Progress In Electromagnetics Research C, Vol. 5, 71–82, 2008

IMPROVEMENT IN THE ANALYTICAL CALCULATION OF THE MAGNETIC FIELD PRODUCED BY PERMANENT MAGNET RINGS

S. I. Babic

École Polytechnique
Département de Génie Physique
C.P. 6079 Succ. Centre Ville, QC H3C 3A7, Montréal, Canada

C. Akyel

École Polytechnique
Département de Génie Électrique
C.P. 6079 Succ. Centre Ville, QC H3C 3A7, Montréal, Canada

Abstract—In this paper, we present an improved Coulombian-based analytical calculation of magnetic fields created by permanent-magnetic rings. The 3 dimensional (3D) components of two types of magnetized rings (axially and radially) were analytically evaluated. The obtained components of the magnetic field are expressed over complete elliptical integrals of the first and second kind, as well as by Heuman's Lambda function. These expressions permit fast and accurate calculations of the magnetic field at any point of interest, for both regular and singular cases. The presented method gives an improvement of already known expressions for calculating the magnetic fields of the aforementioned magnetized rings, and we consider that these improved analytical expressions are more extendable to numerical applications.

1. INTRODUCTION

Permanent magnet rings are used in many technical applications, such as magnetic separators, magnetic holding devices, magnetic torque drives, magnetic bearing devices, generators, alternators, loudspeakers, and actuators. In these applications, the magnetic fields created by these magnets need to be accurately calculated to optimize devices used therein. Until now, most analytical and numerical approaches have evaluated two major kinds of applications, such as parallelepipedic and cylindrical magnets, [1–21]. Even though parallelepipedic magnets are easy to produce, are easily magnetized, and create a magnetic field that is easily calculated, we wish to evaluate cylindrical magnets with radial and axial polarizations. The analytical solutions of the magnetic field, based on the Coulumbian approach, created by axially and radially magnetized permanent magnetic rings have been evaluated in [1]. The proposed expressions obtained over the complete elliptical integrals of the first, second, and third kind, as evaluated by Mathematica, produce tedious expressions which are complicated for the numerical treatment. In addition, Mathematica-based solutions contain some magnetic field components with imaginary parts nearly equal to zero (Hz), making these components more difficult to plot with Mathematica, especially in the treatment of singular cases that can be solved analytically. Herein, for the previous reason, we propose purely analytical solutions for these components of the magnetic field for the aforementioned types of magnets. The obtained 3D analytical expressions ameliorate already calculated magnetic fields for permanent magnetic rings, and are more suitable for numerical applications. All results are expressed over the complete elliptical integrals of the first, and second kind, as well by Heuman's Lambda function.

2. BASIC EXPRESSIONS

The solutions for the magnetic fields, produced by permanent magnet rings (See Figs. 1 and 2) can be obtained by using the Coulumbian method [1]. Rings with axial (Fig. 1) and radial (Fig. 2) magnetic polarizations were investigated in this study.

The corresponding magnetic fields are:

a) Axial magnetic polarization

$$\begin{aligned}
\vec{H}^{+}(r,z) &= \frac{\sigma^*}{4\pi\mu_0}\int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{r_1=r_{in}}^{r_1=r_{out}}\frac{P_{1+}\vec{M}}{\left|P_{1+}\vec{M}\right|^3}r_1dr_1d\theta \\
\vec{H}^{-}(r,z) &= \frac{-\sigma^*}{4\pi\mu_0}\int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{r_1=r_{in}}^{r_1=r_{out}}\frac{P_{1-}\vec{M}}{\left|P_{1-}\vec{M}\right|^3}r_1dr_1d\theta \\
\vec{H}(r,z) &= \vec{H}^{+}(r,z)+\vec{H}^{-}(r,z)
\end{aligned} \tag{1}$$

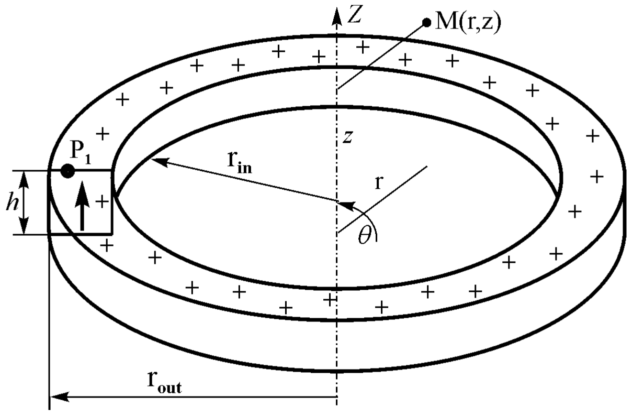

Figure 1. Given geometry: A ring with z axis of symmetry and inner and outer radius r_{in} and r_{out}, respectively. The magnetic polarization is axial [1].

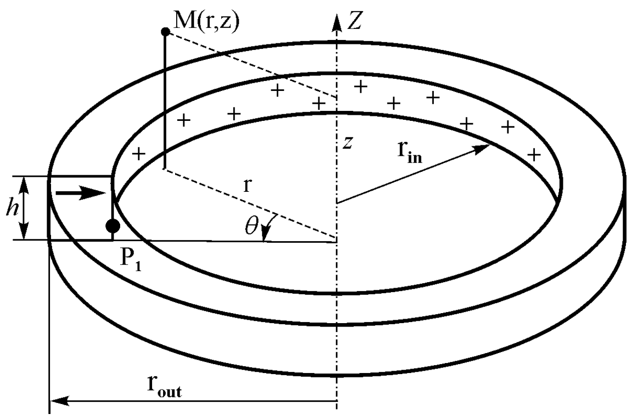

Figure 2. Given geometry: A ring with z axis of symmetry and inner and outer radius r_{in} and r_{out}, respectively. The magnetic polarization is radial, [1].

where

$$P_{1+}\vec{M} = (r - r_1 \cos\theta)\vec{i}_r - r_1 \sin\theta \vec{i}_\theta + (z - h)\vec{i}_k$$
$$P_{1-}\vec{M} = (r - r_1 \cos\theta)\vec{i}_r - r_1 \sin\theta \vec{i}_\theta + (z + h)\vec{i}_k$$

b) Radial magnetic polarization

$$\begin{aligned}
\vec{H}^{+}(r,z) &= \frac{\sigma^*}{4\pi\mu_0}\int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{z_1=0}^{z_2=h}\frac{P_{1+}\vec{M}}{\left|P_{1+}\vec{M}\right|^3}r_{in}dz_1d\theta \\
\vec{H}^{-}(r,z) &= \frac{-\sigma^*}{4\pi\mu_0}\int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{z_1=0}^{z_2=h}\frac{P_{1-}\vec{M}}{\left|P_{1-}\vec{M}\right|^3}r_{out}dz_1d\theta \\
\vec{H}(r,z) &= \vec{H}^{+}(r,z)+\vec{H}^{-}(r,z)
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
P_{1+}\vec{M} &= (r-r_{in}\cos\theta)\vec{i}_r - r_{in}\sin\theta\vec{i}_\theta + (z-z_1)\vec{i}_k \\
P_{1-}\vec{M} &= (r-r_{out}\cos\theta)\vec{i}_r - r_{out}\sin\theta\vec{i}_\theta + (z-z_1)\vec{i}_k
\end{aligned}$$

$\mu_0 = 4\pi\times10^{-7}$ H/m — the magnetic permeability of vacuum. $\sigma^* = 1T$ — surface magnetic pole density, i_r, i_θ, i_z — unit vectors in the cylindrical coordinate system.

3. CALCULATION METHOD

a) Axial magnetic polarization

By substituting the variable θ with $\theta = \pi - 2\beta$ in (1), integrating twice, and respecting angle variable symmetry, we obtained the 3D components of the magnetic field at the point $M(r, 0, z)$. In this case all integrals will be transformed on the interval of integration $0 \le \beta \le \pi/2$ so that all results will be expressed over complete elliptical integrals.

$$\begin{aligned}
H_r(r,z) &= H_r^+(r,z) + H_r^-(r,z) \\
H_\theta(r,z) &= H_\theta^+(r,z) + H_\theta^-(r,z) = 0 \\
H_z(r,z) &= H_z^+(r,z) + H_z^-(r,z)
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
H_r^+(r,z) &= \frac{\sigma^*}{\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\frac{1}{k_n^+}\sqrt{\frac{r_n}{r}}\left[E(k_n^+) - (1-\frac{k_n^{+2}}{2})K(k_n^+)\right] \\
H_r^-(r,z) &= \frac{-\sigma^*}{\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\frac{1}{k_n^-}\sqrt{\frac{r_n}{r}}\left[E(k_n^-) - (1-\frac{k_n^{-2}}{2})K(k_n^-)\right]
\end{aligned}$$

$$H_\theta^+(r,z) = H_\theta^-(r,z) = 0$$

$$\begin{aligned}
H_z^+(r,z) \;=\; & \frac{\sigma^*}{4\pi\mu_0(z-h)}\sum_{n=1}^{2}(-1)^{n-1}\frac{k_n^+}{\sqrt{rr_n}} \\
& \Big\{\Big[\sqrt{r^2+(z-h)^2}-r\Big]\Big[\sqrt{r^2+(z-h)^2}-r_n\Big]\Pi(h_1^+,k_n^+) \\
& +\Big[\sqrt{r^2+(z-h)^2}+r\Big]\Big[\sqrt{r^2+(z-h)^2}+r_n\Big]\Pi(h_2^+,k_n^+)\Big\} \\
H_z^-(r,z) \;=\; & \frac{-\sigma^*}{4\pi\mu_0 z}\sum_{n=1}^{2}(-1)^{n-1}\frac{k_n^-}{\sqrt{rr_n}} \\
& \Big\{\Big[\sqrt{r^2+z^2}-r\Big]\Big[\sqrt{r^2+z^2}-r_n\Big]\Pi(h_1^-,k_n^-) \\
& +\Big[\sqrt{r^2+z^2}+r\Big]\Big[\sqrt{r^2+z^2}+r_n\Big]\Pi(h_2^-,k_n^-)\Big\}
\end{aligned}$$

$$\begin{aligned}
k_n^{+2} &= \frac{4rr_n}{(r+r_n)^2+(z-h)^2}, & k_n^{-2} &= \frac{4rr_n}{(r+r_n)^2+z^2} \\
h_1^+ &= \frac{2r}{r+\sqrt{r^2+(z-h)^2}}, & h_2^+ &= \frac{2r}{r-\sqrt{r^2+(z-h)^2}} \\
h_1^- &= \frac{2r}{r+\sqrt{r^2+z^2}}, & h_2^- &= \frac{2r}{r-\sqrt{r^2+z^2}} \\
r_1 &= r_{in}, & r_2 &= r_{out}
\end{aligned}$$

Thus, we obtained the 3D components of the magnetic field expressed over the complete elliptical integrals of the first, second, and third kind [22, 23]. Ravaud et al. [1] obtained the same expressions for the component H_r and relatively tedious expressions for the component H_z wherein, the incomplete integral of the third kind appears. They used Mathematica to evaluate this component and obtained its real and the imaginary components of H_z. Obviously this component has to be purely real. Even though the imaginary part is a consequence of 'numerical noise' and nearly equals zero they obtained satisfactory results for this component. In our calculation we obtained the complete elliptical integrals of the third kind. These expressions are easy to program and can be used to evaluate the magnetic field anywhere except in the singular case. In the singular case, it is possible to replace the coplete elliptic integral of the third kind with the elliptic integral of the first kind and with Heuman's Lambda function [22, 23].

After some mathematical modifications the component $H_z(r,\,z)$ which is suitable for either regular or singular cases appears in the following form:

$$H_z^+(r,z) = \frac{\sigma^*}{2\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\left\{\frac{k_n^+(z-h)\sqrt{r^2+(z-h)^2}}{\sqrt{rr_n}(\sqrt{r^2+(z-h)^2}+r)}K(k_n^+)\right.$$
$$+\frac{\pi}{2}sign(z-h)sign(\sqrt{r^2+(z-h)^2}-r_n)[1-\Lambda_0(\theta_{1n}^+,k_n^+)]$$
$$\left.+\frac{\pi}{2}sign(z-h)[1-\Lambda_0(\theta_{2n}^+,k_n^+)]\right\}$$

$$H_z^-(r,z) = \frac{-\sigma^*}{2\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\left\{\frac{k_n^- z\sqrt{r^2+z^2}}{\sqrt{rr_n}(\sqrt{r^2+z^2}+r)}K(k_n^-)\right.$$
$$+\frac{\pi}{2}sign(z)sign(\sqrt{r^2+z^2}-r_n)[1-\Lambda_0(\theta_{1n}^-,k_n^-)]$$
$$\left.+\frac{\pi}{2}sign(z)[1-\Lambda_0(\theta_{2n}^-,k_n^-)]\right\}$$

$$\theta_1^+ = \arcsin\sqrt{\frac{1-h_1^+}{1-k_n^{+2}}},\quad \theta_2^+ = \arcsin\frac{|z-h|}{\sqrt{r^2+(z-h)^2}+r}$$
$$\theta_1^- = \arcsin\sqrt{\frac{1-h_1^+}{1-k_n^{+2}}},\quad \theta_2^+ = \arcsin\frac{|z|}{\sqrt{r^2+z^2}+r}$$

a) Singularity treatment

The component Hr^+ has two singularities (See Fig. 3) for $k_n^{+2} = 1$ $(z = h,\ r = r_{in},\ r = r_{out})$ for which H_r^+ values are $-\infty$ and $+\infty$.

The component H_z^+ has two removable singularities (See Fig. 5) for $k_n^{+2} = 1$ $(z = h,\ r = r_{in},\ r = r_{out})$. Using the L'Hopital rule we obtained $H_z^+ \to 0$ for $k_n^{+2} \to 1$.

$K(k)$ — complete elliptic integral of the first kind [22, 23].

$E(k)$ — complete elliptic integral of the second kind [22, 23].

$\Pi(h,k)$ — complete elliptic integral of the third kind [22, 23].

$\Lambda(\varepsilon,k)$ — Heuman's Lambda function [22, 23].

b) Radial magnetic polarization

By substituting the variable θ with $\theta = \pi - 2\beta$ in (2), integrating twice, and respecting angle variable symmetry,we obtained the 3D components of the magnetic field at the point $M(r,\,0,\,z)$:

$$\begin{aligned} H_r(r,z) &= H_r^+(r,z) + H_r^-(r,z) \\ H_\theta(r,z) &= H_\theta^+(r,z) + H_\theta^-(r,z) = 0 \\ H_z(r,z) &= H_z^+(r,z) + H_z^-(r,z) \end{aligned} \tag{4}$$

where

$$H_r^+(r,z) = -\frac{\sigma^*}{4\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\frac{t_n k_n^+}{r}\sqrt{\frac{r_{in}}{r}}\left[K(k_n^+) + \frac{r-r_{in}}{r+r_{in}}\Pi(h^+, k_n^+)\right]$$

$$H_r^-(r,z) = \frac{\sigma^*}{4\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\frac{t_n k_n^-}{r}\sqrt{\frac{r_{out}}{r}}\left[K(k_n^-) + \frac{r-r_{out}}{r+r_{out}}\Pi(h^-, k_n^-)\right]$$

$$H_\theta^+(r,z) = H_\theta^-(r,z) = 0$$

$$H_z^+(r,z) = \frac{\sigma^*}{2\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}k_n^+\sqrt{\frac{r_{in}}{r}}K(k_n^+)$$

$$H_z^-(r,z) = -\frac{\sigma^*}{2\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}k_n^-\sqrt{\frac{r_{out}}{r}}K(k_n^-)$$

$$k_n^{+2} = \frac{4rr_{in}}{(r+r_{in})^2+t_n^2}, \qquad k_n^{-2} = \frac{4rr_{out}}{(r+r_{out})^2+t_n^2}$$

$$h^+ = \frac{4rr_{in}}{(r+r_{in})^2}, \qquad h^- = \frac{4rr_{out}}{(r+r_{out})^2}$$

$$t_1 = z-h, \qquad t_2 = z$$

Thus, we obtained the 3D components of the magnetic field expressed over the complete elliptical integrals of the first, second, and third kind [22, 23]. Ravaud at al. [1] obtained the same expressions for the component H_z and relatively tedious expressions for the component Hr wherein, the incomplete integrals of the first and third kind appear. Above obtained expressions can be easily used to calculate the magnetic field anywhere except in the singular case. In the singular case, it is possible to replace the elliptic integral of the third kind with the elliptic integral of the first kind and Heumans Lambda function [22, 23]. After some mathematical modifications, the component $H_r(r,\, z)$ suitable for either regular or singular cases appears in the following form:

$$\begin{aligned} H_r^+(r,z) &= -\frac{\sigma^*}{4\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\sqrt{\frac{r_{in}}{r}}\left[2\frac{t_n k_n^+}{r+r_{in}}K(k_n^+)\right. \\ &\quad \left. +\pi\sqrt{\frac{r_{in}}{r}}sign(r-r_{in})sign(t_n)[1-\Lambda_0(\varepsilon_n^+, k_n^+)]\right] \\ H_r^-(r,z) &= \frac{\sigma^*}{4\pi\mu_0}\sum_{n=1}^{2}(-1)^{n-1}\sqrt{\frac{r_{out}}{r}}\left[2\frac{t_n k_n^-}{r+r_{out}}K(k_n^-)\right. \\ &\quad \left. +\pi\sqrt{\frac{r_{out}}{r}}sign(r-r_{out})sign(t_n)[1-\Lambda_0(\varepsilon_n^-, k_n^-)]\right] \end{aligned}$$

where

$$\varepsilon_n^+ = \arcsin\sqrt{\frac{1-h^+}{1-k_n^{+2}}}, \qquad \varepsilon_n^- = \arcsin\sqrt{\frac{1-h^-}{1-k_n^{-2}}}$$

b1) Singularity treatment

The component H_r^+ has one removable singularity (See Figs. 7) for $k_n^{+2} = 1$ ($z = h,\ r = r_{in}$). Using the L'Hopital rule we obtained $H_r^+ \to 0$ for $k_n^{+2} \to 1$.

The component H_z^+ has one singularity (See Figs. 9) for $k_n^{+2} = 1$ ($z = h,\ r = r_{in}$) for which H_z^+ value is $+\infty$.

4. EXAMPLES

a) Axial magnetic polarization

In Figs. 3 and 4 we show the radial and the axial components of the magnetic field (axial magnetic polarization) as a function of the radial distance of the observation point for a given altitude with $z = 3\,\mathrm{mm}$ (component H_r) and $z = 4\,\mathrm{mm}$ (component H_z) for a magnet with dimensions $r_{in} = 25\,\mathrm{mm}$, $r_{out} = 28\,\mathrm{mm}$, $h = 3\,\mathrm{mm}$ and $\sigma^* = 1\,\mathrm{T}$ [1]. We only considered the upper face of the permanent magnet ring to simplify the analytical calculation [1]. If we compare our results to those obtained in [1], we observed the same form for the components

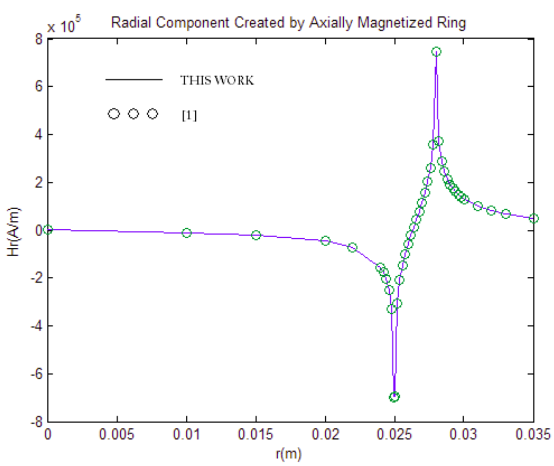

Figure 3. The radial component H_r is a function of the radial distance: the observation height is $z = 3\,\mathrm{mm}$, $r_{in} = 25\,\mathrm{mm}$, $r_{out} = 28\,\mathrm{mm}$.

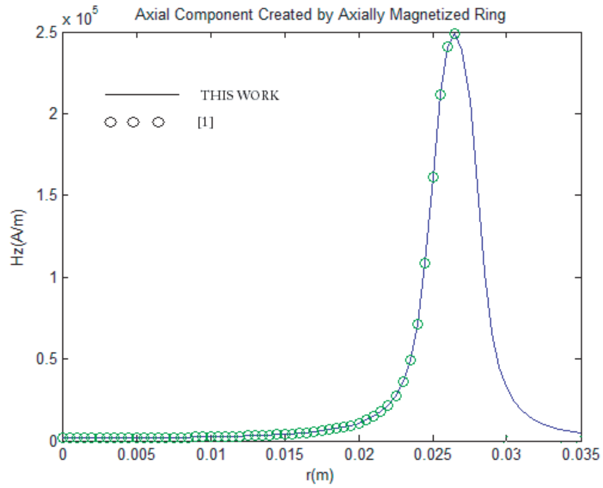

Figure 4. The axial component H_z is a function of the radial distance: the observation height is $z = 4\,\mathrm{mm}$, $r_{in} = 25\,\mathrm{mm}$, $r_{out} = 28\,\mathrm{mm}$.

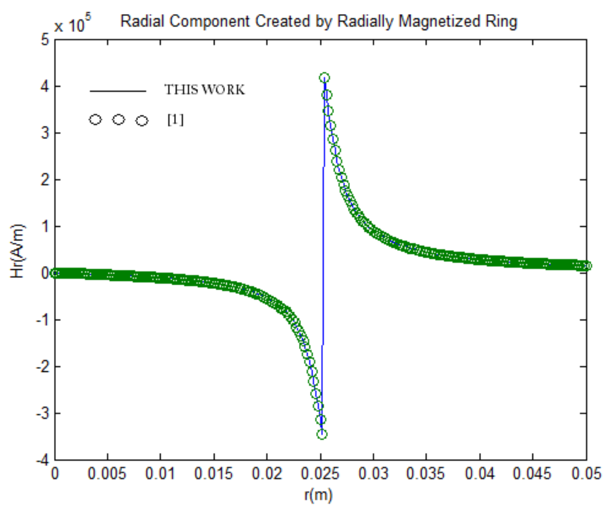

Figure 5. The radial component H_r is a function of the radial distance: the observation height is $z = 1.5\,\mathrm{mm}$, $r_{in} = 25\,\mathrm{mm}$, $r_{out} = 28\,\mathrm{mm}$ [1].

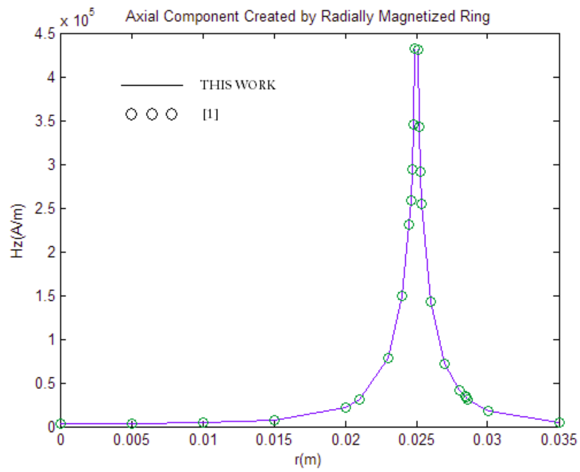

Figure 6. The axial component H_z is a function of the radial distance: the observation height is $z = 3\,\mathrm{mm}$, $r_{in} = 25\,\mathrm{mm}$, $r_{out} = 28\,\mathrm{mm}$.

of H_r (Fig. 3). From the components of H_z (Fig. 4) we can see that there is a difference in the right side where the magnetic field does not exist for $r > 0.025\,\mathrm{m}$ [1]. This component of the magnetic field is difficult to plot with Mathematica as mentioned in [1]. We find that our approach, in which we replaced the elliptic integral of third kind by Heumans Lambda function, can considerably improve the magnetic field calculation near singular points where the elliptic integral of third kind is tedious for numerical evaluation [20].

b) Radial magnetic polarization

In Figs. 5 and 6 we show the radial and the axial components of the magnetic field as a function of the radial distance of the observation point for a given altitude with $z = 1.5\,\mathrm{mm}$ (component H_r) and $z = 3\,\mathrm{mm}$ (component H_z) for given magnet with dimensions $r_{in} = 25\,\mathrm{mm}$, $r_{out} = 28\,\mathrm{mm}$, $h = 3\,\mathrm{mm}$ and $\sigma^* = 1\,\mathrm{T}$ [1]. We only consider the ring inner face of the permanent magnet ring to simplify the analytical calculation [1]. If we compare our results to those obtained in [1] we observe the same form for all of the components of either H_r or H_z.

Thus, we can confirm that obtained expressions (3) and (4) perfectly correspond to those given in [1] but they are more suitable for the calculation of magnetic fields especially in singular cases. The comparative calculation was made using MATLAB programming.

5. CONCLUSION

In this paper, we presented an improved analytical approach, based on a previously developed analytical method, to calculate the magnetic field of axially- and radially-polarized magnetic rings. The proposed analytical expressions are easily extended to the numerical calculation of the magnetic field at any regular or singular point in space.

REFERENCES

1. Ravaud, R., G. Lemarquand, V. Lemarquand, and C. Depollier, "Analytical calculation of the magnetic field created by permanent-magnet rings," *IEEE Trans. Magn.*, Vol. 44, No. 8, 1982–1989, Aug. 2008.
2. Rakotoarison, H., J. Yonnet, and B. Delinchant, "Using coulombian approach for modelling scalar potential and magnetic field of a permanent magnet with radial polarization," *IEEE Trans. Magn.*, Vol. 43, No. 4, 1261–1264, Apr. 2007.
3. Zhilichev, Y., "Calculation of magnetic field of tubular permanent magnet assemblies in cylindrical bipolar coordinates," *IEEE Trans. Magn.*, Vol. 43, No. 7, 3189–3195, Jul. 2007.
4. Lindell, I. V., "Electromagnetic field in self-dual media in different-form representation," *Progress In Electromagnetics Research*, PIER 58, 319–333, 2006.
5. Lemarquand, G. and V. Lemarquand, "Annular magnet position sensor," *IEEE. Trans. Magn.*, Vol. 26, No. 5, 2041–2043, Sep. 1990.
6. Lemarquand, G. and V. Lemarquand, "Variable magnetic torque sensor," *J. Appl. Phys.*, Vol. 70, No. 10, 6630–6632, 1991.
7. Pulyer, Y. and M. Hrovat, "Generation of remote homogeneous magnetic fields," *IEEE Trans. Magn.*, Vol. 38, No. 3, 1553–1563, May 2002.
8. Berkouk, M., V. Lemarquand, and G. Lemarquand, "Analytical calculation of ironless loudspeaker motors," *IEEE Trans. Magn.*, Vol. 37, No. 2, 1011–1014, Mar. 2001.
9. Furlani, E. P., S. Reznik, and A. Kroll, "A three-dimensonal field solution for radially polarized cylinders," *IEEE Trans. Magn.*, Vol. 31, No. 1, 844–851, Jan. 1995.
10. Durand, E., *Electrostatique*, Vol. 1, 248–251, Masson Editeur, Paris, France, 1964.
11. Dehghan, M. and F. Shakeri, "Solution of an integral-differential equation arising oscillating magnetic fields using He's homotopy

peturbation method," *Progress In Electromagnetics Research*, PIER 78, 361–376, 2008.

12. D'Agostino, F., F. Ferrara, C. Gennarelli, R. Guerriero, and G. Riccio, "An effective technique for reducing the truncation error in the near-field-far-field transformation with plane-polar scanning," *Progress In Electromagnetics Research*, PIER 73, 213–238, 2007.
13. Furlani, E. P., "A 3-dimensional field solution for axially polarized multipole disks," *J. Magn. Magn. Mat.*, Vol. 135, No. 2, 205–214, 1994.
14. Furlani, E. P., *Permanent Magnet & Electromechanical Devices: Materials, Analysis, and Applications*, Academic Press, 2001.
15. Azzerboni, B. and E. Cardelli, "Magnetic field evaluation for disk conductors," *IEEE Trans. Magn.*, Vol. 29, No. 6, 2419–2421, 1993.
16. Xu, X. B. and L. Zeng, " Ferromagnetic cylinders in earth's magnetic field: A two-dimentional model of magnetization of submarine," *Progress In Electromagnetics Research*, PIER 19, 319–335, 1998.
17. Babic, S. I. and C. Akyel, " New mutual inductance calculation of the magnetically coupled coils: Thin disk coil-thin wall solenoid," *Journal of Electromagnetic Waves and Applications*, Vol. 20, No. 10, 1661–1669, 2006.
18. Babic, S., C. Akyel, S. Salon, and S. Kincic, "New expressions for calculating the magnetic field created by radial current in massive disks," *IEEE Trans. Magn.*, Vol. 38, No. 2, 497–500, Mar. 2002.
19. Babic, S. and M. Gavrilovic, "New expression for calculating magnetic fields due to current-carrying solid conductors," *IEEE Trans. Magn.*, Vol. 33, No. 5, 4134–4136, Sep. 1997.
20. Babic, S., et al., "Analytical magnetostatic field calculation for the conductor carrying constant current in the longitudinal diretion," *J. Appl. Phys.*, Vol. 67, 5827–5829, May 1990.
21. Blache, C. and G. Lemarquand, "High magnetic field gradients in flux confining permanent magnet structures," *J. Magn. Magn. Mater.*, Vol. 104–107, 1111–1112, 1992.
22. Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics*, Series 55, Washington DC, Dec. 1972.
23. Gradshteyn, I. S. and I. M. Rhyzik, *Tables of Integrals, Series and Products*, Dover, New York, 1972.