Progress In Electromagnetics Research Letters, Vol. 5, 1–12, 2008

COUPLING PROJECTION DOMAIN DECOMPOSITION METHOD AND MESHLESS COLLOCATION METHOD USING RADIAL BASIS FUNCTIONS IN ELECTROMAGNETICS

Y. Duan

School of Applied Mathematics
University of Electronic Sciences and Technology of China
Chengdu 610054, China

S. J. Lai

Institute of Applied Physics
University of Electronic Sciences and Technology of China
Chengdu 610054, China

T. Z. Huang

School of Applied Mathematics
University of Electronic Sciences and Technology of China
Chengdu 610054, China

Abstract—This paper presents an efficient meshless approach for solving electrostatic problems. This novel approach is based on combination of radial basis functions-based meshless unsymmetric collocation method with projection domain decomposition method. Under this new method, we just need to solve a Steklov-Poincaré interface equation and the original problem is solved by computing a series of independent sub-problems. An electrostatic problem is used as an example to illustrate the application of the proposed approach. Numerical results that demonstrate the accuracy and efficiency of the method are stated.

1. INTRODUCTION

The use of a mesh is a basic characteristic of traditional numerical approaches for the solution of partial differential equations (PDEs), such as finite element method(FEM), finite difference method(FDM) and so on. In those approaches, assumptions are made for the local approximation of the primitive variables, which require mesh to support them. The generation of mesh is a complicated work. During the last decade, considerable effort has been given to the development of so-called meshless or meshfree methods. This kind of algorithm is quite different from the traditional ones applied in electromagnetism,such as finite element method(FEM) and finite difference method(FDM) stated in [11–24]. The aim of this type of approach is to eliminate at least the structure of the mesh and approximate the solution entirely using nodal values inside and in the boundary quasi-random distributed in the domain. For instance, the element free Galerkin(EFG) method was given by Belytschko [25]. The meshless local Petrov-Galerkin and generalized finite element methods were given by Atluri et al. [2] and Babuška [7] respectively. The meshless method also consists of the method of foundational solution and meshless method using radial basis functions.

In recent years, the theory of radial basis functions (RBFs) has undergone intensive research and enjoyed considerable success as a technique for interpolating multivariable data and functions [1]. Although most work to date on RBFs relates to scattered data approximation and in general to interpolation theory, there has recently been an increased interest in their use for solving PDEs. This approach, which approximates the whole solution of the PDE by a translates of RBFs, is very attractive due to the fact that it is a truly meshless method and spatial dimension independent, which can easily be extended to solve high dimensional problems. Furthermore, since the RBFs are smooth, it can easily be applied to solve high order differential equations. Collocation method to solve PDEs using radial basis functions was first proposed by Kansa [3] and is extensively studied by Schaback [5]. It has been applied to solve some typical electromagnetic problems [8–10].

In the view of parallelism, The major applied technique is the domain decomposition method. It is nowadays considered as one of the most popular technique that can be applied for numerical solution of partial differential equations. The idea behind the domain decomposition is to divide the considered domain into a number of sub-domains and then try to solve the original problem as a series of sub-problems that interact through an internal interfaces. The numerical solution can be computed either iteratively, using Schwartz method [4], by changing data interfaces between sub-problems or by computing the interfaces data directly using Steklov technique and then use interface solution to solve each sub-problem separately. There exist two different approaches for domain decomposition: overlapping and non overlapping domains. After it has been seen more development with finite element method, it was applied with meshless method in many work. We can cite the work published by Hon et al. [6]. Compared with another domain decomposition method (DDM), the PDM need neither the suitable choice of the acceleration parameters to ensure the convergent rate of the iteration arising in DDM nor the iteration at each subdomains. This property is meaningful for meshless method (using radial basis function (RBF)) coupled with DDM.

The main objective of this paper is to couple the project domain decomposition method with asymmetric collocation method based on radial basis functions to solve Poisson problem. The paper is organized as follows. In Section 2, we use a general elliptic problem to analyze the coupling method of PDM and RBF-based meshless collocation method. In Section 3, several numerical examples, consisting of pure mathematical test and computation of the fields of an infinite square grounding metal slot, are given to validate the proposed method. Conclusions are drawn in the final section, Section 4.

2. MESHLESS PDM USING RBFS

As we known that the governing equation in magneto/electrostatic problems is an elliptic equation. Without a loss of generality,we use a general elliptic boundary value problem (BVP) to illustrate the meshless unsymmetric collocation method using RBFs, projection domain decomposition and the coupling algorithm.

2.1. Elliptic Equation and Domain Decomposition

For the general elliptic BVP:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where Ω is a d-dimensional domain, with a Lipschitz boundary $\partial\Omega$, whose outer unit normal direction is denoted by $\mathbf{n}$, f is a given function of $L^2(\Omega)$. Assuming that Ω is conformally partitioned into two non-overlapping convex subdomains Ω_1 and Ω_2, and denoting by $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$. We denote by u_i the restriction to Ω_i, $i = 1, 2$, of the solution u of Equation (1), and $\mathbf{n}^i$ by the normal direction on $\partial\Omega_i \cap \Gamma$, oriented outward Ω_2. Set $\mathbf{n} = \mathbf{n}^1$.

A standard domain decomposition skill [4] leads to the following equation

$$\begin{cases} \mathcal{L}u_1 = f & \text{in } \Omega_1 \\ u_1 = g & \text{on } \partial\Omega_1 \bigcap \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ \dfrac{\partial u_2}{\partial n} = \dfrac{\partial u_1}{\partial n} & \text{on } \Gamma \\ u_2 = g & \text{on } \partial\Omega_2 \bigcap \partial\Omega \\ \mathcal{L}u_2 = f & \text{in } \Omega_2. \end{cases} \tag{2}$$

Denoting by $\lambda = u_{|\Gamma}$, the unknown trace of the solution of Equation (1) on the interface Γ, $u_i = u_i^* + \mathcal{H}_i\lambda$, with

$$\begin{cases} \mathcal{L}u_i^* = f & \text{in } \Omega_i \\ u_i^* = g & \text{on } \partial\Omega_i \bigcap \partial\Omega \\ u_i^* = 0 & \text{on } \Gamma. \end{cases} \tag{3}$$

$\mathcal{H}_i\lambda$ is the harmonic extension of the trace λ

$$\begin{cases} \mathcal{L}\mathcal{H}_i\lambda = 0 & \text{in } \Omega_i \\ \mathcal{H}_i\lambda = 0 & \text{on } \partial\Omega_i \bigcap \partial\Omega \\ \mathcal{H}_i\lambda = \lambda & \text{on } \Gamma. \end{cases} \tag{4}$$

An additional equation, named Steklov-Poincaré equation, should be satisfied to ensure $u_i^* + \mathcal{H}_i\lambda$ is the solution of Equation (1).

$$S\lambda = \chi \quad \text{on } \Gamma \tag{5}$$

where $\chi = \frac{\partial u_2^*}{\partial n} - \frac{\partial u_1^*}{\partial n}$. S is the *Steklov-Poincaré operator*

$$S\lambda = \frac{\partial \mathcal{H}_1\lambda}{\partial n} - \frac{\partial \mathcal{H}_2\lambda}{\partial n} \tag{6}$$

2.2. RBF-based Meshless Unsymmetric Collocation Method

For a given set of distinct centers $X = \{x_1, x_2, \ldots, x_N\} \subset \overline{\Omega}$ and some positively definite RBF $\phi(r)$, let the approximated solution of Equation (1) be $u = \sum\limits_i \lambda_i \phi(\|x - x_i\|)$, where $\|\cdot\|$ is the Euclidean norm. The RBF-based meshless unsymmetric collocation method reads:

$$\begin{cases} \mathcal{L}u(x_i) = f(x_i) & x_i \in \Omega \\ u(x_j) = g(x_j) & x_j \in \partial\Omega \end{cases} \tag{7}$$

Denoted by

$$A = \left[\mathcal{L}\phi\left(\|x_i - x_j\|\right), \mathcal{B}\phi\left(\|x_i - x_k\|\right)\right]^T = [A_1, A_2]^T$$

the linear system induced by collocation method using RBFs is

$$A\vec{\lambda} = \left[\vec{f}, \vec{g}\right]^T$$

We can obtain the numerical solution of Equation (1) by solving the linear system above.

2.3. Meshless PDM Using RBFs

For the set of distinct centers $X \subset \overline{\Omega}$ and positively definite RBF $\phi(r)$, we can define the finite dimensional space

$$\begin{aligned} V_h^i &= span\left\{\phi\left(\left\|x - x_j^i\right\|\right)\chi_{\Omega_i}\right\}, \quad j = 1, 2, \ldots, N_i \\ \Lambda_h &= span\left\{\phi\left(\|x - x_j\|\right),\ x_j \in X_\Gamma\right\} \end{aligned}$$

the space spanned by the RBFs with centers in $\overline{\Omega_i}$, N_i is the number of centers in $\overline{\Omega_i}$, χ_{Ω_i} is the characteristic function of subdomain Ω_i, X_Γ is the set of centers on Γ. Assuming that the density in Ω_i are equivalent to $h_i = \sup_{x\in\Omega_i} \min_{x_i\in X} \|x - x_i\|$, $h_2 = \sup_{x\in\Gamma} \min_{x_i\in X_\Gamma} \|x - x_i\|$ are the density of the distinct centers $X = \{x_1, x_2, \ldots, x_N\}$ and X_Γ respectively. The coupling algorithm reads:

STEP 1. Solving Equation (4) by taking $\lambda = \phi(x - x_k)$, we can obtain $\mathcal{H}_i\phi(x - x_k)$, where $x_k \in \Gamma$. i.e., Solve linear system

$$\begin{cases} \mathcal{L}\mathcal{H}_i\phi(x_j - x_k) = 0 & x_j \in \Omega_i \\ \mathcal{H}_i\phi(x_j - x_k) = 0 & x_j \in \partial\Omega_i \bigcap \partial\Omega \\ \mathcal{H}_i\phi(x_j - x_k) = \phi(x_j - x_k) & x_j \in \Gamma. \end{cases} \tag{8}$$

This step is to solve N_Γ, the number of centers on interface Γ, independent problems in each subdomain. It can be easily to compute in parallel.

STEP 2. Solving Equation (3) to get $u_i^* = \sum_k C_k\phi(x - x_k)$, with $x_k \in \overline{\Omega_i}$, i.e.,

$$\begin{cases} \mathcal{L}u_i^*(x_j) = f(x_j) & x_j \in \Omega_i \\ u_i^*(x_j) = g(x_j) & x_j \in \partial\Omega_i \bigcap \partial\Omega \\ u_i^*(x_j) = 0 & x_j \in \Gamma. \end{cases} \tag{9}$$

This step can also be computed in parallel.

STEP 3. Assuming that $\mathcal{H}_i\lambda = \sum_j c_j \mathcal{H}_i \phi(x - x_j)$ with $x_j \in \Gamma$, we can obtain c_j by solving the Steklov-Poincaré Equation (5). i.e.,

$$S\lambda(x_j) = \chi(x_j) \quad x_j \in \Gamma \tag{10}$$

STEP 4. The approximated solution is obtained by $u_i = u_i^* + \sum_j c_j \mathcal{H}_i \phi(x - x_j)$.

3. NUMERICAL EXAMPLES

3.1. Numerical Validation

In order to validate the proposed method, a model Poisson equation with analytic solutions is solved by using two kinds of radial basis functions, namely, TPS (Thin plate spline) $\phi(r) = r^5$ and Sobolev Spline $\phi(r) = e^{-r}(3 + 3r + 3r^2)$. Of course, others RBFs can also be selected as basis, Wendland's functions [1], Gaussian function, for example. The problem is to find the solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

with $\Omega = \Omega_1 \cup \Omega_2 = [-1,1;-1,0] \cup [-1,1;0,1]$, $\Gamma = \{(x,y) \mid x \in [-1,1], y = 0\}$. The right-hand sides are

$$\begin{aligned} f_1(x,y) &= e^x \left(y^2 - 1\right)\left(x^2 + 4x + 1\right) + 2e^x\left(x^2 - 1\right) \\ f_2(x,y) &= -4\left(x^2 + y^2\right)\sin\left(x^2 - 1\right)\sin\left(y^2 - 1\right) \\ &\quad +2\sin\left(y^2 + x^2 - 2\right) \end{aligned}$$

respectively and the corresponding exact solution are

$$\begin{aligned} u_1(x,y) &= \left(x^2 - 1\right)\left(y^2 - 1\right)e^x \\ u_2(x,y) &= \sin\left(x^2 - 1\right)\sin\left(y^2 - 1\right) \end{aligned}$$

Let the relative error of $L^2 - norm$ in Ω_i be

$$\left(L^2\left(\Omega_i\right)_{error}\right)^2 = \frac{\int\limits_{\Omega_i} \left(\widetilde{u}_i - u_{exact}\right)^2}{\int\limits_{\Omega_i} \left(u_{exact}\right)^2}$$

where $\widetilde{u}_i$ is the numerical solution in Ω_i, and the maximum pointwise relative error at centers be

$$L^{\infty}(\Omega_i)_{error} = \sup_{\Omega_i} \frac{|\widetilde{u}_i - u_{exact}|}{|u_{exact}|}$$

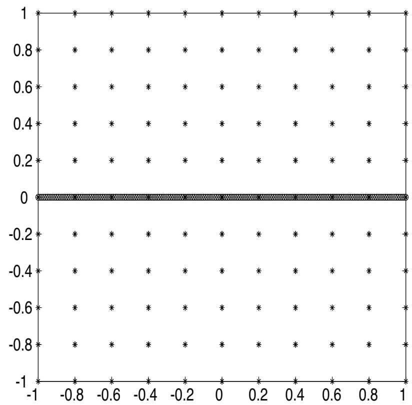

Figure 1. The distribution of centers and two decomposition regions in the computation region.

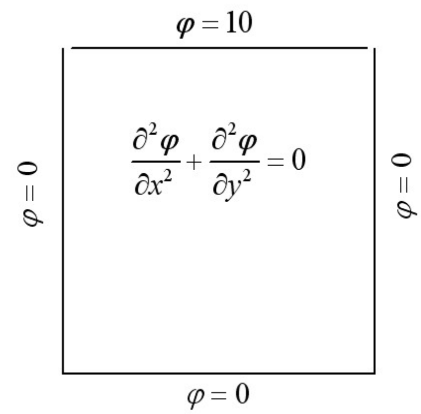

Figure 2. An infinite square grounding slot.

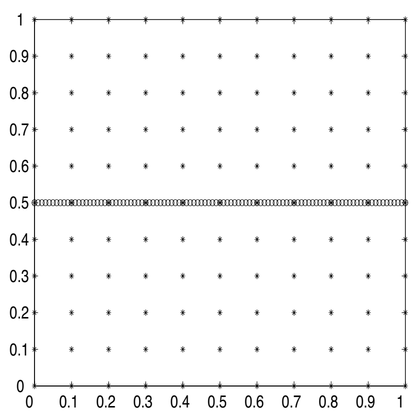

Figure 3. The distribution of centers and two decomposition regions in the computation region.

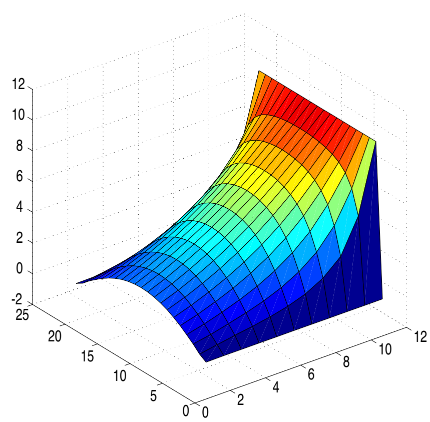

Figure 4. The distribution of field with $h = 0.1$ using TPS.

The set of distinct centers is uniformly distributed in Ω with density $h = 0.2$ (Fig. 1). This choice is based on two reasons: one is that it is easy to chose the distinct center for a regular domain; another is that a better distribution can ensure the convergence of the proposed method [26–28]. Tables 1–4 are the numerical results.

3.2. Application

The proposed method is used to study 2-D field problem, i.e., to determine the fields of an infinite square grounding metal slot (Fig. 2). In the numerical implementation, TPS and Sobolev spline are chosen as the basis functions, different density of the set of distinct centers are used Fig. 3, the distribution of fields is shown in Fig. 4. The relative error between the proposed numerical solution and analytical one and The equipotential contours obtained by using the proposed method are shown in Figs. 5 and 6. From these results it can be seen that the proposed method is efficient.

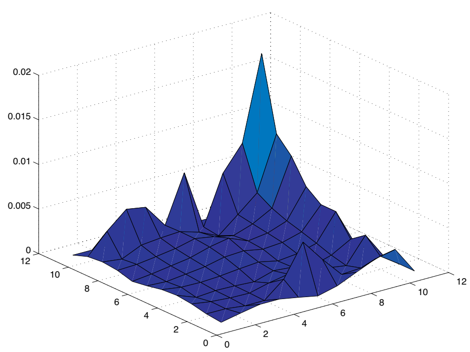

Figure 5. The relative error between numerical solution and analytical one of the ground metal slot with $h = 0.1$ using TPS.

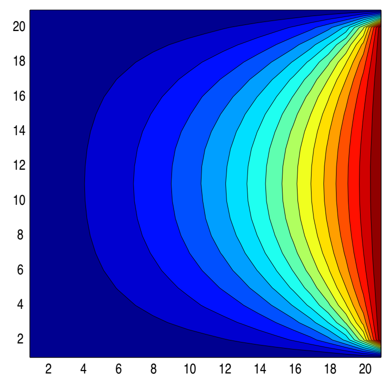

Figure 6. The computed equipotential contours of the ground metal slot with $h = 0.05$ using sobolev spline.

Table 1. TPS.

$f_1(x,y)$	
$L^2(\Omega_1)_{error} = 0.0082$	$L^2(\Omega_2)_{error} = 0.0082$
$L^\infty(\Omega_1)_{error} = 0.0463$	$L^\infty(\Omega_2)_{error} = 0.0466$

Table 2. TPS.

$f_2(x,y)$	
$L^2(\Omega_1)_{error} = 0.0053$	$L^2(\Omega_2)_{error} = 0.0053$
$L^\infty(\Omega_1)_{error} = 0.0142$	$L^\infty(\Omega_2)_{error} = 0.0141$

Table 3. Sobolev spline.

$f_1(x,y)$	
$L^2(\Omega_1)_{error} = 0.0071$	$L^2(\Omega_2)_{error} = 0.0071$
$L^\infty(\Omega_2)_{error} = 0.0332$	$L^\infty(\Omega_1)_{error} = 0.0320$

Table 4. Sobolev spline.

$f_2(x,y)$	
$L^2(\Omega_1)_{error} = 0.0027$	$L^2(\Omega_2)_{error} = 0.0027$
$L^\infty(\Omega_1)_{error} = 0.0093$	$L^\infty(\Omega_2)_{error} = 0.0088$

4. CONCLUSION

In this paper, a coupling algorithm of the RBFs-based unsymmetric method and projection domain decomposition for solving magneto/electrostatic problems is given. Unlike the FEM which interpolates the solution by using low order piecewise continuous polynomials or the FDM where the derivatives of the solution are approximated by finite quotients, the RBF-based meshless method provides a global interpolation formula not only for the solution but also for the derivatives of the solution. The proposed method can easily be computed in parallel. For the purpose of illustration, an infinite square grounding metal slot problem is solved. Good results can be seen clearly from the figure, which verifies the accuracy and efficiency of the method.

ACKNOWLEDGMENT

This work was supported by Research Foundation for Young scholar of UESTC under JX0646, 973 Programs (2008CB317110), NSFC (10771030), the Scientific and Technological Key Project of the Chinese

Education Ministry (107098), the Ph.D. Programs Fund of Chinese Universities (20070614001), Sichuan Province Project for Applied Basic Research (2008JY0052) and the Project for Academic Leader and Group of UESTC.

REFERENCES

1. Wendland, H., *Scattered Data Approximation*, Cambridge University Press, 2005.
2. Atluri, S. and T. Zhu, "New concepts in meshless methods," *Int. J. Numer. Meth. Eng.*, Vol. 47, 537–556, 2000.
3. Kansa, E. J., "Multlqudrms — A scattered data approximation scheme with apphcations to computational fluiddynamics, II. Solutions to parabolic, hyperbolic and elliptic partial differential equations," *Computers Math. Applic.*, Vol. 19, No. 8/9, 147–161, 1990.
4. Quarteroni, A. and A. Valli, *Domain Decomposition Method for Partial Differential Equation*, Oxford University Press, England, 1999.
5. Schaback, R., "Convergence of unsymmetric kernal-based meshless collocation methods," *SIAM J. Numer. Anal.*, Vol. 45, No. 1, 333–351, 2007.
6. Li, J. and Y. C. Hon, "Domain decomposition for radial basis meshless method," *Numer. Method for Partial Differential Equations*, Vol. 20, No. 3, 450–462, 2004.
7. Babuška, I., U. Banerjee, and J. E. Osborn, "Survey of meshless and generalized finite element methods: A unified approach," *Acta. Numer.*, 1–125, Cambridge University Press, 2003.
8. Viana, S. A., D. Rodger, and H. C. Lai, "Application of the local radial point interpolation method to solve eddy-current problems," *IEEE Trans. Magn.*, Vol. 42, No. 4, 591–594, 2006.
9. Ho, S. L., S. Y. Yang, H. C. Wong, and G. Ni, "A meshless collocation method based on radial basis functions and wavelets," *IEEE Trans. Magn.*, Vol. 40, No. 2, 1021–1024, 2004.
10. Ala, G., E. Francomano, A. Tortaorici, E. Toscano, and F. Viola, "A smoothed particle interpolation scheme for transient electromagnetic simulation," *IEEE Trans. Magn.*, Vol. 42, No. 4, 647–650, 2006.
11. Angell, A. J. and C. M. Rappaport, "Computational modeling analysis of radar scattering by clothing covered arrays of metallic

body-worn explosive devices," *Progress In Electromagnetics Research*, PIER 76, 285–298, 2007.

12. Hussein, K. F. A., "Fast computational algoritm for EFIE applied to arbitrarily-shaped conducting surfaces," *Progress In Electromagnetics Research*, PIER 68, 339–357, 2007.
13. Sun, J., C.-F. Wang, L.-W. Li, and M.-S. Leong, "Mixed potential spatial domain Green's functions in fast computational form for cylindrically stratified media," *Progress In Electromagnetics Research*, PIER 45, 181–199, 2004.
14. Bossavit, A., "Generalized finite differences in computational electromagnetics," *Progress In Electromagnetics Research*, PIER 32, 45–64, 2001.
15. Marrone, M., "Computational aspects of the cell method in electrodynamics," *Progress In Electromagnetics Research*, PIER 32, 317–356, 2001.
16. Tarhasaari, T. and L. Kettunen, "Topological approach to computational electromagnetism," *Progress In Electromagnetics Research*, PIER 32, 189–206, 2001.
17. Gross, P. W. and P. R. Kotiuga, "Finite element-based algorithms to make cuts for magnetic scalar potentials: Topological constraints and computational complexity," *Progress In Electromagnetics Research*, PIER 32, 207–245, 2001.
18. Marliani, F. and A. Ciccolella, "Computationally efficient expressions of the dyadic Green's function for rectangular enclosures," *Progress In Electromagnetics Research*, PIER 31, 195–223, 2001.
19. Mukerji, S. K., M. George, M. B. Ramamurthy, and K. Asaduzzaman, "Eddy currents in laminated rectangular cores," *Progress In Electromagnetics Research*, PIER 83, 435–445, 2008.
20. Cacciola, M., F. C. Morabito, D. Polimeni, and M. Versaci, "Fuzzy characterization of flawed metallic plates with eddy current tests," *Progress In Electromagnetics Research*, PIER 72, 241–252, 2007.
21. Shi, X., F. Wei, Q. Huang, L. Wang, and X.-W. Shi, "Novel binary search algorithm of backtracking for RFID tag anti-collision," *Progress In Electromagnetics Research B*, Vol. 9, 97–104, 2008.
22. Kazemi, S., H. R. Hassani, G. Dadashzadeh, and F. G. Gharakhili, "Performance improvement in amplitude synthesis of unequally spaced array using least mean square method," *Progress In Electromagnetics Research B*, Vol. 1, 135–145, 2008.
23. Guo, S.-X., Y. Gao, and L.-X. Zhang, "Simulation research on turbo equlization algorithm based on microwave fading channel,"

Progress In Electromagnetics Research Letters, Vol. 4, 99–107, 2008.

24. Su, D., D.-M. Fu, and D. Yu, "Genetic algorithms and method of moments for the design of PIFAS," *Progress In Electromagnetics Research Letters*, Vol. 1, 9–18, 2008.
25. Belytschko, T., Y. Y. Lu, and L. Gu, "Element-free Galerkin methods," *International Journal for Numerical Methods in Engineering*, Vol. 37, 229–256, 1994.
26. Hon, Y. C. and R. Schaback, "On unsymmetric collocation by radial basis functions," *Appl. Math. Comput.*, Vol. 119, 177–186, 2001.
27. Ling, L., R. Opfer, R. Schaback, "Results on meshless collocation technique," *Eng. Anal. Bound. Elem.*, Vol. 30, No. 4, 247–253, 2006.
28. Ling, L. and R. Schaback, "Stable and convergent unsymmetric meshless collocation methods," *SIAM J. Numer. Anal.*, Vol. 46, No. 3, 1097–1115, 2008.