Progress In Electromagnetics Research M, Vol. 4, 33–46, 2008

A RECURRENCE TECHNIQUE FOR COMPUTING THE EFFECTIVE INDEXES OF THE GUIDED MODES OF COUPLED SINGLE-MODE WAVEGUIDES

T. A. Ramadan †

Department of Physics, Faculty of Science
Kuwait University
P. O. Box 5969, Safat 13060, State of Kuwait

Abstract—The recurrence dispersion equation of coupled single-mode waveguides is modified by eliminating redundant singularities from the dispersion function. A recurrence zero-bracketing (RZB) technique is proposed in which the zeros of the dispersion function at one recurrence step bracket those of the next recurrence step. Numerical examples verify the utility of the RZB technique in computing the roots of the dispersion equation of the TE and TM modes of both uniform and non-uniform arrays.

1. INTRODUCTION

Single-mode waveguide arrays are widely used in many photonic devices, including directional couplers, modulators, switches, arrayed waveguide gratings, modal and power splitters [1–7]. In most of these applications the device functionality depends primarily on the interaction of guided, as opposed to leaky or radiation, modes [8].

Many methods have been used to determine the modal properties of waveguide arrays by solving for the roots of the dispersion function, e.g., in [9–18]. While all of these methods require initial guess for each root, most of them do not specify a rule to identify this guess. The argument principle method [15–18] uses the roots of a polynomial as initial guess and then continues to use traditional zero-search techniques [19] to get the actual roots of the dispersion equation. However, the computer implementation of this method is not easy as it involves numerical integration along closed contour in the complex plane.

† Permanent Address: Electronics and Electrical Communication Engineering Department, Ain Shams University, Abbasia 11517, Cairo, Egypt.

While the complexity of most of the above mentioned approaches stems, in part, from its applicability to general multilayer structures, it remains desirable to trade this generality in favor of simplicity for more widely used structures such as coupled single-mode waveguides. Therefore, it is required to develop a simple and efficient zero-search technique, which enables locating the roots of the dispersion equation of these coupled waveguides without using extensive and/or complex numerical computations.

In this paper, the dispersion equation of coupled single-mode waveguides is modified to remove singularities from the dispersion function. A recursive zero-bracketing (RZB) technique is proposed for the computation of the roots of the dispersion equation. It is applied to find the effective indexes of the TE and TM guided modes of both uniform and non-uniform arrays. The paper is organized as follows. The recurrence dispersion relation of coupled waveguide arrays is introduced in Section 2. The problem of bracketing singularities of the dispersion function is outlined in Section 3.1. The modifications made to eliminate these singularities are carried out in Sections 3.2 and 3.3. The diversity of dispersion functions is discussed in Section 4. The proposed RZB technique as well as numerical examples which utilize this technique are introduced in Section 5. Finally, Section 6 presents the conclusion.

2. RECURRENCE DISPERSION RELATION

Different recurrence approaches have been used to express the dispersion relation of waveguide arrays, e.g., in [14] and [20]. The main advantage of these approaches is that they are simple to implement on computers and require fewer computational steps compared to other more complex approaches. Also, they enable monitoring the evolution of modal spectrum with each recurrence step, which not only gives an insight to the physics of modal formation in the array but also is the basis of the RZB technique proposed in this paper. In this work we follow the approach in [14] which applies for both uniform and non-uniform arrays. According to this approach the dispersion relation of an array of M coupled waveguides (see Fig. 1) is given by,

$$\varepsilon_M = 0 \tag{1a}$$

where, ε_M, is an implicit dispersion function of the modal effective index, N, and the free-space propagation constants, k_o. It satisfies the following recurrence relation,

$$\varepsilon_{i+1} = J_{i+1}\varepsilon_i - \mathrm{K}_{i+1}\varepsilon_{i-1}. \tag{1b}$$

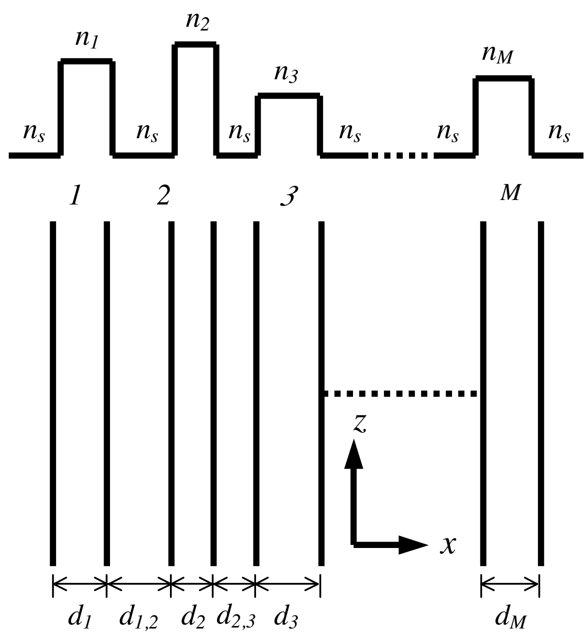

Figure 1. Schematic of an array of M-coupled waveguides (bottom) with the corresponding refractive index profile (top).

Here, i is the recurrence index which is incremented in steps from $i = 1$ to $i = M - 1$. The recurrence parameters J_{i+1} and K_{i+1} are given by,

$$J_{i+1} = F_{i+1} + \frac{C_{i,i+1}\left(1 - p_{i+1}F_{i+1}\right)\left\{2p_i + F_i\left(1 - p_i^2\right)\right\}}{\left(1 - p_iF_i\right)} \tag{1c}$$

and

$$\mathrm{K}_{i+1} = \frac{C_{i,i+1}\left(1 + F_i^2\right)\left(1 - p_{i+1}F_{i+1}\right)}{\left(1 - p_iF_i\right)}, \tag{1d}$$

with

$$p_i = \frac{h_i^2 - \eta_i^2\gamma^2}{2\eta_i\gamma h_i}, \tag{1e}$$

$$C_{i,i+1} = \frac{4\eta_i\eta_{i+1}h_ih_{i+1}\gamma^2}{\left(h_i^2 + \eta_i^2\gamma^2\right)\left(h_{i+1}^2 + \eta_{i+1}^2\gamma^2\right)}e^{-2\gamma d_{i,i+1}}, \tag{1f}$$

and

$$F_i = \tan\left(2\tan^{-1}\left(\eta_i\gamma/h_i\right) - h_id_i\right), \tag{1g}$$

while the dispersion functions, $\varepsilon_0 = 1$ and $\varepsilon_1 = F_1$. Here, $h_i = k_o\sqrt{n_i^2 - N^2}$ and $\gamma = k_o\sqrt{N^2 - n_s^2}$ are the transverse propagation constant and the evanescent-field decay constant, respectively. The parameters, n_s and n_i, are the substrate refractive index and the core refractive index of the i^{th} waveguide, respectively. The quantity, η_i, depends on the modal polarization. It equals $(n_i/n_s)^2$ for TM modes and unity for TE modes. In most of photonic devices the effective indexes of the guided modes of the array satisfy the condition [14],

$$n_s \leq N \leq \min_i n_i. \tag{2}$$

This condition means that the modal fields are oscillatory in the waveguide cores and evanescent outside these cores. As will become clear shortly, it is a key condition in simplifying the search of the roots of the dispersion equation of coupled single-mode waveguides.

3. MODIFIED DISPERSION FUNCTION

3.1. The Problem of Bracketing Singularities

Even under single-mode condition of the isolated waveguides in the array, the dispersion function ε_M has singularities in the effective index, N. As will be shown below, these singularities result from the poles of ε_1, J_{i+1}, and K_{i+1}. They set up a fundamental zero-bracketing problem [19]. For example, the opposite signs of the dispersion function between two successive values of N, may bracket a pole instead of a zero due to the discontinuity of that function. In this case, the zero search algorithm may end up returning incorrect roots of the dispersion equation. An example of such an algorithm is the built-in FZERO function in MATLAB, which is a widely used software package [21]. While, it is straightforward to discover the incorrect roots by simply evaluating the function at these roots, the absence of a clear bracketing technique implies continuing to search for the correct roots, which may potentially add to the overall computational time. Therefore, it is desirable to remove singularities of the dispersion function and to develop an efficient zero-bracketing technique to locate and compute the correct roots.

3.2. Normalization of Dispersion Function

As a preliminary step to remove singularities from the dispersion function, we use normalization. The standard normalization parameters used are [22],

$$b = \frac{N^2 - n_s^2}{n_f^2 - n_s^2}, \tag{3a}$$

$$V_i = k_o d_i \sqrt{n_f^2 - n_s^2}, \tag{3b}$$

$$V_{i,i+1} = k_o d_{i,i+1} \sqrt{n_f^2 - n_s^2}. \tag{3c}$$

Here, n_f is the core refractive index of an arbitrarily chosen waveguide in the array. In terms of these normalized parameters the recurrence parameters, J_{i+1} and K_{i+1}, are given by,

$$J_{i+1} = \frac{(\cot(\Phi_{i+1}) - p_{i+1}) + \mu_{i,i+1}(\cot(\Phi_i) + p_i)\, e^{-2V_{i,i+1}\sqrt{b}}}{(1 + p_{i+1}\cot(\Phi_{i+1}))} \tag{4a}$$

and

$$\mathrm{K}_{i+1} = \mu_{i,i+1} \frac{\csc^2(\Phi_i)}{(1 + p_i \cot(\Phi_i))(1 + p_{i+1}\cot(\Phi_{i+1}))} e^{-2V_{i,i+1}\sqrt{b}}, \tag{4b}$$

with

$$p_i = \frac{a_i - \left(1 + \eta_i^2\right) b}{2\eta_i \sqrt{b(a_i - b)}} \quad , \tag{4c}$$

and

$$\mu_{i,i+1} = \frac{\eta_i}{\eta_{i+1}} \sqrt{\frac{a_i - b}{a_{i+1} - b}} \frac{a_{i+1} + \left(\eta_{i+1}^2 - 1\right) b}{a_i + \left(\eta_i^2 - 1\right) b}, \tag{4d}$$

while $\varepsilon_0 = 1$ and $\varepsilon_1 = \frac{\cot(\Phi_1) - p_1}{1 + p_1 \cot(\Phi_1)}$. Here, $\Phi_i = V_i\sqrt{a_i - b}$ and $a_i = \frac{n_i^2 - n_s^2}{n_f^2 - n_s^2}$. By combining (4) and (1b), ε_M becomes a function of b and the parameters, V_i, $V_{i,i+1}$, a_i, and η_i. In order to identify the singularities of ε_M it is required to identify the possible values that these parameters may take. Under single-mode condition of the isolated waveguides in the array, $V_i\sqrt{a_i} < \pi$. The parameter $V_{i,i+1}$ takes any value in the range, $0 < V_{i,i+1} < \infty$. The parameter a_i equals unity (for all i) in the case of uniform arrays and depends on the choice of n_f in the case of non-uniform arrays. The mode-polarization parameter η_i is always greater than, or equal to, unity. The search of guided modal effective indexes satisfying condition (2) limits b between 0 and 1 for uniform arrays, and between 0 and $a_{\min} \equiv \min\limits_i a_i$ for non-uniform arrays.

3.3. Removal of Singularities from the Dispersion Function

The choice of $n_f = \min_i n_i$ ensures that $a_i \geq 1$ for all the waveguides in the array. In this case, b becomes limited between 0 and 1 for both uniform and non-uniform arrays. Both of these constraints on a_i and b, and the above constraint on η_i, are sufficient to eliminate any poles in p_i and $\mu_{i,i+1}$; see (4c) and (4d). Also the single-mode condition $V_i\sqrt{a_i} < \pi$ ensures that $\cot(\Phi_i)$ has no poles in the range, $0 < b < 1$. Further inspection of ε_1, J_{i+1} and K_{i+1} in (4), shows that the remaining poles are only due to the zeros of $(1 + p_i \cot(\Phi_i))$ and $(1 + p_{i+1} \cot(\Phi_{i+1}))$, which appear in the denominators of these parameters. It can be shown that neither of $1/(1 + p_i \cot(\Phi_i))$ nor $1/(1 + p_i \cot(\Phi_i))$ have zeros in the range $0 < b < 1$, under the above constraints on V_i, a_i, and η_i. Thus, these quantities result in redundant poles which may safely be removed without changing the zeros of the original dispersion function, ε_M. An induction proof for removing these poles is carried out in Appendix A. The result of removing the poles is that the dispersion equation reduces to,

$$\chi_M = 0, \tag{5a}$$

where the modified dispersion function, χ_M, satisfies the recurrence relation,

$$\chi_{i+1} = D_{i+1}\chi_i - E_{i+1}\chi_{i-1}, \tag{5b}$$

with,

$$D_{i+1} = (\cot(\Phi_{i+1}) - p_{i+1}) + \mu_{i,i+1}(\cot(\Phi_i) + p_i)\, e^{-2V_{i,i+1}\sqrt{b}} \tag{5c}$$

and

$$E_{i+1} = \mu_{i,i+1}\csc^2(\Phi_i)\, e^{-2V_{i,i+1}\sqrt{b}}. \tag{5d}$$

These recurrence parameters take more simple forms compared to J_{i+1} and K_{i+1}, in (4a) and (4b). The dispersion functions, $\chi_0 = 1$ and $\chi_1 = (\cot(\Phi_1) - p_1)$. Unlike ε_M, the modified dispersion function, χ_M, has no singularities in the case of single-mode waveguide arrays satisfying condition (2), i.e., in the range $0 < b < 1$. Thus, the sign change of χ_M around any point b in that range, only implies a zero at that point. This continuity of χ_M simplifies searching for the roots of the dispersion equation. The next section shows that there is no unique form of the dispersion function. It can take various forms, all satisfying a recurrence relation similar to (1b) but with different combinations of recurrence parameters.

4. DIVERSITY OF DISPERSION FUNCTIONS

Thus far, the dispersion function has been modified by eliminating redundant poles from the original dispersion function. Further modification of the dispersion function may be carried out by first writing the dispersion function (5b) in terms of the determinant of a tri-diagonal matrix, as shown in Appendix B. Next by noting that E_{i+1} in (5d) does not have any zeros in the entire range, $0 < b < 1$, which allows dividing the jth raw of this determinant by E_{M-j+1} or $\sqrt{E_{M-j+1}}$ (see Appendix B), without changing the zeros of the dispersion function. This modification allows generating different dispersion functions which are all continuous and share the same zeros in the range, $0 < b < 1$. For example, one form of the dispersion equation is given by,

$$\delta_M = 0, \tag{6a}$$

where the dispersion function, δ_M, satisfies the recurrence relation,

$$\delta_{i+1} = A_{i+1}\delta_i - B_{i+1}\delta_{i-1}, \tag{6b}$$

with the following recurrence parameters,

$$\begin{aligned} A_{i+1} = &\left\{(\cot(\Phi_{i+1}) - p_{i+1})/\sqrt{\mu_{i,i+1}}\right\} \sin(\Phi_i)\, e^{V_{i,i+1}\sqrt{b}} \\ &+ \sqrt{\mu_{i,i+1}}\,(\cos(\Phi_i) + p_i \sin(\Phi_i))\, e^{-V_{i,i+1}\sqrt{b}} \end{aligned} \tag{6c}$$

and

$$B_{i+1} = \left(\sqrt{\mu_{i,i+1}/\mu_{i-1,i}}\right)(\sin(\Phi_{i-1})/\sin(\Phi_i))\, e^{-(V_{i,i+1}-V_{i-1,i})\sqrt{b}}, \tag{6d}$$

and the dispersion functions, $\delta_0 = 1$ and $\delta_1 = \{(\cos(\Phi_1) - p_1 \sin(\Phi_1))/\sqrt{\mu_{1,2}}\}e^{V_{1,2}\sqrt{b}}$. Indeed, δ_1, χ_1, and ε_1 all have the same zeros in the range, $0 < b < 1$. The particular form of the dispersion function defined by (6b)–(6d) has an advantage over other forms in the case of uniform arrays where $B_{i+1} = 1$ and,

$$A_{i+1} = (\cos(\Phi_1) - p_1 \sin(\Phi_1))e^{V_{1,2}\sqrt{b}} + (\cos(\Phi_1) + p_1 \sin(\Phi_1))e^{-V_{1,2}\sqrt{b}}. \tag{6e}$$

becomes the single recurrence parameter required to compute the dispersion function. The following section presents numerical examples of computing the roots of the dispersion Equation (6a).

5. COMPUTATIONS OF THE ROOTS OF THE DISPERSION EQUATION

5.1. RZB Technique

The continuity of δ_M in the range, $0 < b < 1$, eliminates the problem of bracketing singularities in that range and allow the zeros of the dispersion function at one recurrence step to bracket its zeros at the next recurrence step. One of the primary goals of this work is to use this RZB technique to find the roots of the dispersion equation. The following numerical examples apply this technique to compute the effective indexes of the TE and TM guided modes of both uniform and non-uniform waveguide arrays.

Table 1. Computed values of modal effective indexes of TE and TM modes at a freespace wavelength of $\lambda_o = 1.3\,\mu\text{m}$ for a non-uniform array of $M = 4$ single-mode waveguides. The array parameters are given in the text.

mode	N		
	RZB + Eq.(6a)	BPM	RZB + Eq.(1a)
TE_0	1.529001	1.529000	1.528142
TE_1	1.527431	1.527431	1.521509
TE_2	1.516728	1.516727	1.511549
TE_3	1.513257	1.513259	1.500719
TM_0	1.527733	1.527734	1.528142
TM_1	1.526582	1.526587	1.521174
TM_2	1.516066	1.516068	1.512809
TM_3	1.512804	1.512808	1.502936

5.2. Numerical Examples

5.2.1. Non-uniform Waveguide Array

This example uses a non-uniform array of $M = 4$ single-mode waveguides with the following design parameters. The substrate refractive index $n_s = 1.5$. The core index of the isolated waveguides are $n_1 = 1.55$, $n_2 = 1.54$, $n_3 = 1.56$, and $n_4 = 1.53$. The core widths are, $d_1 = 1.3\,\mu\text{m}$, $d_2 = 1.1\,\mu\text{m}$, $d_3 = 1.0\,\mu\text{m}$, and $d_4 = 1.5$, while the separation between the waveguides are, $d_{1,2} = 2\,\mu\text{m}$, $d_{2,3} = 3\,\mu\text{m}$, and $d_{3,4} = 1\,\mu\text{m}$. Table 1 shows the result of computations of the effective indexes of TE and TM modes at a free-space wavelength, $\lambda_o = 1.3\,\mu\text{m}$. In these computations the RZB

Table 2. Computed values of modal effective indexes of TE and TM modes at a freespace wavelength of $\lambda_o = 1.3\,\mu\text{m}$ for a uniform array of $M = 8$ single-mode waveguides. The array parameters are given in the text.

mode	N		
	RZB + Eq.(6a)	BPM	RZB + Eq.(1a)
TE_0	1.528774	1.528751	1.549664
TE_1	1.528533	1.528541	1.546658
TE_2	1.528151	1.528154	1.528778
TE_3	1.527658	1.527659	1.525703
TE_4	1.527100	1.527099	1.504627
TE_5	1.526537	1.526535	1.501843
TE_6	1.526047	1.526041	1.500448
TE_7	1.525710	1.525725	1.500050
TM_0	1.527990	1.527974	1.549802
TM_1	1.527738	1.527751	1.548616
TM_2	1.527337	1.527345	1.542330
TM_3	1.526819	1.526825	1.532339
TM_4	1.526229	1.526234	1.520848
TM_5	1.525633	1.525636	1.507846
TM_6	1.525111	1.525111	1.500872
TM_7	1.524749	1.524769	1.500046

technique was applied with the bisection method [19] to compute the zeros of both, δ_M and ε_M, given by (6b) and (1b), respectively. The results of computations are compared with those obtained by a beam propagation method (BPM) simulator which employs an iterative technique with transparent boundary conditions [23]. The BPM computations used a computational window of 40 μm, a grid size $\Delta x = 0.001\,\mu\text{m}$, a step size in the propagation direction $\Delta z = 0.5\,\mu\text{m}$, and an overall propagation length of 5 mm. It is shown that the RZB technique is successful in computing the zeros of δ_M, and consequently the roots of the dispersion equation (6a), for both TE and TM modes. It fails to compute the roots of (1a) because of the presence of singularities in the dispersion function, ε_M.

5.2.2. Uniform Waveguide Array

This example uses a uniform array of $M = 8$ single-mode waveguides each of core refractive index 1.55, core thickness 1 μm, substrate refractive index 1.5, and waveguide separation 2 μm. As before, the effective indexes of TE and TM modes were computed using the RZB at a free-space wavelength, $\lambda_o = 1.3\,\mu\text{m}$, with both the modified and conventional dispersion functions, δ_M and ε_M. In computing the zeros of δ_M only one recurrence parameter was used; see (6e). The results of Table 2 show excellent agreement between the modal indexes computed using the modified dispersion function, δ_M, and those obtained by the iterative BPM technique. Again, the RZB technique fails in computing the zeros of ε_M due to the discontinuity of this function. The BPM computations used a computational window of $70\,\mu\text{m}$, $\Delta x = 0.01\,\mu\text{m}$, $\Delta z = 1\,\mu\text{m}$, and a propagation length of 5 mm. The small difference between the modal indexes of the successive modes verify the utility of the RZB technique in resolving closely spaced roots of the dispersion equation.

6. CONCLUSION

The recurrence dispersion function of coupled waveguide arrays has been modified by removing redundant poles from this function. The removal of these poles eliminates the possibility of bracketing singularities, instead of zeros, of the modified dispersion function in the case of coupled single-mode waveguides satisfying condition (2). It has been shown that various forms of this functions may be generated which all share the same zeros. The continuity of these modified dispersion functions allows employing a simple RZB technique to locate their zeros. Numerical examples verify the utility of this technique in computing closely spaced roots of the dispersion equation.

ACKNOWLEDGMENT

This study was supported by Kuwait University research grant SP01/05.

APPENDIX A. DERIVATION OF THE MODIFIED DISPERSION FUNCTION, χ_M

In this Appendix the modified dispersion function χ_M is derived following an induction proof. The base step is carried out by substituting with the dispersion functions $\varepsilon_0 = 1$ and $\varepsilon_1 = \chi_1 \Big/ \prod_{j=1}^{1} (1 + p_j \cot(\Phi_j))$, and the recurrence parameters, K_2 and J_2 from (4) into (1b). The result of substitution is the dispersion function of an array of two coupled waveguides,

$$\varepsilon_2 = \chi_2 \Big/ \prod_{j=1}^{2} (1 + p_j \cot(\Phi_j)), \tag{A1}$$

with $\chi_2 = D_2\chi_1 - E_2\chi_0$. Here, $\chi_1 = (\cot(\Phi_1) - p_1)$, $\chi_0 = 1$, $E_2 = \mu_{1,2} \csc^2(\Phi_1) e^{-2V_{1,2}\sqrt{b}}$, and $D_2 = (\cot(\Phi_2) - p_2) + \mu_{1,2}(\cot(\Phi_1) + p_1) e^{-2V_{1,2}\sqrt{b}}$.

In order to generalize this result for an array of M waveguides, an induction step is carried out by substituting with the dispersion functions $\varepsilon_{M-2} = \chi_{M-2} \Big/ \prod_{j=1}^{M-2} (1 + p_j \cot(\Phi_j))$ and $\varepsilon_{M-1} = \chi_{M-1} \Big/ \prod_{j=1}^{M-1} (1 + p_j \cot(\Phi_j))$ and the recurrence parameters K_{i+1} and J_{i+1} from (4) into (1b). The results of substitution is,

$$\varepsilon_M = \chi_M \Big/ \prod_{j=1}^{M} (1 + p_j \cot(\Phi_j)). \tag{A2}$$

with $\chi_M = D_M\chi_{M-1} - E_M\chi_{M-2}$ where D_M and E_M are given in (5c) and (5d). Since the quantity, $1 \Big/ \prod_{j=1}^{M} (1 + p \cot(\Phi_j))$, have no zeros in the interval, $0 < b < 1$, both of ε_M and χ_M share the same zeros in that interval.

APPENDIX B. MATRIX FORMULATION OF THE MODIFIED DISPERSION FUCNTION

The modified dispersion function, χ_M, defined by the recurrence Equation (5b) may also be defined, at each point b, by the determinant of an $M \times M$ tri-diagonal matrix. It can be shown using determinant properties [24] that there is no unique form of this matrix. In this Appendix we choose a symmetric form so that, χ_M, becomes,

$$\chi_M = \det \begin{pmatrix} D_M & \sqrt{E_M} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{E_M} & D_{M-1} & \sqrt{E_{M-1}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{E_{M-1}} & D_{M-2} & \sqrt{E_{M-2}} & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 0 & 0 & \sqrt{E_3} & D_2 & \sqrt{E_2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{E_2} & \chi_1 \end{pmatrix} \tag{B1}$$

Since E_{i+1} given by (5d) is nonzero in the entire range, $0 < b < 1$, we can divide each raw of the above determinant by $\sqrt{E_{M-j+1}}$, where j is the raw index ($E_1 \equiv E_2$), without changing the zeros of that function. This division results in the following from of the dispersion function,

$$\delta_M = \det \begin{pmatrix} D_M/\sqrt{E_M} & 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{E_M/E_{M-1}} & D_{M-1}/\sqrt{E_{M-1}} & 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{E_{M-1}/E_{M-2}} & D_{M-2}/\sqrt{E_{M-2}} & 1 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 0 & 0 & \sqrt{E_3/E_2} & D_2/\sqrt{E_2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \chi_1/\sqrt{E_2} \end{pmatrix} \tag{B2}$$

which satisfies the recurrence relation, $\delta_{i+1} = A_{i+1}\delta_i - B_{i+1}\delta_{i-1}$. For this form, the recurrence parameters, $A_{i+1} = D_{i+1}\big/\sqrt{E_{i+1}}$ and $B_{i+1} = \sqrt{E_{i+1}/E_i}$, and the dispersion functions, $\delta_0 = 1$ and $\delta_1 = \chi_1\big/\sqrt{E_2}$. These recurrence parameters are different from D_{i+1} and E_{i+1} in (5c) and (5d) due to the change in the off-diagonal matrix elements in (B2) compared to (B1). In the case of uniform waveguide arrays the lower off-diagonal elements in (B2) are all ones and the original symmetry of the matrix in (B1) is restored.

Different forms of the recurrence parameters may also be obtained following a similar approach. For each selection of these recurrence parameters the dispersion function itself changes, which verifies the diversity of the dispersion functions describing the same waveguide array. e.g., we may use the recurrence parameters, $R_{i+1} = D_{i+1}/E_{i+1}$ and $Q_{i+1} = 1/E_i\,(Q_2 \equiv 1)$, with the dispersion functions, $\rho_0 = 1$ and $\rho_1 = \chi_1$ to generate the dispersion function, $\rho_M = R_M\rho_{M-1} - Q_M\rho_{M-2}$. All of the functions, χ_M, δ_M, and ρ_M, have the same zeros. In addition, they are all continuous for single-mode-waveguide arrays satisfying condition (2).

REFERENCES

1. Kaplan, A. and S. Ruschin, "Optical switching and power control in $LiNbO_3$ coupled waveguide arrays," *IEEE J. Quantum Electron.*, Vol. 37, 1562–1573, 2001.
2. Lee, M.-H., Y. H. Min, S. Park, J. J. Ju, J. Y. Do, and S. K. Park, "Fully packages polymeric four arrayed 2×2 digital optical switch," *IEEE Photon. Technol. Lett.*, Vol. 14, 615–617, 2002.
3. Kawakita, Y., T. Saitoh, S. Shimotaya, and K. Shimomura, "A novel straight arrayed waveguide grating with linearly varying refractive-index distribution," *IEEE Photon. Technol. Lett.*, Vol. 16, 144–146, 2004.
4. Kawakita, Y., S. Shimotaya, A. Kawai, D. Machida, and K. Shimomura, "Wavelength demultiplexer using GaInAs-InP MQW-based variable refractive index arrayed waveguides fabricated by selective MOVPE," *IEEE J. Select. Quantum. Electron.*, Vol. 11, 211–216, 2005.
5. Tangdiongga, E., Y. Liu, J. H. den Besten, M. van Geemert, T. van Dongen, J. J. M. Binsma, H. de Waardt, G. D. Khoe, M. K. Smit, and H. J. S. Dorren, "Monolithically integrated 80-Gb/s AWG-based all-optical wavelength converter," *IEEE Photon. Technol. Lett.*, Vol. 18, 1627–1629, 2006.
6. Choi, C.-G., S.-P. Han, B. C. Kim, S.-H. Ahn, and M.-Y. Jeong, "Fabrication of large-core 1×16 optical power splitters in polymers using hot-embossing process," *IEEE Photon. Technol. Lett.*, Vol. 15, 825–827, 2003.
7. Olivero, M. and M. Svalgaard, "Fabrication of 2×8 power splitters in silica-on-silicon by direct UV writing technique," *IEEE Photon. Technol. Lett.*, Vol. 18, 802–804, 2006.
8. Marcuse, D., *Theory of Dielectric Optical Waveguides*, Academic, New York, 1991.
9. Chilwell, J. and I. Hodgkinson, "Thin-films field transfer matrix theory of planar multiplayer waveguides and reflection from prism-loaded waveguides," *J. Opt. Soc. Am. A*, Vol. 1, 742–753, 1984.
10. Walpita, L. M., "Solutions for planar optical waveguide equations by selecting zero elements in a characteristic matrix," *J. Opt. Soc. Am. A*, Vol. 2, 595–602, 1985.
11. Li, Y.-F. and J. W. Y. Lit, "General formulas for the guiding properties of a multiplayer slab waveguide," *J. Opt. Soc. Am. A*, Vol. 4, 671–677, 1987.
12. Li, Y.-F. and J. W. Y. Lit, "Generalized dispersion properties of a multilayer dielectric planar waveguide," *J. Opt. Soc. Am. A*,

Vol. 9, 121–129, 1992.

13. Yeh, P., "Resonant tunneling of electromagnetic radiation in superlattice structures," *J. Opt. Soc. Am. A*, Vol. 2, 568–571, 1985.
14. Chiang, K. S., "Coupled-zigzag-wave theory for guided waves in slab waveguide arrays," *J. Lightwave Technol.*, Vol. 10, 1380–1387, 1992.
15. Anemogiannis, E. and E. N. Glytsis, "Multilayer waveguides: Efficient numerical analysis of general structures," *J. Lightwave Technol.*, Vol. 10, 1344–1351, 1992.
16. Smith, R. E., S. N. Houde-Walter, and G. W. Forbes, "Mode determination of planar waveguides using the four-sheeted dispersion relation," *IEEE J. Quantum. Electron.*, Vol. 28, 1520–1526, 1992.
17. Chen, C., P. Berini, D. Feng, and V. P. Tozolov, "Efficient and accurate numerical analysis of multilayer planar optical waveguides," *Proc. SPIE*, Vol. 3797, 676–686, July 1999.
18. Chen, C., P. Berini, D. Feng, S. Tanev, and V. P. Tozolov, "Efficient and accurate numerical analysis of multilayer planar optical waveguides," *Opt. Express*, Vol. 7, 260–272, 2000.
19. Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edition, Cambridge University Press, New York, 1996.
20. Ma, C., "Coupling properties in periodic waveguides and in multiple quantum-well waveguides," *IEEE J. Quantum. Electron.*, Vol. 30, 2811–2816, 1994.
21. The MathWorks, Inc., Natick, MA, USA.
22. Kogelnik, H., "Theory of optical waveguides," *Guided-wave Optoelectronics*, T. Tamir (ed.), Ch. 2, Springer-Verlag, New York, 1990.
23. Hadley, G. R. and R. E. Smith, "Full-vector waveguide modeling using an iterative finite difference method with transparent boundary conditions," *IEEE J. Quantum. Electron.*, Vol. 13, 465–469, 1995.
24. Golub, G. H. and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins Univ. Press, Baltimore, MD, 1983.