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# APPROXIMATE ANALYTICAL SOLUTIONS TO NONLINEAR OSCILLATIONS OF NON-NATURAL SYSTEMS USING HE'S ENERGY BALANCE METHOD

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**Abstract**—This paper applies He's Energy balance method (EBM) to study periodic solutions of strongly nonlinear systems such as nonlinear vibrations and oscillations. The method is applied to two nonlinear differential equations. Some examples are given to illustrate the effectiveness and convenience of the method. The results are compared with exact solutions which lead us showing a good accuracy. The method can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other science.

## 1. INTRODUCTION

Nonlinear oscillation systems are such phenomena that mostly nonlinearly occur. These systems are important in engineering because many practical engineering components consist of vibrating systems that can be modeled using oscillator systems such as elastic beams supported by two springs or mass-on-moving belt or nonlinear pendulum and vibration of a milling machine [1, 2]. Hence solving of governing equations and due to limitation of existing exact solutions have been one of the most time-consuming and difficult affairs among researchers of vibrations problems. If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Recently, considerable attention has been directed towards the analytical solutions for nonlinear equations without possible small parameters. The traditional perturbation

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methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have been appeared in open literature [3–14], such as Non–perturbative methods [3], homotopy perturbation method [4–7, 33–35], perturbation techniques [8], Lindstedt-Poincaré method [9, 10], parameter–expansion method [11, 12] and Parameterized perturbation method [13, 14].

Recently, some approximate variational methods, including approximate energy method [15–17, 37, 38], variational iteration method [18–22] and variational approach [23–29, ] and other methods [39], to solution, bifurcation, limit cycle and period solutions of nonlinear equations have been paid much attention.

This paper presents energy balance method (EBM) to study periodic solutions of strongly nonlinear systems. In this method, a variational principle for the nonlinear oscillation is established, then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones. Some examples reveal that even the lowest order approximations are of high accuracy.

## 2. ENERGY BALANCE METHOD

In the present paper, we consider a general nonlinear oscillator in the form [17]:

$$u'' + f = 0 \tag{1}$$

in which $u$ and $t$ are generalized dimensionless displacement and time variables, respectively, and $f = f(u, u', t)$.

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left( -\frac{1}{2} {u'}^2 + F(u) \right) dt \tag{2}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2} {u'}^2 + F(u) = F(A) \tag{3}$$

Or:

$$R(t) = \frac{1}{2} {u'}^2 + F(u) - F(A) = 0 \tag{4}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency $\omega$ and the amplitude of oscillation, $A$. So let us consider such initial conditions:

$$u(0) = A, \quad u'(0) = 0 \tag{5}$$

Assume that its initial approximate guess can be expressed as:

$$u(t) = A\cos(\omega t) \tag{6}$$

Substituting Eq. (6) into $u$ term of Eq. (4), yield:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A\cos\omega t) - F(A) = 0 \tag{7}$$

If by any chance, the exact solution had been chosen as the trial function, then it would be possible to make $R$ zero for all values of $t$ by appropriate choice of $\omega$. Since Eq. (5) is only an approximation to the exact solution, $R$ cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{\frac{2\left(F(A) - F(A\cos(\pi/4))\right)}{A^2 \sin^2(\pi/4)}} \tag{8}$$

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{\dfrac{2\left(F(A) - F(A\cos(\pi/4))\right)}{A^2 \sin^2(\pi/4)}}} \tag{9}$$

## 3. APPLICATIONS OF STRONGLY NONLINEAR VIBRATION SYSTEMS

In this section, we will present three examples to illustrate the applicability, accuracy and effectiveness of the proposed approach.

**Example 1.** The motion of a particle on a rotating parabola. The governing equation of motion and initial conditions can be expressed as [30]:

$$\left(1 + 4q^2u^2\right)\frac{d^2u}{dt^2} + 4q^2u\left(\frac{du}{dt}\right)^2 + \Delta u = 0, \;\; u(0) = A, \;\; \frac{du}{dt}(0) = 0. \tag{10}$$

where $q > 0$ and $\Delta > 0$ are known positive constants [31]. For this problem, $f(u) = 4q^2u^2\frac{d^2u}{dt^2} + 4q^2u\left(\frac{du}{dt}\right)^2 + \Delta u$ and $F(u) = -2q^2u^2u'^2 + \frac{1}{2}\Delta u^2$. Its variational and Hamiltonian formulations can be readily obtained as follows:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 - 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2\right) dt, \tag{11}$$

$$H = \frac{1}{2}u'^2 + 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2 = \frac{1}{2}\Delta A^2, \tag{12}$$

$$R(t) = \frac{1}{2}u'^2 + 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2 - \frac{1}{2}\Delta A^2 = 0, \tag{13}$$

Substituting Eq. (6) into Eq. (13), we obtain:

$$\begin{aligned} R(t) &= \frac{1}{2}A^2\omega^2\sin^2(\omega t) + 2q^2\omega^2A^4\cos^2(\omega t)\sin^2(\omega t) \\ &\quad + \frac{1}{2}\Delta A^2\cos^2(\omega t) - \frac{1}{2}\Delta A^2 = 0, \end{aligned} \tag{14}$$

If we collocate at $\omega t = \pi/4$, we obtain the following result:

$$\omega = \sqrt{\frac{\Delta}{(4A^2q^2\cos^2(\pi/4) + 1)}}, \tag{15}$$

with $T = \frac{2\pi}{\omega}$, yield:

$$T = \frac{2\pi}{\sqrt{\frac{\Delta}{(4A^2q^2\cos^2(\pi/4) + 1)}}}, \tag{16}$$

Simplifying Eq. (16), gives:

$$\omega_{\text{EBM}} = \sqrt{\frac{\Delta}{(2A^2q^2 + 1)}}, \tag{17}$$

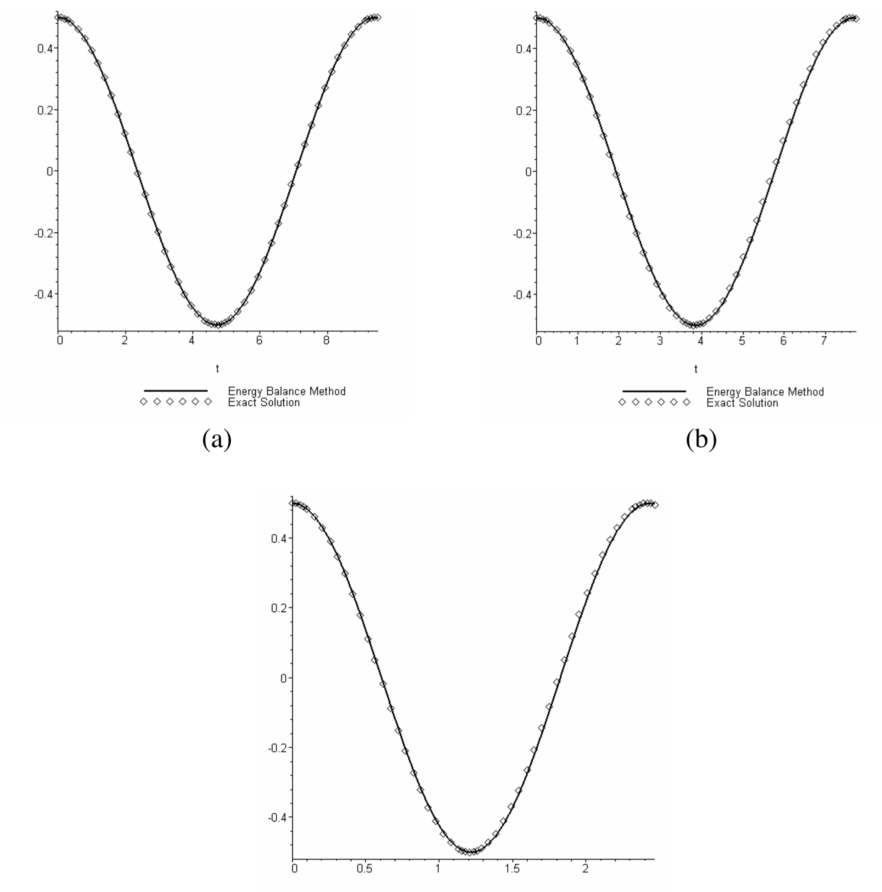
with $T = \frac{2\pi}{\omega}$, yield:

$$T_{\text{EBM}} = \frac{2\pi}{\sqrt{\frac{\Delta}{(2A^2q^2 + 1)}}}, \tag{18}$$

The exact period is [30]:

$$T_{ex} = 4\Delta^{\frac{-1}{2}} \int_0^{\frac{\pi}{2}} \left(1 + 4q^2 A^2 \cos^2 \varphi\right)^{\frac{1}{2}} d\varphi. \tag{19}$$

For comparison, the exact periodic solutions $u_{ex}(t)$ achieved by Eqs. (6) and (20), the approximate analytical periodic solutions $u_{\mathrm{EBM}}(t)$ computed by Eqs. (6) and (19), are plotted in Figs. 1(a)–(c).




(c)

**Figure 1.** Comparison of the approximate solution (EBM) with the exact solution for *Example 1*, (a) $q = 0.5$, $\Delta = 0.5$, $A = 0.5$, (b) $q = 1$, $\Delta = 1$, $A = 0.5$, (c) $q = 1$, $\Delta = 10$, $A = 0.5$.

**Example 2.** The motion of a rigid rod rocking back and forth on the circular surface without slipping. The governing equation of motion can be expressed as [30]:

$$\left(\frac{1}{12}+\frac{1}{16}u^2\right)\frac{d^2u}{dt}+\frac{1}{16}u\left(\frac{du}{dt}\right)^2+\frac{g}{4l}u\cos u=0,$$
$$u(0)=A,\quad \frac{du}{dt}(0)=0, \tag{20}$$

where $g>0$ and $l>0$ are known positive constants [31].

For the problem, its variational formulation can be obtained as follows:

$$J(u)=\int_0^t\left(-\frac{1}{2}u'^2-\frac{3}{8}u^2u'^2+\frac{3g(\cos u+\sin u)}{l}\right)dt, \tag{21}$$

By a similar manipulation as illustrated in previous example by using Eq. (6) and with $T=\frac{2\pi}{\omega}$, if we substituting $\omega t=\pi/4$, we obtain the following result:

$$\begin{aligned} R(t) &= \frac{1}{2}A^2\omega^2\sin^2(\omega t)+\frac{3}{8}A^4\omega^2\cos^2(\omega t)\sin^2(\omega t) \\ &\quad+\frac{\begin{array}{c}3g\left(\cos\left(A\cos(\omega t)\right)+A\cos(\omega t)\sin\left(A\cos(\omega t)\right)\right.\\ \left.-\cos(A)-A\sin(A)\right)\end{array}}{l}=0, \end{aligned} \tag{22}$$

$$\omega = \frac{\sqrt{\begin{array}{c}2\left(-6\,lg\left(3A^2\cos^2(\pi/4)+4\right)\left(\cos(A\cos(\pi/4))\right.\right.\\ \left.\left.+A\cos(\pi/4)\sin(A\cos(\pi/4))-\cos(A)-A\sin(A)\right)\right)\end{array}}}{\left(lA\left(3A^2\cos^2(\pi/4)+4\right)\sin(\pi/4)\right)}, \tag{23}$$

$$T = \frac{2\pi\left(lA\left(3A^2\cos^2(\pi/4)+4\right)\sin(\pi/4)\right)}{\sqrt{\begin{array}{c}2\left(-6\,lg\left(3A^2\cos^2(\pi/4)+4\right)\left(\cos(A\cos(\pi/4))\right.\right.\\ \left.\left.+A\cos(\pi/4)\sin(A\cos(\pi/4))-\cos(A)-A\sin(A)\right)\right)\end{array}}}, \tag{24}$$

Substituting $\omega t=\pi/4$ into (24), (25), we have:

$$\omega_{\text{EBM}} = \frac{4\sqrt{-3\,lg\left(8+3A^2\right)(\eta)}}{lA\left(3A^2+8\right)}, \tag{25}$$

$$T_{\text{EBM}} = \frac{2\pi lA\left(3A^2+8\right)}{4\sqrt{-3\,lg\left(8+3A^2\right)(\eta)}}. \tag{26}$$

where $\eta$ is:

$$\eta=2\cos\left(\frac{A\sqrt{2}}{2}\right)+A\sqrt{2}\sin\left(\frac{A\sqrt{2}}{2}\right)-2\cos(A)-2A\sin(A) \tag{27}$$

$$p^3 : -\delta \frac{\partial^2 v_3(x,t)}{\partial x^2} + v_2(x,t)\frac{\partial v_0(x,t)}{\partial x} + \frac{\partial v_2(x,t)}{\partial t} + v_1(x,t)\frac{\partial v_1(x,t)}{\partial x} + v_0(x,t)\frac{\partial v_2(x,t)}{\partial x} = 0.$$

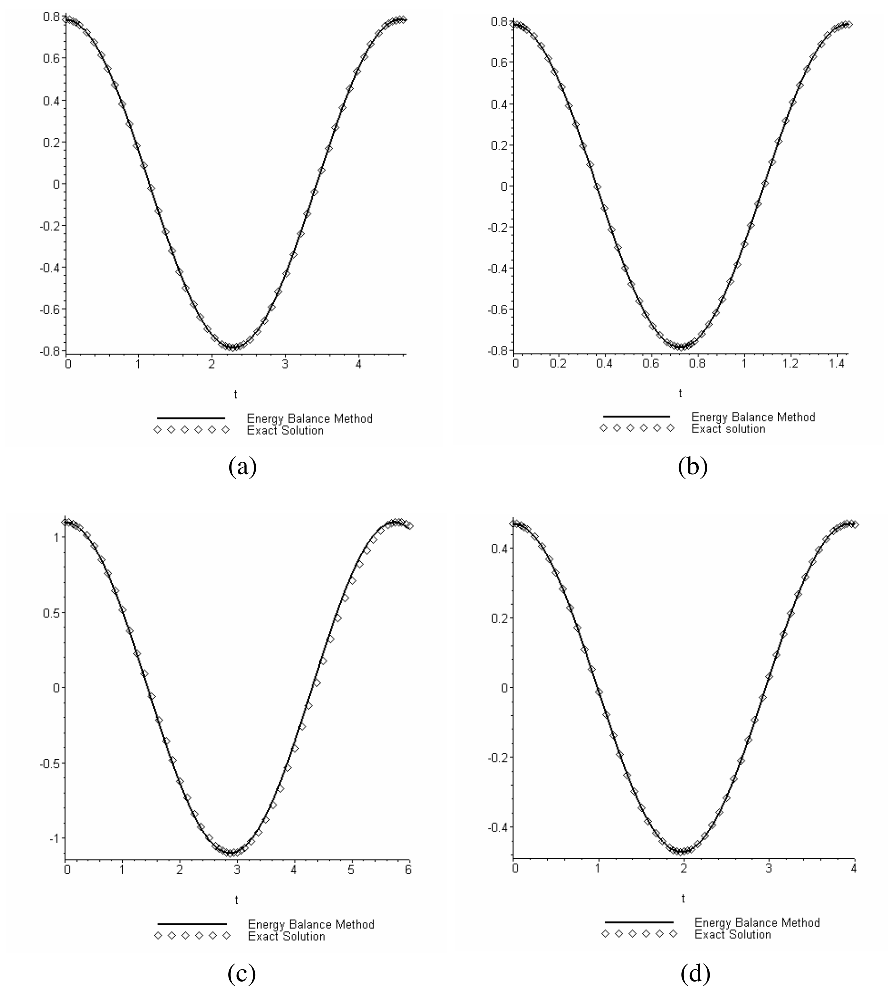



**Figure 2.** Comparison of the approximate solution (EBM) with the exact solution for *Example 2*, (a) $g = l = 1$, $A = 0.25\pi$, (b) $g = 10$, $l = 1$, $A = 0.25\pi$, (c) $g = 1$, $l = 1$, $A = 0.35\pi$, (d) $g = 1$, $l = 1$, $A = 0.15\pi$.

The exact period of (20) is:

$$T_{ex} = 4\Delta^{\frac{-1}{2}} \int_0^{\pi/2} \left( \frac{\left(4 + 3A^2 \sin^2 \varphi\right) A^2 \cos^2 \varphi}{8 \left[A \sin A + \cos A - A \sin \varphi \sin(A \sin \varphi) - \cos\right] (A \sin \varphi)} \right)^{\frac{1}{2}} d\varphi. \quad (28)$$

The exact period $T_{ex}$ achieved by Eq. (28), the approximate period $T_{\text{EBM}}$ calculated by Eq. (26), are shown in Table 1. Note that for the problem, the maximum amplitude of oscillation should satisfy $A < \pi/2$.

Table 1 indicates that Eq. (25) can give an excellent approximate period for oscillation amplitude except those near $A = \pi/2$.

**Table 1.** Comparison of approximate periods with exact period for Example 2.

| $A$ | $T_{\text{EBM}}$ | $T_{ex}$ | **Error percentage** |
|---|---|---|---|
| $\mathbf{0.05\pi}$ | 3.66129 | 3.66109 | 0.0054 |
| $\mathbf{0.10\pi}$ | 3.76397 | 3.76397 | 0.0008 |
| $\mathbf{0.15\pi}$ | 3.94064 | 3.94086 | 0.0056 |
| $\mathbf{0.20\pi}$ | 4.20181 | 4.20292 | 0.02642 |
| $\mathbf{0.25\pi}$ | 4.56432 | 4.56948 | 0.1129 |
| $\mathbf{0.30\pi}$ | 5.05831 | 5.07728 | 0.37348 |
| $\mathbf{0.35\pi}$ | 5.73741 | 5.79770 | 1.0399 |
| $\mathbf{0.40\pi}$ | 6.70586 | 6.89564 | 2.7521 |
| $\mathbf{0.45\pi}$ | 8.60226 | 8.94333 | 3.8136 |

The exact periodic solution $u_{ex}(t)$ obtained by Eqs. (6) and (28), and the approximate analytical periodic solutions $u_{\text{EBM}}(t)$ computed by Eqs. (6) and (25) are plotted in Figs. 2(a)–(d).

## 4. CONCLUSION

The Energy Balance Method (EBM) is used to obtain two approximate frequencies for two nonlinear oscillatory systems. Excellent agreement between approximate frequencies and the exact one is demonstrated and discussed. We think that the method has a great potential and can be applied to other strongly nonlinear oscillators.

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