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VARIATIONAL APPROACH METHOD FOR NONLINEAR OSCILLATIONS OF THE MOTION OF A RIGID ROD ROCKING BACK AND CUBIC-QUINTIC DUFFING OSCILLATORS

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Abstract—This paper deals with Approximate Analytical Solutions to nonlinear oscillations of a conservative, non-natural, single-degree-of-freedom system with odd nonlinearity. By extending the Variational approach proposed by He, we established approximate analytical formulas for the period and periodic solution.

To illustrate the applicability and accuracy of the method, two examples are presented: (i) the motion of a rigid rod rocking back and forth on the circular surface without slipping, and (ii) Cubic-Quintic Duffing Oscillators. Comparison of the result which is obtained by this method with the obtained result by the Exact solution reveals that the He's Variational approach is very effective and convenient and can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other sciences.

1. INTRODUCTION

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. Many new techniques have appeared in the literature, such as perturbation techniques [1–13], harmonic balance method [14–23], energy balance method [24–28], He's variational iteration method [29–32], and variational approach [33–36]. In this paper, we apply the variational approach to the Nonlinear Oscillators.

2. DESCRIPTION OF HE'S VARIATIONAL METHOD

In 2007, He [35] suggested a variational approach which is different from the known variational methods in open literature. Hereby we give a brief introduction of the method:

$$u'' + f(u) = 0 \tag{1}$$

Its variational principle can be established using the semi-inverse method [38]:

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2} u'^2 + F(u) \right) dt \tag{2}$$

where T is period of the nonlinear oscillator, $\partial F/\partial u = f$.

Assume that its solution can be expressed as:

$$u(t) = A\cos(\omega t), \tag{3}$$

where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting (3) into (2) results in:

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4} \left(-\frac{1}{2} A^2\omega^2 \sin^2 \omega t + F(A\cos\omega t) \right) dt \\ &= \frac{1}{\omega} \int_0^{\pi/2} \left(-\frac{1}{2} A^2\omega^2 \sin^2 t + F(A\cos t) \right) dt \\ &= -\frac{1}{2} A^2\omega \int_0^{\pi/2} \sin^2 t\, dt + \frac{1}{\omega} \int_0^{\pi/2} F(A\cos\omega t) dt \end{aligned} \tag{4}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial A} = 0 \tag{5}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{6}$$

But with a careful inspection, for most cases we find that:

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2} A^2 \int_0^{\pi/2} \sin^2 t\, dt - \frac{1}{\omega^2} \int_0^{\pi/2} F(A\cos t)\, dt < 0 \tag{7}$$

Thus, we modify conditions (5) and (6) into a simpler form:

$$\frac{dJ}{d\omega} = 0 \tag{8}$$

from which the relationship between the amplitude and frequency of the oscillator can be obtained.

3. THE MOTION OF A RIGID ROD ROCKING BACK

In this section, we present the motion example of a rigid rod rocking back and forth on the circular surface without slipping. To illustrate the applicability, accuracy and effectiveness of the proposed approach, the governing equation of motion can be expressed as [39, 40]:

$$\left(\frac{1}{12}+\frac{1}{16}u^2\right)\frac{d^2u}{dt}+\frac{1}{16}u\left(\frac{du}{dt}\right)^2+\frac{g}{4l}u\cos u=0,$$
$$u(0)=\beta,\quad \frac{du}{dt}(0)=0, \tag{9}$$

where $g>0$ and $l>0$ are known positive constants.

For its variational form reads:

$$J(u)=\int_0^{T/4}\left(-\frac{1}{2}u'^2-\frac{3}{8}u^2u'^2+\frac{3g\left(\cos u+u\sin u\right)}{l}\right)dt. \tag{10}$$

Substituting $u(t)=\beta\cos\omega t$ into (10), we obtain:

$$\begin{aligned}J(\beta) \;=\; &\int_0^{T/4}\frac{1}{8l}\left(4\beta^2\omega^2 l\sin^2\omega t+3\beta^4\omega^2 l\left(\cos^4\omega t-\cos^2\omega t\right)\right.\\ &+24g\left(\cos\left(\beta\cos\omega t\right)+\beta\cos\omega t\sin\left(\beta\cos\omega t\right)\right.\\ &\left.\left.-\;\cos(\beta)-\beta\sin(\beta)\right)\right)dt.\end{aligned} \tag{11}$$

The stationary condition with respect to β reads:

$$\begin{aligned}\frac{dJ}{d\beta} \;=\; &\int_0^{T/4}\frac{1}{2l}\beta\left(-2\omega^2 l\sin^2\omega t+3\beta^4\omega^2 l\left(\cos^4\omega t-\cos^2\omega t\right)\right.\\ &\left.+\,6g\left(\cos^2\omega t\cos\left(\beta\cos\omega t\right)-\cos(\beta)\right)\right)dt=0,\\ =\; &\int_0^{\pi/2}\frac{1}{2l}\beta\left(-2\omega^2 l\sin^2 t+3\beta^4\omega^2 l\left(\cos^4 t-\cos^2 t\right)\right.\\ &\left.+\,6g\left(\cos^2 t\cos\left(\beta\cos t\right)-\cos(\beta)\right)\right)dt=0.\end{aligned} \tag{12}$$

Solving (12), we have:

$$\omega \;=\; \sqrt{\frac{6g\left(1/4ABesselJ(0,\beta)-1/4BesselJ(1,\beta)\right)}{\beta l\left(1/4+3/32\beta^2\right)}}. \tag{13}$$

$$T \;=\; 2\pi\sqrt{\frac{\beta l\left(1/4+3/32\beta^2\right)}{6g\left(1/4\beta BesselJ(0,\beta)-1/4BesselJ(1,\beta)\right)}}. \tag{14}$$

4. CUBIC-QUINTIC DUFFING EQUATIONS

Now, we consider the nonlinear cubic–quintic Duffing equations, which read [41]:

$$x'' + f(x) = 0, \qquad f(x) = \alpha x + \beta x^3 + \gamma x^5 \tag{15}$$

With the boundary conditions of:

$$x(0) = A \qquad x'(0) = 0 \tag{16}$$

Its variational formulation is:

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}x'^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 + \frac{1}{6}\gamma x^6\right) dt. \tag{17}$$

Proceeding in a similar way as before, we have:

$$\begin{aligned} J(A) \;=\; & \int_0^{T/4} -\frac{1}{12}A^2 \left(\left(6\omega^2 \sin^2 \omega t\right) + 6\alpha \cos^2 \omega t \right. \\ & \left. + 3\beta A^2 \cos^4 \omega t + 3\gamma A^4 \cos^6 \omega t\right) dt. \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{dJ}{dA} =& \int_0^{T/4} -\frac{1}{6}A^2 \left(\left(6\omega^2 \sin^2 t\right) + 6\alpha \cos^2 t + 3\beta A^2 \cos^4 t + 3\gamma A^4 \cos^6 t\right) \\ & + \frac{1}{12}A^2 \left(6\beta A \cos^4 t + 8\gamma A^3 \cos^6 t\right) dt. \\ =& \int_0^{\pi/2} -\frac{1}{6}A^2 \left(\left(6\omega^2 \sin^2 \omega t\right) + 6\alpha \cos^2 \omega t + 3\beta A^2 \cos^4 \omega t + 3\gamma A^4 \cos^6 \omega t\right) \\ & + \frac{1}{12}A^2 \left(6\beta A \cos^4 \omega t + 8\gamma A^3 \cos^6 \omega t\right) dt = 0. \end{aligned} \tag{19}$$

From (17), we obtain the following approximate frequency:

$$\omega \;=\; \sqrt{\alpha + 3/4A^2\beta + 5/8\gamma A^4}. \tag{20}$$

$$T \;=\; \frac{2\pi}{\sqrt{\alpha + 3/4A^2\beta + 5/8\gamma A^4}}. \tag{21}$$

5. DISCUSSION

In order to compare, we write the exact solutions for previous examples governed by Eqs. (9) and (15) that can be derived as shown in Eqs. (22) and (23), respectively [40, 42].

The Exact period T_{ex} for (9) is:

$$T_{ex} = 4\left(\frac{l}{3g}\right)^{1/2} \int_0^{\pi/2} \cdot \left(\frac{\left(4+3\beta^2 \sin^2\varphi\right)\beta^2\cos^2\varphi}{8\left[\beta\sin\beta+\cos\beta-\beta\sin\varphi\sin(\beta\sin\varphi)-\cos(\beta\sin\varphi)\right]}\right)^{1/2} d\varphi. \quad (22)$$

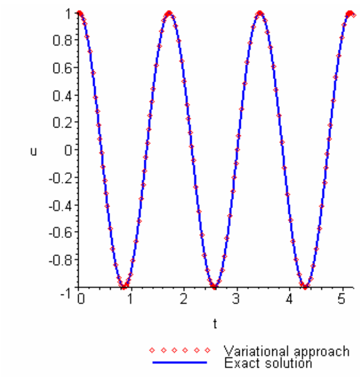
The Exact frequency ω_{ex} for the Cubic-Quintic Duffing oscillator is:

$$\begin{aligned} \omega_e(A) &= \frac{\pi k_1}{2\int_0^{\pi/2}\left(1+k_2\sin^2 t+k_3\sin^4 t\right)^{-1/2} dt}, \\ k_1 &= \sqrt{\alpha+\frac{\beta A^2}{2}+\frac{\gamma A^4}{3}}, \\ k_2 &= \frac{3\beta A^2+2\gamma A^4}{6\alpha+3\beta A^2+2\gamma A^4}, \\ k_3 &= \frac{2\gamma A^4}{6\alpha+3\beta A^2+2\gamma A^4}. \end{aligned} \quad (23)$$

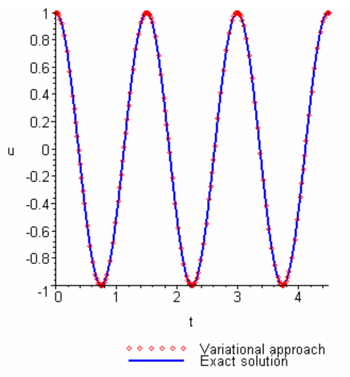
The above results are in good agreement with the results obtained by the Exact solution in [40] as illustrated in Figs. 1 and 2. Comparison between analytical Variational approach and the Exact solutions for previous nonlinear oscillators are given in Tables 1 and 2, respectively.

Table 1. Comparison between analytical variational approach and exact solutions for the motion equation (15), when $g = l = 1$.

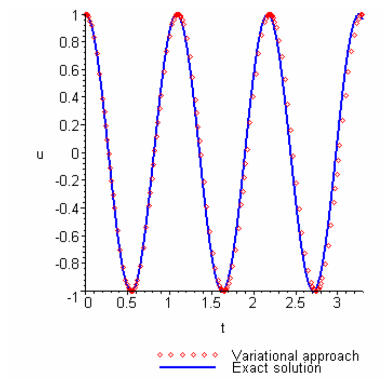
β	T	T_{ex}	**Error percentage**
0.05 π	3.66129	3.66109	0.0054
0.10 π	3.76394	3.76397	0.0008
0.15 π	3.94064	3.94086	0.0056
0.20 π	4.20116	4.20292	0.04187
0.25 π	4.56246	4.56948	0.15363
0.30 π	5.05355	5.07728	0.46738
0.35 π	5.72584	5.79770	1.23946
0.40 π	6.67785	6.89564	3.1584



(a) g=l=1, A=0.05π



(b) g=l=1, A=0.20π

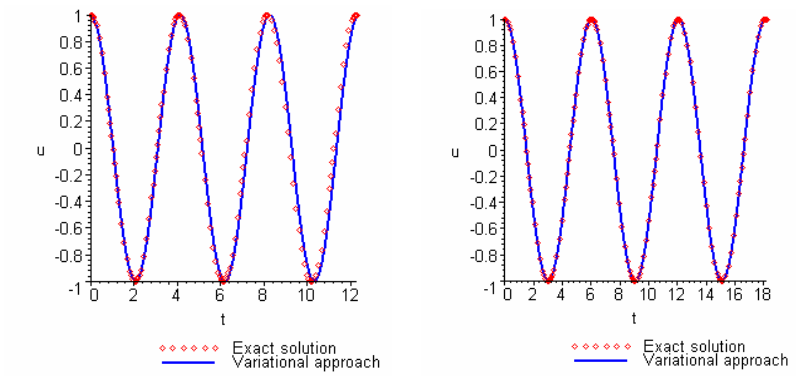


(c) g=l=1, A=0.35π

Figure 1. Comparison of the approximate solution with the Exact solution of The motion of a rigid rod rocking back (9).

Table 2. Comparison between analytical variational approach and exact solutions for the Cubic-Quintic Duffing oscillator.

A	$\alpha=\beta=\gamma=1$			$\alpha=1,\ \beta=10,\ \gamma=100$		
	ω	ω_{ex} **[42]**	**Error percentage**	ω	ω_{ex} **[42]**	**Error percentage**
0.1	1.00377	1.00377	0.0	1.03983	1.03970	0.01250
0.5	1.10750	1.10654	0.06757	2.60408	2.52469	3.14468
1	1.54110	1.52359	1.14926	8.42615	8.01005	5.19472
5	20.2577	19.1815	5.61061	198.119	187.199	5.83318
10	79.5361	75.1774	5.79795	791.044	747.323	5.85038
50	1976.90	1867.57	5.85413	19764.71	18671.34	5.85587
100	7906.17	7468.83	5.85553	79057.42	74683.91	5.85602
500	197642.83	186709.04	5.85606	1976424.01	1867085.99	5.85608
1000	790569.89	746834.69	5.85608	7905694.62	7468342.49	5.85608



(a) α=1, β=1, γ=1 (b) α=0.1, β=10, γ=100

Figure 2. Comparison of the approximate solution with the Exact solution of the Cubic-Quintic Duffing oscillator.

6. CONCLUSION

In this paper, we applied He's Variational approach to the Motion of a Rigid Rod Rocking Back and Cubic–Quintic Duffing Oscillators. We conclude from the results obtained that Variational approach is extremely simple in its principle, easy to apply, and gives good accuracy even with the first-order approximation and the simplest trial functions. Comparison made with the Exact solutions shows that the method provides a powerful mathematical tool to the determination of more complex nonlinear systems.

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