Progress In Electromagnetics Research Letters, Vol. 5, 23–34, 2008

RESEARCH ON THE COHERENT PHASE NOISE OF MILLIMETER-WAVE DOPPLER RADAR

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Abstract—The phase noise is a very important index to wireless system, especially in millimeter-wave continuous wave radar systems. The phase noise of the signal, which is firstly leaked from transmitter and then mixed to intermediate frequency band by the local oscillator (Tx-IF), will worsen the sensitivity of superheterodyne radar system used for Doppler velocity detection. In this paper, the coherent analysis is applied on the phase noise after nonlinear process, which shows that the phase noise of the Tx-IF is affected by those factors: the magnitude of the phase noise of the transmitter and that of the local oscillator, and the correlationship between each other. In practice, by reducing the phase noise of the transmitter and that of the local oscillator and ameliorating the correlationship of the two phase noises, the phase noise of the Tx-IF can be improved greatly. Such proposition is successfully applied in the design of a millimeter-wave Doppler radar working at 95 GHz. The experimental measurement shows that the sensitivity of this radar is better than −70 dBm.

1. INTRODUCTION

As the simplest one of radars, continuous-wave (CW) Doppler radar is used widely. Operating at a 100% duty cycle, it maximizes the use of transmitting power and eliminates issues of Doppler ambiguity. According to Doppler theory, Doppler frequency is proportional to working frequency of the transmitter. Therefore, millimeter-wave frequency band is the preferable choice for low-speed detection, within which the two atmosphere windows, i.e., Ka band and W band, are widely used in practice [1, 2].

But the phase noise, one of the key characteristics of frequency sources [3, 4], will deteriorate sharply along with the rise of operating frequency, which will result in the fact that the performance of millimeter-wave radar is not so good as that of the microwave one [5–7]. Especially, the sensitivity of millimeter-wave radar will sharply deteriorate. The sensitivity of radar is defined as the lowest signal power level which can be detected by radar, which is dependent on the noises from outer space or radar itself. In our study, it shows that the influence of the phase noise cannot be ignored in millimeter-wave radar. In this paper, the effect of the phase noise on the performance of a millimeter-wave Doppler radar system is analyzed in detail, and the improvement on it is proposed.

2. THE EFFECT OF PHASE NOISE ON THE SENSITIVITY OF MILLIMETER-WAVE DOPPLER RADAR

Figure 1 shows a schematic diagram of the front-end of millimeter-wave superheterodyne Doppler radar. The millimeter-wave transmitter is connected with the receiver by a circulator. The operating frequency of the transmitter signal (Tx) is f_{Tx}, with the emitting power P_{Tx}. There are two signals coming into the receiver. One is the echo signal (Rx) whose power is P_{Rx} and the frequency is $f_{\mathrm{Tx}}+f_d$ with f_d being the Doppler frequency shift. The other is the signal leaked from the Tx to

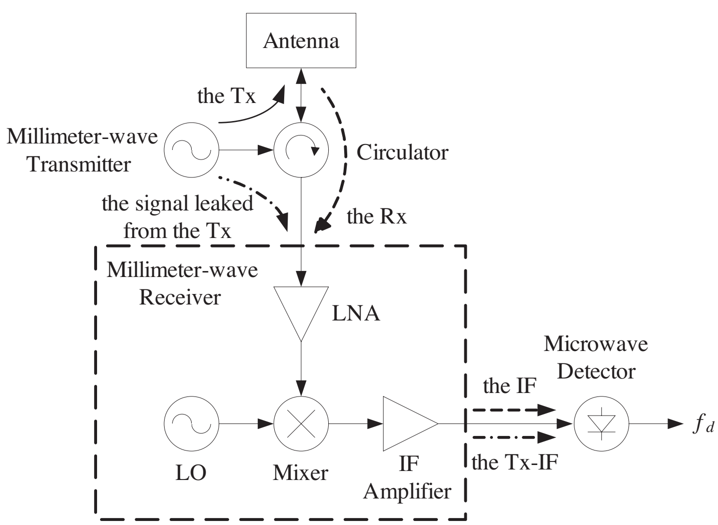


Figure 1. The front-end of millimeter-wave superheterodyne Doppler radar.

the receiver, whose power is $P_{\mathrm{Tx}} \times L_P$ with L_P being the isolation of the circulator, whose frequency is still f_{Tx}. The millimeter-wave receiver can be regard as the frequency down-conversion network. After the two input signals mixed with the local oscillator (LO), another two signals, i.e., the IF and the Tx-IF appear at the output of the receiver. Here the denotation of the IF is used to represent the signal which is the output of the Rx mixed to intermediate frequency band by the LO, and the denotation of the Tx-IF is used to represent the signal which is firstly leaked from transmitter and then mixed to intermediate frequency band by the LO. Supposing that the receiver is working within its dynamic range, and then the receiver can be described with its noise figure F and the total gain G. Thus, the power of the IF is $P_{\mathrm{IF}} = P_{\mathrm{Rx}}G$, with the frequency $f_{\mathrm{IF}} = f_{\mathrm{Tx}} + f_d - f_{\mathrm{LO}}$, and the power of the Tx-IF is $P_{\mathrm{Tx-IF}} = P_{\mathrm{Tx}}L_P G$, with the frequency $f_{\mathrm{Tx-IF}} = f_{\mathrm{Tx}} - f_{\mathrm{LO}}$, where the supposition of $f_{\mathrm{LO}} < f_{\mathrm{Tx}}$ is made. The frequency spectrum of the output of the receiver is shown in Fig. 2.

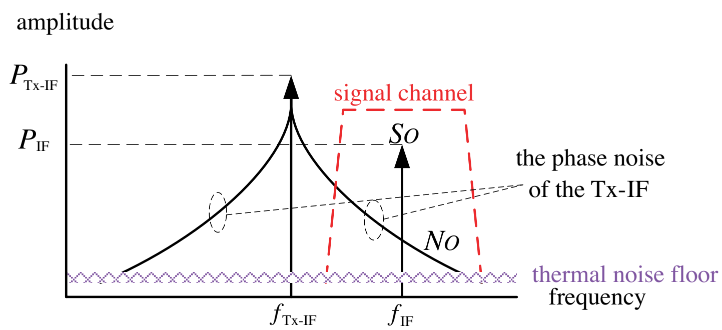


Figure 2. The output of the millimeter-wave receiver.

In the signal channel of the receiver, the output has two parts including the signal with power S_O and the noise with power N_O. The IF is the signal needed for velocity detection to radar, so the signal power in the signal channel is

$$S_O = P_{\mathrm{IF}} = P_{\mathrm{Rx}}G \tag{1}$$

As can be seen from Fig. 2, the noise in the signal channel consists of two parts, i.e., the thermal noise floor and the phase noise of the Tx-IF. The thermal noise floor can be written as $kTBFG$, where k is the Boltzmann constant, T the temperature in Kelvins, and B the signal channel bandwidth. The single side band of the power spectrum density L at f_m in 1 Hz is used here to denote the magnitude of the phase noise, where f_m is the offset frequency from the carrier. Both of them are additive in the working frequency band, so the total output noise is

$$\begin{aligned} N_O &= \int\limits_B P_{\mathrm{Tx-IF}} L_{\mathrm{TX-IF}}\left(f_m\right) df_m + kTBFG \\ &= \int\limits_B P_{\mathrm{Tx}} L_P G L_{\mathrm{TX-IF}}\left(f_m\right) df_m + kTBFG \end{aligned} \tag{2}$$

In order to improve the sensitivity of radar system, the N_O should be reduced. On the other hand, the reduction of the N_O is also advantageous to the improvement of the output Signal to Noise Ratio (SNR) of the radar.

$$\mathrm{SNR} = \frac{S_O}{N_O} \tag{3}$$

3. IMPROVEMENT ON THE PHASE NOISE OF THE TX-IF

Due to the fact that the phase noise of the Tx-IF is related with the phase noise of the transmitter and that of the LO, the mathematical relationship among those phase noises will be derived for analysis in the following.

Firstly, without losing generality, we assume that a continuous wave with side band noise in the time domain is represented by

$$v_O\left(t\right) = V_O \cos\left(2\pi f_O t + \Delta\phi \sin 2\pi f_m t\right) \tag{4}$$

where, V_O is the amplitude of the signal, f_O the frequency, and f_m is the modulation (also known as offset) frequency. Here the modulation index $\Delta\phi$, a function of f_m, is used to represent small changes in the oscillator frequency. Assuming that the phase deviation is small, i.e., $\Delta\phi \ll 1$. The following Equation (5) can be get after some simple mathematical manipulations.

$$v_O(t) \approx V_O \left\{ \cos 2\pi f_O t - \frac{\Delta\phi}{2} [\cos 2\pi(f_O + f_m)t - \cos 2\pi(f_O - f_m)t] \right\} \tag{5}$$

(5) shows that $v_O(t)$ includes two parts: the signal $V_O \cos 2\pi f_O t$ whose power is $P_s = 1/2V_O^2$ and the noise $V_O\Delta\phi/2[\cos 2\pi(f_O + f_m)t - \cos 2\pi(f_O - f_m)t]$. Small phase or frequency deviation in the output of an oscillator results in the modulation sidebands at $f_O \pm f_m$, located on each side of the carrier frequency f_O with the power $P_n = 1/2(V_O\Delta\phi/2)^2$ for each side band during in 1 Hz at f_m. Here the load resistance is unitary.

According to the definition of the phase noise, i.e., the ratio of noise power in a single sideband P_n to the carrier power P_s, the output waveform of (5) has a corresponding single sideband phase noise L is [6]

$$L = \frac{P_n}{P_s} = \frac{\frac{1}{2}\left(\frac{V_O \Delta\phi}{2}\right)^2}{\frac{1}{2}V_O^2} = \frac{\Delta\phi^2}{4} \tag{6}$$

As can be seen from (6), $\Delta\phi$ can be used to describe the phase noise of the signal at the offset frequency f_m in the time domain. Therefore, the relationship of the expression of the phase noise in the time domain and the frequency domain is found.

Based on the analysis above, the transmitter and the LO signals are supposed in the time domain, which are

$$v_{\mathrm{Tx}}(t) = V_{\mathrm{Tx}} \cos\left(2\pi f_{\mathrm{Tx}} t + \Delta\phi_{\mathrm{Tx}} \sin 2\pi f_m t\right) \tag{7}$$

$$v_{\mathrm{LO}}(t) = V_{\mathrm{LO}} \cos\left(2\pi f_{\mathrm{LO}} t + \Delta\phi_{\mathrm{LO}} \sin 2\pi f_m t\right) \tag{8}$$

Here, we use the modulation index $\Delta\phi_{\mathrm{Tx}}$ or $\Delta\phi_{\mathrm{LO}}$ to denote the phase noise level of the transmitter or the LO, respectively.

As seen from Fig. 1, the product of the transmitter and LO signals, i.e., $v_{\mathrm{Tx}}(t) \times v_{\mathrm{LO}}(t)$, is obtained after both signals are inputted into the mixer. Then, only the Tx-IF is kept due to the filtering of the lowpass filter. Thus, with (5), (7) and (8) taken into consideration, we can get the mathematical expression of the Tx-IF as follows.

$$\begin{aligned} v_{\mathrm{Tx-IF}}(t) &= V_{\mathrm{Tx-IF}} \cos 2\pi f_{\mathrm{Tx-IF}} t + \Delta\phi_{\mathrm{Tx}} \sin 2\pi f_m t - \Delta\phi_{\mathrm{LO}} \sin 2\pi f_m t \\ &\approx V_{\mathrm{Tx-IF}} \left\{ \cos 2\pi f_{\mathrm{Tx-IF}} t - \frac{\Delta\phi_{\mathrm{Tx}}}{2} \cos 2\pi (f_{\mathrm{Tx-IF}} + f_m) t \right. \\ &\quad + \frac{\Delta\phi_{\mathrm{LO}}}{2} \cos 2\pi (f_{\mathrm{Tx-IF}} + f_m) t + \frac{\Delta\phi_{\mathrm{Tx}}}{2} \cos 2\pi (f_{\mathrm{Tx-IF}} - f_m) t \\ &\quad \left. - \frac{\Delta\phi_{\mathrm{LO}}}{2} \cos 2\pi (f_{\mathrm{Tx-IF}} - f_m) t \right\} \end{aligned} \tag{9}$$

Here, $V_{\mathrm{Tx-IF}}$ is amplitude of the Tx-IF. (9) shows that both the phase noise of the transmitter (L_{Tx}) and the phase noise of the LO (L_{LO}) at the same offset frequency f_m, are mixed together to form the phase noise of the Tx-IF ($L_{\mathrm{Tx-IF}}$) in modulation sidebands at $f_{\mathrm{Tx-IF}} \pm f_m$, located on each side of the frequency of the Tx-IF. The mixer of the receiver can be regarded as a correlator here. Therefore, it is necessary to analyze the correlationship of the two phase noises, to determine the phase noise of the Tx-IF. Here, the correlation coefficient ρ is used to define the correlationship between the two phase noises. According to the definition [9, 10], ρ presents the dependence between L_{Tx} and L_{LO}. For the two phase noises themselves, there is only positive dependence between each other, so $0 \le \rho \le 1$. Especially, ρ will be zero if there is no relationship between the two phase noises. Otherwise, ρ will be one if there is a perfect positive match.

From (9), the upper sideband noise of the Tx-IF is

$$\begin{aligned} x(t) &= V_{\mathrm{Tx-IF}} \left\{ -\frac{\Delta\phi_{\mathrm{Tx}}}{2} \cos 2\pi \left(f_{\mathrm{Tx-IF}} + f_m \right) t \right. \\ &\quad \left. + \frac{\Delta\phi_{\mathrm{LO}}}{2} \cos 2\pi \left(f_{\mathrm{Tx-IF}} + f_m \right) t \right\} \end{aligned} \tag{10}$$

Because the self correlation function is defined as

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t) x(t+\tau)\, dt \tag{11}$$

then (10) can be substituted into (11) to obtain the single sideband noise power at f_m as follows.

$$P_n = R_x(\tau)\big|_{\tau=0} = \frac{V_{\mathrm{Tx-IF}}^2}{8} \left(\Delta\phi_{\mathrm{Tx}}^2 + \Delta\phi_{\mathrm{LO}}^2 - 2\rho\Delta\phi_{\mathrm{Tx}}\Delta\phi_{\mathrm{LO}} \right) \tag{12}$$

In addition, the carrier power of the Tx-IF is:

$$P_s = \frac{1}{2} V_{\mathrm{Tx-IF}}^2 = \frac{1}{2} V_{\mathrm{Tx}}^2 L_P G = P_{\mathrm{Tx}} L_P G \tag{13}$$

after (12) and (13) are substituted into (6), the single sideband phase noise spectrum density of the Tx-IF at f_m is

$$\begin{aligned} L_{\mathrm{Tx-IF}} &= \frac{P_n}{P_s} = \frac{1}{4} \left(\Delta\phi_{\mathrm{Tx}}^2 + \Delta\phi_{\mathrm{LO}}^2 - 2\rho\Delta\phi_{\mathrm{Tx}}\Delta\phi_{\mathrm{LO}} \right) \\ &= L_{\mathrm{Tx}} + L_{\mathrm{LO}} - 2\rho\sqrt{L_{\mathrm{Tx}} L_{\mathrm{LO}}} \end{aligned} \tag{14}$$

As for the phase noise at the lower sideband, it can also be discussed in the same way as the one at the upper sideband, and the result is also the same.

The parameter ρ is hard to be measured directly, but after L_{Tx}, L_{LO} and $L_{\mathrm{Tx-IF}}$ are measured, ρ can be calculated by the following from (14)

$$\rho = \frac{L_{\mathrm{Tx}} + L_{\mathrm{LO}} - L_{\mathrm{Tx-IF}}}{2\sqrt{L_{\mathrm{Tx}} L_{\mathrm{LO}}}} \tag{15}$$

Finally, by substituting (14) into (2), the total output noise is

$$N_O = \int_B P_{\mathrm{Tx}} L_P G \left(L_{\mathrm{Tx}} + L_{\mathrm{LO}} - 2\rho\sqrt{L_{\mathrm{Tx}} L_{\mathrm{LO}}} \right) df_m + kTFGB \tag{16}$$

and the output SNR is

$$\begin{aligned}\mathrm{SNR} &= \frac{S_O}{N_O}\\ &= \frac{P_R G}{\int_B P_{\mathrm{Tx}} L_P G(L_{\mathrm{Tx}}+L_{\mathrm{LO}}-2\rho\sqrt{L_{\mathrm{Tx}}L_{\mathrm{LO}}})df_m+kTFGB}\end{aligned} \tag{17}$$

From (16) and (17), we can draw the conclusion that we can reduce the phase noise of the Tx-IF or the thermal noise floor, or reduce both of them, to make the total output noise low, according to actual systems requirement. For example, if the thermal noise floor is dominant at the output which always happens in microwave systems, the sensitivity of radar will be improved by reducing the F of the receiver. In this case, enhancing the P_{Tx} can enlarge the P_{Rx} to improve the output SNR of radar. However, when the contribution of the phase noise is dominant, enhancing the P_{Tx} will not improve the SNR of radar, because the noise is enhanced either. In this case, the sensitivity will be improved by improving the L_P, or reducing the L_{Tx} and L_{LO}, or improving the ρ between them. In other cases, both might be reduced through the methods discussed above at the same time.

In practice, the L_P is hard to be adjusted, which is rarely used in millimeter-wave band and thus will not be discussed in this paper. In addition, $L_{\mathrm{Tx-IF}}$ is influenced by several factors: L_{Tx} and L_{LO}, and ρ between each other. As for how to reduce L_{Tx} and L_{LO}, some effective methods can be found in the literatures [11–15]. Thus, we only discuss how to improve ρ in the following part.

Assuming that $L_{\mathrm{Tx}} = L_{\mathrm{LO}}$, which means that the phase noise of the transmitter and that of the LO have the same magnitude. Calculating by (14), the relationship between $L_{\mathrm{Tx-IF}}$ and L_{Tx} is shown in Fig. 3.

Figure 3 shows that, if there is no correlationship between L_{Tx} and L_{LO}, $L_{\mathrm{Tx-IF}}$ is the double of L_{Tx}. With the ρ increasing, $L_{\mathrm{Tx-IF}}$ is significantly improved. Especially when $\rho = 1$, the $L_{\mathrm{Tx-IF}}$ will be 0. This case only happens when the transmitter and the LO are the same signals thus the radar system will be zero intermediate frequency system and Direct Current (DC) appears at the output port. But in practice, there is always some frequency difference between the transmitter and the LO in the superheterodyne Doppler radar system, so for better $L_{\mathrm{Tx-IF}}$, there should be a strong positive relationship between L_{Tx} and L_{LO}.

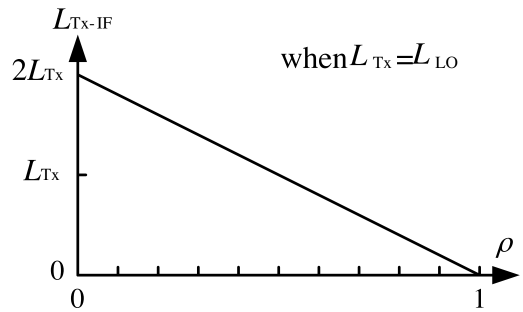


Figure 3. The relationships between $L_{\mathrm{Tx-IF}}$ and L_{Tx}.

4. PROJECT DESIGN AND VERIFICATION

In order to verify the discussion above, we designed a frequency synthesizer of millimeter-wave Doppler front-end to generate the transmitter and the LO signals at W band. The schematic is shown in Fig. 4.

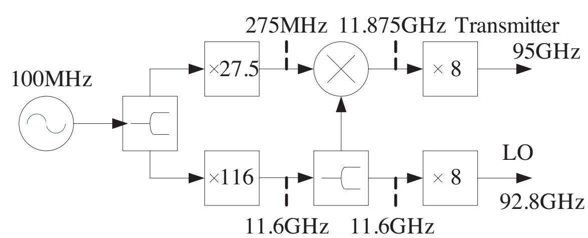


Figure 4. A schematic of W band frequency synthesizer of millimeter-wave Doppler radar.

In this schematic, f_{Tx} is designed at 95 GHz and f_{LO} is 92.8 GHz. Therefore, $f_{\mathrm{Tx-IF}} = f_{\mathrm{Tx}} - f_{\mathrm{LO}}$ is 2.2 GHz. In Fig. 4, all signals are coherent because they depend on the same low phase noise 100 MHz reference signal from crystal oscillator (Xtal). A 116 times multiplication of the Xtal is used to generate 11.6 GHz frequency signal, and a further 8 times multiplication of one part of 11.6 GHz signal is used to generate the LO signal at 92.8 GHz. Another part of 11.6 GHz signal mixed with 275 MHz signal, the 27.5 times of the Xtal, generates 11.875 GHz signal after passing through an up-conversion mixer. Then, another 8 times multiplication of 11.875 GHz is used to generate transmitter at 95 GHz. Thus, the phase noise of 11.6 GHz signal will be counteracted at the Tx-IF, and the phase noise of Tx-IF

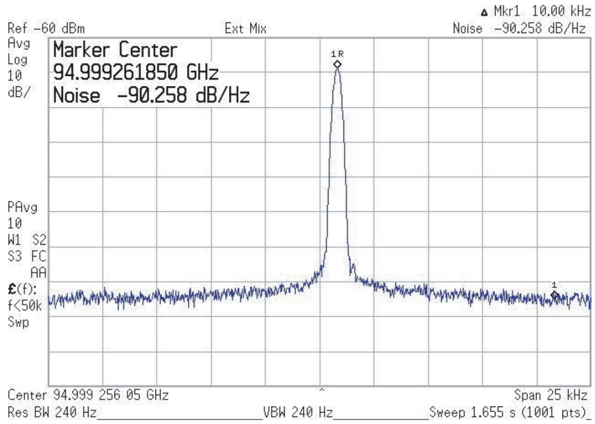


Figure 5. The phase noise of the transmitter.

should only be related with that of the 275 MHz signal, and be equal to the phase noise of 275 MHz multiplied by $8^2 = 64$ or added by $20 \log 8 = 18$ in dB.

According to Fig. 4, the W-band coherent radar front-end is designed and the phase noises of the key signals are measured. Firstly, the phase noise of 275 MHz signal is measured by using the signal analyzer, and the results are shown in Table 1.

Table 1. The phase noise of the 275 MHz signal.

f_m (Hz)	1 K	10 K	100 K	1 M
$L_{\text{Tx}-\text{IF}}$ (dBc/Hz)	-127.32	-136.13	-140.90	-149.12

The frequencies of the X band and the W band signals exceed the upper frequency measurement limit of the signal analyzer, so the phase noises of those signals are measured by the frequency spectrum analyzer. In order to compare the phase noises of different frequency signals, the f_m is fixed to 10 KHz. The transmitter and the LO have the same phase noise, which is -90.26 dBc/Hz@10 KHz measured by the frequency spectrum analyzer of Agilent E4440A with an external harmonic mixer. And the output spectrum is shown in Fig. 5. The output powers of those signals are measured by the power meter, which are about 10 dBm.

As can be seen from Fig. 3, $L_{\text{Tx}-\text{IF}}$ lies within the range of $2L_{\text{Tx}}$ and 0, and its value depends on correlation coefficient ρ. In our design, the theoretical value of $L_{\text{Tx}-\text{IF}}$ is -118 dBc/Hz@10 KHz, calculated by

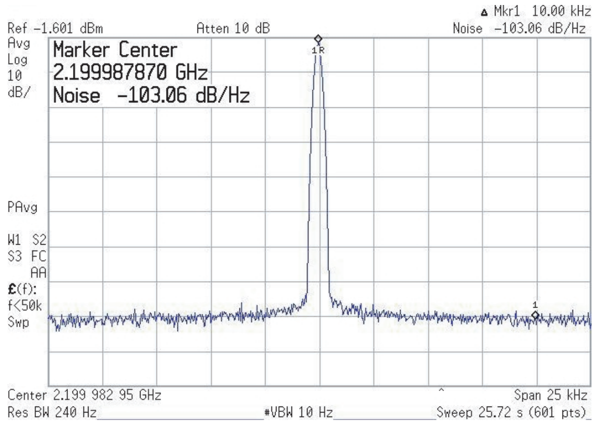


Figure 6. The phase noise of the Tx-IF.

$L_{275\,\mathrm{MHz}}$@$10\,\mathrm{KHz} + 20\log 8$. In the actual experimental measurement, if there is no target to be detected, only the Tx-IF appears at the output port. Measured by the frequency spectrum analyzer, it can be found out that $L_{\mathrm{Tx-IF}} = -103.06\,\mathrm{dBc/Hz}$@$10\,\mathrm{KHz}$ as shown in Fig. 6.

Other parameters of the W band frequency synthesizer are measured too, the measurements show that the $F = 19\,\mathrm{dB}$, $G = 25\,\mathrm{dB}$, $B = 1.5\,\mathrm{MHz}$, and $L_P = 35\,\mathrm{dB}$. According to (15), the correlation coefficient ρ of the two signals should be 0.9992. In our experiment, ρ is 0.9738, slightly lower than the theoretical value. One possible reason to cause the difference is that there is a certain time delay difference between the two signals, which results from the different flow paths. Another possible reason is the thermal noise of the other nonlinear components. Both of them will cause additive noises and worsen the correlationship between the two signals.

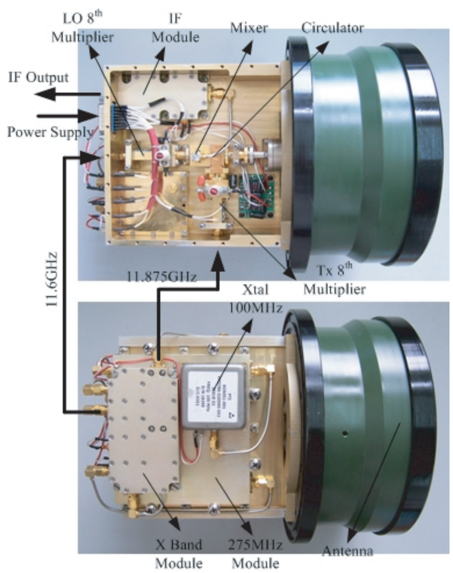
5. CONCLUSION

Through the analysis, the noise model of the output of the millimeter-wave superheterodyne Doppler radar is build. It consists two parts, i.e., the thermal noise floor and the phase noise of the signal which is firstly leaked from transmitter and then mixed to intermediate frequency band by the LO. In order to improve the sensitivity of millimeter-wave Doppler radar, the methods to reduce the phase noise of the Tx-IF is proposed. Then, we apply those methods in the analysis of a W band frequency synthesizer of Doppler radar to improve its performance. The experimental measurement shows that the phase noise of W band transmitter and that of the LO are as low as −90 dBc/Hz@10 KHz. And the correlation coefficient of the two W band signals is as high as 0.9738. Thus, the phase noise of Tx-IF is −103 dBc/Hz@10 KHz. With this front-end, the W band Doppler radar is designed for the detection of target velocity vary from 20 m/s to 1000 m/s, and the sensitivity of the radar is about −70 dBm. The W band front-end is shown in appendix.

ACKNOWLEDGMENT

This work was supported by the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20070614034).

APPENDIX A. THE PICTURE OF THE FRONT-END OF THE W BAND DOPPLER RADAR



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