Progress In Electromagnetics Research M, Vol. 4, 143–154, 2008

HE'S ENERGY BALANCE METHOD TO EVALUATE THE EFFECT OF AMPLITUDE ON THE NATURAL FREQUENCY IN NONLINEAR VIBRATION SYSTEMS

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Abstract—This paper presents a new approach for solving accurate approximate analytical solution for strong nonlinear oscillators. The new algorithm offers a promising approach by Hamiltonian for the nonlinear oscillator. We find that these attained solutions are not only with high degree of accuracy, but also uniformly valid in the whole solution domain.

1. INTRODUCTION

This paper considers the following general nonlinear oscillators [1]:

$$u'' + \omega_0^2 u + \varepsilon f(u) = 0 \tag{1}$$

With initial conditions:

$$u(0) = A, \quad u'(0) = 0 \tag{2}$$

where f is a nonlinear function of u'', u', u, in this preliminary report, we suppose the simplest case, i.e., f depends upon only the function of u.

If there is no small parameter in the equation, traditional perturbation methods cannot be applied directly. Recently, considerable attention has been paid to the analytical solutions for nonlinear equations without possible small parameters. Traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have appeared in open literature, for example, delta-perturbation method [2, 3], variational iteration method (VIM) [4–9], homotopy perturbation method [10–17, 23] and bookkeeping parameter perturbation method [18], just to name a few, a detailed review on some recently developed nonlinear analytical methods can be found in [19–20, 24–34]. And especially in He's methods [21, 22].

In energy balance method, a variational principle for the nonlinear oscillation is established, then the corresponding Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones. Some examples reveal the lowest order approximations which benefit high accuracy [1].

2. BASIC IDEA

First we consider the Duffing equation [1, 21]:

$$u'' + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{3}$$

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 \right\} dt \tag{4}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 = \frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4 \tag{5}$$

Or:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 - \frac{1}{2}A^2 - \frac{1}{4}\varepsilon A^4 = 0 \tag{6}$$

In Eq. (5) and Eq. (6) the kinetic energy (E) and potential energy (T) can be respectively expressed as: $u'^2/2$, $u^2/2 + \varepsilon u^4/4$ throughout the oscillation, it holds $H = E + T$ constant.

We use the following trial function to determine the angular frequency ω.

$$u = A \cos \omega t, \tag{7}$$

Substituting Eq. (7) into Eq. (6), we obtain the following residual equation:

$$R(t) = A^2\omega^2 \sin^2 \omega t + \cos^2 \omega t + \frac{1}{2}\varepsilon A^2 \cos^4 \omega t - 1 - \frac{1}{2}\varepsilon A^2, \tag{8}$$

If the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of

ω. Since Eq. (7) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2} \tag{9}$$

We can apply various other techniques, for examples, least Square Method, Galerkin method, to identify the constant ω.

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\varepsilon A^2}} \tag{10}$$

The approximate period obtained by the traditional perturbation method reads (Nayfeh, 1985).

$$T_{pert} = 2\pi\left(1 - \frac{3}{8}\varepsilon A^2\right) \tag{11}$$

So our theory, in case $\varepsilon \ll 1$, gives exactly the same result with those obtained by perturbation method.

What is rather surprising about the remarkable range of validity of Eq. (10) is that the actual asymptotic period as $\varepsilon \to \infty$ is also of high accuracy.

$$\lim_{\varepsilon\to\infty} \frac{T_{ex}}{T} = \frac{2\sqrt{3/4}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = 0.9294 \tag{12}$$

The lowest order approximation given by Eq. (10) is actually within 7.6% of the exact frequency regardless of the magnitude of εA^2.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly [1].

3. APPLICATIONS

In order to assess the advantages and the accuracy of the energy balance method, we should consider the following three examples:

3.1. Example 1

We consider the following nonlinear oscillator [23]:

$$u'' - c^2 u + \varepsilon u^3 = 0 \tag{13}$$

where the coefficient of linear term is negative.

With initial conditions of:

$$u(0) = A, \quad u'(0) = 0 \tag{14}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 - \frac{1}{2}c^2u^2 + \frac{1}{4}\varepsilon u^4 = -\frac{1}{2}c^2A^2 + \frac{1}{4}\varepsilon A^4 \tag{15}$$

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2\omega t - \frac{1}{2}c^2A^2\cos^2\omega t + \frac{1}{4}\varepsilon A^4\cos^4\omega t + \frac{1}{2}c^2A^2 - \frac{1}{4}\varepsilon A^4 = 0 \tag{16}$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \sqrt{\frac{3}{4}\varepsilon A^2 - c^2} \tag{17}$$

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{\frac{3}{4}\varepsilon A^2 - c^2}} \tag{18}$$

So there exists a periodic solution when:

$$\varepsilon A^2 > \frac{4}{3}c^2 \tag{19}$$

In order to compare with Linstedt — Poincare method solution, we write J. H. He's result [23]:

$$\omega = \sqrt{\frac{3}{4}\varepsilon A^2 - c^2} \tag{20}$$

3.2. Example 2

We consider the simple mathematical pendulum which can be written in the form [4]:

$$u'' + \Omega^2\sin u = 0 \tag{21}$$

When u designates the deviation angle from the vertical equilibrium position, $\Omega^2 = \frac{g}{l}$ where g is the gravitational acceleration, l the length of the pendulum.

With initial conditions of:

$$u(0) = A, \quad u'(0) = 0 \tag{22}$$

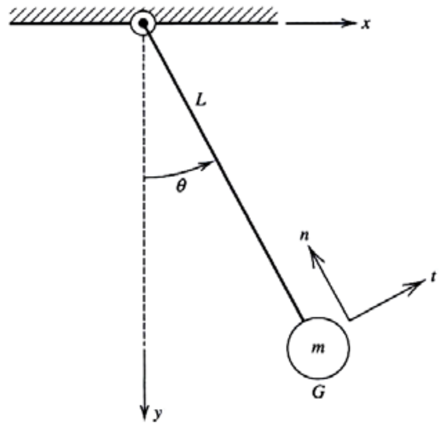

Figure 1. The simple pendulum.

The approximation $\sin(u) \approx u - \frac{1}{6}u^3 + \frac{1}{120}u^5$ is used:
Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \Omega^2\frac{1}{2}u^2 - \Omega^2\frac{1}{24}u^4 + \Omega^2\frac{1}{720}u^6 = \Omega^2\frac{1}{2}A^2 - \Omega^2\frac{1}{24}A^4 + \Omega^2\frac{1}{720}A^6 \tag{23}$$

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$\begin{aligned} R(t) = & \frac{1}{2}A^2\omega^2\sin^2\omega t + \Omega^2\frac{1}{2}A^2\cos^2\omega t - \Omega^2\frac{1}{24}A^4\cos^4\omega t \\ & + \Omega^2\frac{1}{720}A^6\cos^6\omega t - \Omega^2\frac{1}{2}A^2 + \Omega^2\frac{1}{24}A^4 - \Omega^2\frac{1}{720}A^6 = 0 \end{aligned} \tag{24}$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \Omega\sqrt{1 - \frac{1}{8}A^2 + \frac{7}{2880}A^4} \tag{25}$$

In order to compare with homotopy perturbation method solution, we write J. H. He's result [4]:

$$\omega = \Omega\sqrt{1 - \frac{1}{8}A^2 + \frac{1}{192}A^4} \tag{26}$$

We can obtain the following approximate solution:

$$u = A\cos\Omega\sqrt{1 - \frac{1}{8}A^2 + \frac{7}{2880}A^4}t \tag{27}$$

Table 1. Comparison of energy balance frequency with homotopy perturbation frequency ($\Omega = 1$).

A (rad)	Energy balance frequency	homotopy perturbation frequency [24]
0.01	1.0000	1.0000
0.1	0.9994	0.9994
0.2	0.9975	0.9975
0.3	0.9944	0.9944
0.4	0.9900	0.9900
0.5	0.9843	0.9844
0.6	0.9774	0.9776
0.7	0.9692	0.9695
0.8	0.9597	0.9603
0.9	0.9489	0.9498
1.0	0.9367	0.9382
1.5	0.8550	0.8632

3.3. Example 3

We consider the following nonlinear oscillator [24]:

$$u'' + u + \varepsilon u^5 = 0, \tag{28}$$

With initial conditions:

$$u(0) = A, \quad u'(0) = 0 \tag{29}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \varepsilon\frac{1}{6}u^6 = \frac{1}{2}A^2 + \varepsilon\frac{1}{6}A^6 \tag{30}$$

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{2}A^2\cos^2\omega t + \varepsilon\frac{1}{6}A^6\cos^6\omega t - \frac{1}{2}A^2 - \varepsilon\frac{1}{6}A^6 = 0 \tag{31}$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \sqrt{\frac{7}{12}\varepsilon A^4 + 1} \tag{32}$$

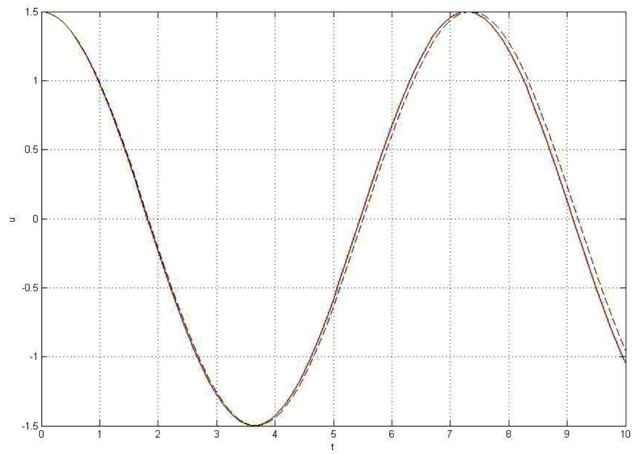

Figure 2. Comparison of the Energy balance solution with the Homotopy perturbation solution: Dashed line: Energy balance and solid line: The Homotopy perturbation solution ($A = 1.5$).

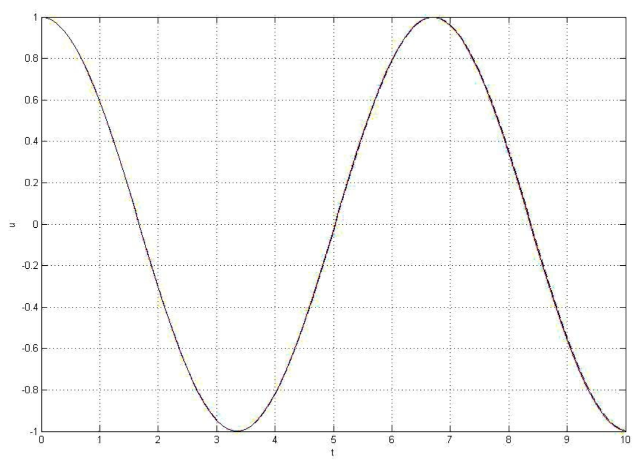

Figure 3. Comparison of the Energy balance solution with the Homotopy perturbation solution: Dashed line: Energy balance and solid line: The Homotopy perturbation solution ($A = 1.0$).

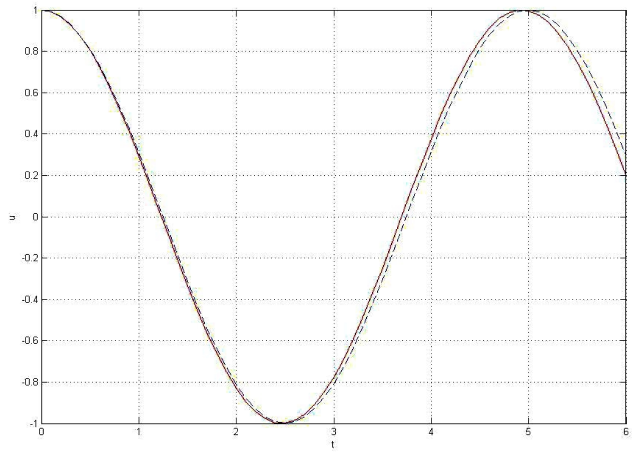

Figure 4. Comparison of the Energy balance solution with the Homotopy perturbation solution: Dashed line: Energy balance and solid line: The Homotopy perturbation solution ($A = 1.0$).

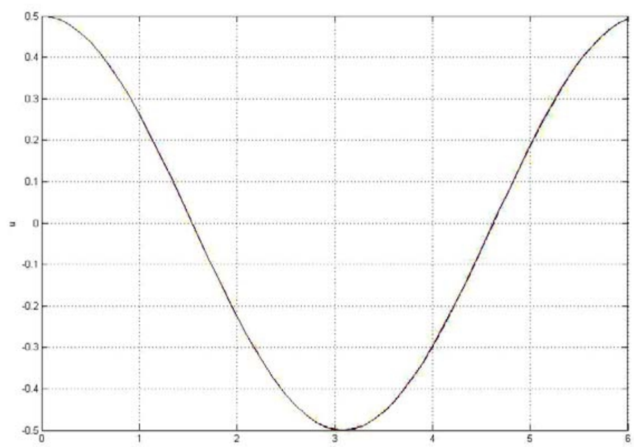

Figure 5. Comparison of the Energy balance solution with the Homotopy perturbation solution: Dashed line: Energy balance and solid line: The Homotopy perturbation solution ($A = 0.5$).

In order to compare with homotopy perturbation method solution, we write J. H. He's result [24]:

$$\omega = \sqrt{\frac{5}{8}\varepsilon A^4 + 1} \tag{33}$$

We can obtain the following approximate solution:

$$u = A\cos\sqrt{\frac{7}{12}\varepsilon A^4 + 1}t \tag{34}$$

Table 2. Comparison of energy balance frequency with homotopy perturbation frequency ($\varepsilon = 1$).

A	Energy balance frequency	homotopy perturbation frequency [24]
0.01	1.00000	1.00000
0.1	1.00003	1.00003
0.2	1.00047	1.00050
0.3	1.00236	1.00253
0.4	1.00744	1.00797
0.5	1.01807	1.01934
1	1.25831	1.27475
5	19.1202	19.7895
10	76.3828	79.0633

4. CONCLUSIONS

He's energy balance method, was applied to nonlinear oscillators which are useful to all oscillators and vibrations in so many branches of sciences such as: fluid mechanics, electromagnetic and waves, telecommunication, civil and its structures and all so-called majors applications and etc. The energy balance method is a well-established method for the analysis of nonlinear systems, can be easily extended to any nonlinear equation. We demonstrated the accuracy and efficiency of the method presenting some examples. Energy balance provides an easy and direct procedure for determining approximations to the periodic solutions. It suggests that the energy balance method is accurate, reliable and easy to use.

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