+\frac{T_1(-\alpha)\gamma(-\alpha)\sinh\varkappa(-\alpha)\sinh\varkappa(-k)\widetilde{C}_2}{2\sqrt{2k}}\\ &\left.\left.+\frac{T_2(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\sqrt{2k}\widetilde{C}_1}{2}\right)e^{i\alpha p}\right\}\\ &+\frac{1}{2\pi K(\alpha)}\left\{\frac{-ik\sin\theta_0\cosh\varkappa(\alpha)\cosh\varkappa(k\cos\theta_0)}{\alpha - k\cos\theta_0}e^{i(\alpha-k\cos\theta_0)q}\right.\\ &-ik\sin\theta_0 R_1(\alpha)\cosh\varkappa(\alpha)e^{i\alpha q - ik\cos\theta_0 p}\\ &+\frac{ik\sin\theta_0\sinh\varkappa(\alpha)\gamma(\alpha)\sinh\varkappa(k\cos\theta_0)}{(\alpha - k\cos\theta_0)\gamma(k\cos\theta_0)}e^{i(\alpha-k\cos\theta_0)q}\end{aligned}$$

$$
\begin{aligned}
&+ik\sin\theta_0 R_3(\alpha)\gamma(\alpha)\sinh\varkappa(\alpha)e^{i\alpha q-k\cos\theta_{0p}q}\\
&+\left(\frac{T(\alpha)\cosh\varkappa(-k)}{2}\left(-\gamma(\alpha)\sinh\varkappa(\alpha)C_1-\cosh\varkappa(\alpha)C_2\right)\right.\\
&+\frac{T_1(\alpha)\gamma(\alpha)\sinh\varkappa(\alpha)\sinh\varkappa(-k)C_2}{2\sqrt{2k}}\\
&\left.+\frac{T_2(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)\sqrt{2k}C_1}{2}\right)e^{i\alpha q}\Bigg\}\\
&+\frac{1}{2\pi}\Bigg\{\frac{-ik\sin\theta_0\sinh\varkappa(-\alpha)\cosh\varkappa\left(-k\cos\theta_0\right)}{\gamma(-\alpha)\left(\alpha-k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)p}\\
&+\frac{ik\sin\theta_0\cosh\varkappa(-\alpha)\sinh\varkappa\left(-k\cos\theta_0\right)}{\left(\alpha-k\cos\theta_0\right)\gamma\left(-k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)p}\\
&+\frac{ik\sin\theta_0 R_2(-\alpha)\sinh\varkappa(-\alpha)e^{i\alpha p-k\cos\theta_0 q}}{\gamma(-\alpha)}\\
&-ik\sin\theta_0 R_4(-\alpha)\cosh\varkappa(-\alpha)e^{i\alpha p-k\cos\theta_0 q}\\
&+\left(\frac{-T(-\alpha)\cosh\varkappa(-k)}{2}\left(\cosh\varkappa(-\alpha)\widetilde{C}_1+\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\widetilde{C}_2\right)\right.\\
&+\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}}\\
&\left.+\ \frac{T_2(-\alpha)\sinh\varkappa(-k)\sinh\varkappa(-\alpha)\sqrt{2k}\widetilde{C}_1}{2}\right)e^{i\alpha p}\Bigg\}\\
&+\frac{1}{2\pi}\Bigg\{\frac{ik\sin\theta_0\sinh\varkappa(\alpha)\cosh\varkappa\left(k\cos\theta_0\right)}{\gamma(\alpha)\left(\alpha-k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)q}\\
&-\frac{ik\sin\theta_0\cosh\varkappa(\alpha)\sinh\varkappa\left(k\cos\theta_0\right)}{\left(\alpha-k\cos\theta_0\right)\gamma\left(-k\cos\theta_0\right)}e^{i(\alpha-k\cos\theta_0)q}\\
&-ik\sin\theta_0 R_3(\alpha)\cosh\varkappa(\alpha)e^{i\alpha q-k\cos\theta_0 p}\\
&+\frac{ik\sin\theta_0 R_1(\alpha)\sinh\varkappa(\alpha)e^{i\alpha q-k\cos\theta_0 p}}{\gamma(\alpha)}\\
&-\left(\frac{-T(-\alpha)\cosh\varkappa(-k)}{2}\left(\cosh\varkappa(\alpha)C_1+\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}C_2\right)\right.\\
&+\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_2}{2\sqrt{2k}}\\
&\left.+\ \frac{T_2(\alpha)\sinh\varkappa(-k)\sinh\varkappa(\alpha)\sqrt{2k}C_1}{2\gamma(\alpha)}\right)e^{i\alpha q}\Bigg\}\Bigg]. \qquad (88)
\end{aligned}
$$

Since $A(\alpha)$ has been determined, the scattered field $\psi(x, y)$ can now be determined by substituting $A(\alpha)$ into Eq. (15) and taking the inverse Fourier transform, we shall arrive at

$$\psi(x,y) = \int_{-\infty}^{\infty} A(\alpha) e^{iK(\alpha)y - i\alpha x} d\alpha, \tag{89}$$

where $A(\alpha)$ is defined in Eq. (88). The scattered field $\psi(x, y)$ can be split up into two components as follows:

$$\psi(x,y) = \psi_{sep}(x,y) + \psi_{int}(x,y), \tag{90}$$

where

$$\begin{aligned}
&\psi_{sep}(x,y) = \\
&\int_{-\infty}^{\infty} \left[\frac{1}{2\pi K(\alpha)} \left\{ \left(\begin{array}{c} ik \sin\theta_0 \cosh\varkappa(-\alpha) \cosh\varkappa(-k\cos\theta_0) \\ -\dfrac{ik\sin\theta_0 \sinh\varkappa(-\alpha)\gamma(-\alpha)\sinh\varkappa(-k\cos\theta_0)}{\gamma(-k\cos\theta_0)} \end{array} \right) \right. \right. \\
&+\frac{1}{2\pi}\left(\frac{-ik\sin\theta_0 \sinh\varkappa(-\alpha)\cosh\varkappa(-k\cos\theta_0)}{\gamma(-\alpha)} \right. \\
&\left.\left. +\frac{ik\sin\theta_0\cosh\varkappa(-\alpha)\sinh\varkappa(-k\cos\theta_0)}{\gamma(-k\cos\theta_0)} \right) \right\} \frac{e^{i(\alpha-k\cos\theta_0)p}}{\alpha-k\cos\theta_0} \\
&+\frac{1}{2\pi K(\alpha)} \{(-ik\sin\theta_0\cosh\varkappa(\alpha)\cosh\varkappa(k\cos\theta_0) \\
&\left. +\frac{ik\sin\theta_0\sinh\varkappa(\alpha)\gamma(\alpha)\sinh\varkappa(k\cos\theta_0)}{\gamma(k\cos\theta_0)} \right) \\
&+\frac{1}{2\pi}\left(\frac{ik\sin\theta_0\sinh\varkappa(\alpha)\cosh\varkappa(k\cos\theta_0)}{\gamma(\alpha)} \right. \\
&\left.\left. -\frac{ik\sin\theta_0\cosh\varkappa(\alpha)\sinh\varkappa(k\cos\theta_0)}{\gamma(k\cos\theta_0)} \right) \right\} \left. \frac{e^{i(\alpha-k\cos\theta_0)q}}{\alpha-k\cos\theta_0} \right] e^{iK(\alpha)y-i\alpha x} d\alpha,
\end{aligned} \tag{91}$$

and

$$\begin{aligned}
\psi_{int}(x,y) &= \int_{-\infty}^{\infty} \frac{1}{2\pi K(\alpha)} [\{(-ik\sin\theta_0\cosh\varkappa(\alpha)R_1(\alpha) \\
&+ik\sin\theta_0\sinh\varkappa(\alpha)\gamma(\alpha)R_3(\alpha))\, e^{i\alpha q - ik\cos\theta_0 p} \\
&+\left(\frac{T(\alpha)\cosh\varkappa(-k)}{2} (-\gamma(\alpha)\sinh\varkappa(\alpha)C_1 - \cosh\varkappa(-\alpha)C_2) \right.
\end{aligned}$$

$$+\frac{T_1(\alpha)\gamma(\alpha)\sinh\varkappa(\alpha)\sinh\varkappa(-k)C_2}{2\sqrt{2k}}$$
$$+\left.\left.\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_1}{2}\right)e^{i\alpha q}\right\}$$
$$+\frac{1}{2\pi K(\alpha)}\left\{(-ik\sin\theta_0\cosh\varkappa(-\alpha)R_2(-\alpha)\right.$$
$$+ik\sin\theta_0\sinh\varkappa(-\alpha)\gamma(-\alpha)R_4(-\alpha))\,e^{i\alpha p-ik\cos\theta_0 q}$$
$$+\left(\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(-\gamma(-\alpha)\sinh\varkappa(-\alpha)\widetilde{C}_1-\cosh\varkappa(-\alpha)\widetilde{C}_2\right)\right.$$
$$+\frac{T_1(-\alpha)\gamma(-\alpha)\sinh\varkappa(-\alpha)\sinh\varkappa(-k)\widetilde{C}_2}{2\sqrt{2k}}$$
$$+\left.\left.\frac{T_2(-\alpha)\sqrt{2k}\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_1}{2}\right)e^{i\alpha p}\right\}$$
$$+\frac{1}{2\pi}\left\{\left(-ik\sin\theta_0\cosh\varkappa(\alpha)R_3(\alpha)+ik\sin\theta_0\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}R_1(\alpha)\right)\right.$$
$$\times e^{i\alpha q-ik\cos\theta_0 p}-\left(\frac{-T(\alpha)\cosh\varkappa(-k)}{2}\left(C_1\cosh\varkappa(\alpha)+C_2\frac{\sinh\varkappa(\alpha)}{\gamma(\alpha)}\right)\right.$$
$$+\frac{T_1(\alpha)\sinh\varkappa(-k)\cosh\varkappa(\alpha)C_2}{2\sqrt{2k}}$$
$$+\left.\left.\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(\alpha)\sinh\varkappa(-k)C_1}{2}\right)e^{i\alpha q}\right\}$$
$$+\frac{1}{2\pi}\left\{\left(\frac{ik\sin\theta_0\sinh\varkappa(-\alpha)R_2(-\alpha)}{\gamma(-\alpha)}-ik\sin\theta_0\cosh\varkappa(-\alpha)R_4(-\alpha)\right)\right.$$
$$\times e^{i\alpha p-ik\cos\theta_0 q}$$
$$-\left(\frac{T(-\alpha)\cosh\varkappa(-k)}{2}\left(-\widetilde{C}_1\cosh\varkappa(-\alpha)-\widetilde{C}_2\frac{\sinh\varkappa(-\alpha)}{\gamma(-\alpha)}\right)\right.$$
$$+\frac{T_1(-\alpha)\sinh\varkappa(-k)\cosh\varkappa(-\alpha)\widetilde{C}_2}{2\sqrt{2k}}$$
$$+\left.\left.\left.\frac{T_2(\alpha)\sqrt{2k}\sinh\varkappa(-\alpha)\sinh\varkappa(-k)\widetilde{C}_1}{2}\right)e^{i\alpha p}\right\}\right]e^{iK(\alpha)y-i\alpha x}d\alpha, \qquad (92)$$

where $\psi_{sep}(x,y)$ gives the diffracted field produced by the edges at $x=p$ and at $x=q$ respectively and $\psi_{int}(x,y)$ gives the interaction of one edge upon the other edge.

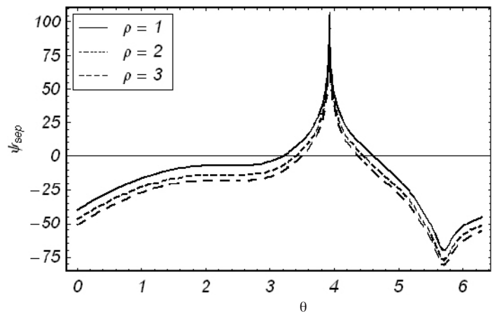

Figure 2. Variation of the separated field ψ_{sep} with observation angle θ at $\theta_0 = \frac{\pi}{4}$, $k = 1$ and $l = 1$.

2.1. Far-Field Solution

The calculations carried out for the three part boundary value problem formulated in terms of matrix WH equations are quite laborious and delicate at the same time, so we report the far field only for the case of $y > 0$ only, (i.e., we determine the unknown coefficient $A(\alpha)$ only), the far field for the case of $y < 0$ can be calculated in a similar manner. Therefore, in order to solve the integral appearing in Eq. (89) we introduced the following substitutions

$$x = \rho\cos\theta, \quad y = \rho\sin\theta \quad \text{and} \quad \alpha = -k\cos\left(\theta + it_1\right), \tag{93}$$

in Eq. (89), omitting the computational details, and using the method of steepest descent, the field at the large distance from a slit in an infinite soft-hard plane is given as

$$\psi(x, y) \simeq \sqrt{\frac{2\pi}{k\rho}} i\sin\theta A(-k\cos\theta)e^{ik\rho+i\frac{\pi}{4}}. \tag{94}$$

where $A(-k\cos\theta)$ can be evaluated from Eq. (88).

3. GRAPHICAL RESULTS

In this section we will present some graphs showing the effects of various parameters on the diffracted field produced by the two edges of the slit in an infinite soft-hard plane.

Figs. 2 and 3 show the variation of separated field ψ_{sep} with observation angle θ at $\theta_0 = \pi/4$, $k = 1$ and $\rho = 1$, 2, 3 for $l = 1$ and 5, respectively. It is observed that by increasing the parameter ρ the overall amplitude of the separated field decreases. The effect of slit

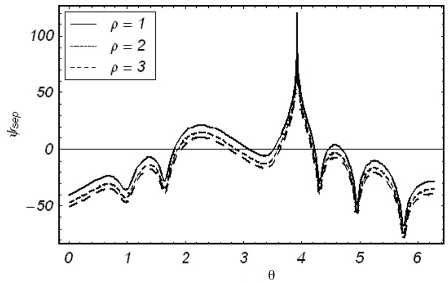

Figure 3. Variation of the separated field ψ_{sep} with observation angle θ at $\theta_0 = \frac{\pi}{4}$, $k = 1$ and $l = 5$.

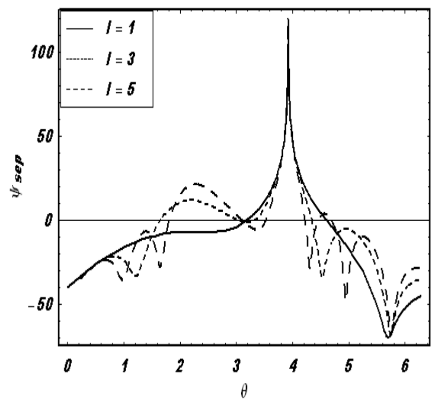

Figure 4. Variation of the separated field ψ_{sep} with observation angle θ at $\theta_0 = \frac{\pi}{4}$, $\rho = 1$ and $k = 1$.

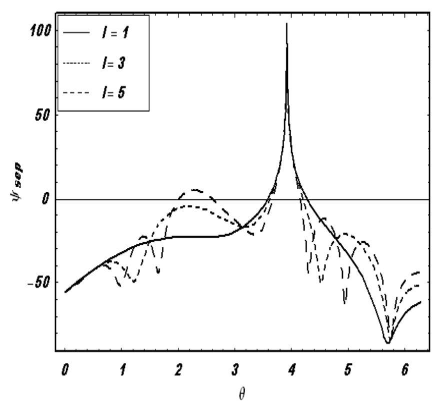

Figure 5. Variation of the separated field ψ_{sep} with observation angle θ at $\theta_0 = \frac{\pi}{4}$, $\rho = 5$ and $k = 1$.

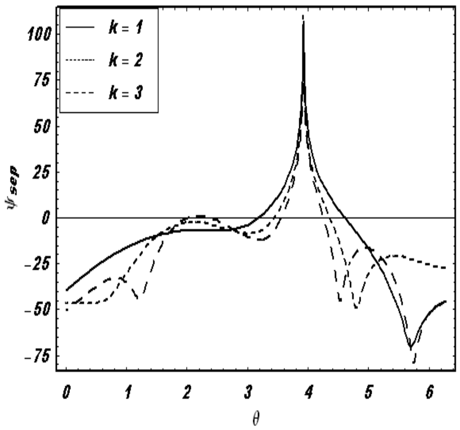

Figure 6. Variation of the separated field ψ_{sep} with observation angle θ at $\theta_0 = \frac{\pi}{4}$, $\rho = 1$ and $l = 1$.

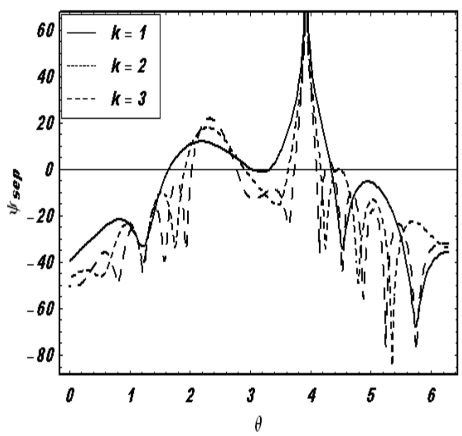

Figure 7. Variation of the separated field ψ_{sep} with observation angle θ at $\theta_0 = \frac{\pi}{4}$, $\rho = 1$ and $l = 5$.

width parameter l is observed through the Figures 4 and 5 in which $\theta_0 = \pi/4, k = 1$ and $l = 1,\ 3,\ 5$ for $\rho = 1$ and 5. It is noted that by keeping the other parameters fixed and increasing the parameter l causes more oscillations in the separated field and its amplitude decreases. Finally in order to see the effects of wave number parameter k figures 6 and 7 are plotted for $\theta_0 = \pi/4$, $\rho = 1$ and $k = 1, 2, 3$ for $l = 1$ and 5. These graphs depict that increasing the parameter k results in increasing oscillations in the separated field and the amplitude of the separated field decreases.

4. CONCLUSION

In this paper the diffraction of a plane acoustic wave by a slit in an infinite soft hard plane is investigated rigorously with the help of integral transform, Wiener-Hopf technique and the method of steepest descent. Further the consideration of slit in an infinite soft-hard plane will help understand acoustic diffraction and will go a step further to complete the discussion for the soft-hard half plane. The two edges of the slit give rise to two diffracted fields (one from each edge) and the interaction of one edge upon the other edge. The diffracted field is presented for the far-field situation and some graphs showing the effects of various parameters on the separated field are also plotted.

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