

**NEW DIRECT METHOD TO SOLVE NONLINEAR
VOLTERRA-FREDHOLM INTEGRAL AND
INTEGRO-DIFFERENTIAL EQUATIONS USING
OPERATIONAL MATRIX WITH BLOCK-PULSE
FUNCTIONS**

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Abstract—A new and effective direct method to determine the numerical solution of specific nonlinear Volterra-Fredholm integral and integro-differential equations is proposed. The method is based on vector forms of block-pulse functions (BPFs). By using BPFs and its operational matrix of integration, an integral or integro-differential equation can be transformed to a nonlinear system of algebraic equations. Some numerical examples are provided to illustrate accuracy and computational efficiency of the method. Finally, the error evaluation of this method is presented. The benefits of this method are low cost of setting up the equations without applying any projection method such as Galerkin, collocation, Also, the nonlinear system of algebraic equations is sparse.

1. INTRODUCTION

Over several decades, numerical methods in Electromagnetics have been the subject of extensive researches [1–11]. On the other hand, many problems in Electromagnetics can be modeled by integral and integro-differential equations (see [12–21]); for example, electric field integral equation (EFIE) and magnetic field integral equation (MFIE). In recent years, several numerical methods for solving linear and nonlinear integro-differential equations have been presented. Some authors use decomposition method [22, 23]. In most methods, a set of basis functions and an appropriate projection method such as Galerkin, collocation, . . . or a direct method have been applied [24–29]. These methods often transform an integral or integro-differential equation to a linear or nonlinear system of algebraic equations which can be solved by direct or iterative methods. In general, generating this system needs calculation of a large number of integrations.

This paper considers specific cases of Volterra-Fredholm integral and integro-differential equations of the forms

$$x(s) + \lambda_1 \int_0^s k_1(s, t) F(x(t)) dt + \lambda_2 \int_0^1 k_2(s, t) G(x(t)) dt = y(s), \quad (1)$$

$$0 \leq s < 1,$$

and

$$\begin{cases} x'(s) + q(s)x(s) + \lambda_1 \int_0^s k_1(s, t) F(x(t)) dt \\ \quad + \lambda_2 \int_0^1 k_2(s, t) G(x(t)) dt = y(s), \\ x(0) = x_0, \quad 0 \leq s < 1, \end{cases} \quad (2)$$

where the functions $F(x(t))$ and $G(x(t))$ are polynomials of $x(t)$ with constant coefficients. For convenience, we put $F(x(t)) = [x(t)]^{n_1}$ and $G(x(t)) = [x(t)]^{n_2}$ where, n_1, n_2 are positive integers. Note that the method presented in this article can be easily extended and applied to any nonlinear integral and integro-differential equations of the forms Eqs. (1) and (2). It is clear that for $n_1, n_2 = 1$, Eqs. (1) and (2) are linear integral and integro-differential equations respectively. Also, without loss of generality, it is supposed that the interval of integration is $[0, 1)$, since any finite interval $[a, b)$ can be transformed to interval $[0, 1)$ by linear maps [26].

For solving these equations, this paper uses the orthogonal block-pulse functions (BPFs). By using vector forms of BPFs and its

operational matrix of integration, Eqs. (1) and (2) can be easily reduced to a nonlinear system of algebraic equations.

Finally, we apply the proposed method on some examples to show its accuracy and efficiency. Also, the error evaluation of this method is presented.

2. REVIEW OF BLOCK-PULSE FUNCTIONS

Block-pulse functions are studied by many authors and applied for solving different problems; for example, see [30, 31].

2.1. Definition

An m -set of block-pulse functions (BPFs) is defined over the interval $[0, T)$ as

$$\phi_i(t) = \begin{cases} 1, & \frac{iT}{m} \leq t < \frac{(i+1)T}{m}, \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

where, $i = 0, 1, \dots, m-1$, with a positive integer value for m . Also, consider $h = T/m$, and ϕ_i is the i th block-pulse function.

In this paper, it is assumed that $T = 1$, so BPFs are defined over $[0, 1)$, and $h = 1/m$.

There are some properties for BPFs, the most important properties are disjointness, orthogonality, and completeness.

The disjointness property can be clearly obtained from the definition of BPFs:

$$\phi_i(t)\phi_j(t) = \begin{cases} \phi_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \quad (4)$$

where $i, j = 0, 1, \dots, m-1$.

The other property is orthogonality. It is clear that

$$\int_0^1 \phi_i(t)\phi_j(t)dt = h \delta_{ij}, \quad (5)$$

where, δ_{ij} is the Kronecker delta.

The third property is completeness. For every $f \in \mathcal{L}^2([0, 1))$, when m approaches to the infinity, Parseval's identity holds:

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\phi_i(t)\|^2, \quad (6)$$

where,

$$f_i = \frac{1}{h} \int_0^1 f(t) \phi_i(t) dt. \quad (7)$$

2.2. Vector Forms

Consider the m terms of BPFs and write them concisely as m -vector:

$$\Phi(t) = [\phi_0(t), \phi_1(t), \dots, \phi_{m-1}(t)]^T, \quad t \in [0, 1]. \quad (8)$$

Above representation and disjointness property, follows:

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} \phi_0(t) & 0 & \dots & 0 \\ 0 & \phi_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \phi_{m-1}(t) \end{pmatrix}, \quad (9)$$

$$\int_0^1 \Phi(t)\Phi^T(t) dt = \begin{pmatrix} h & 0 & \dots & 0 \\ 0 & h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h \end{pmatrix} = hI_{m \times m} = D, \quad (10)$$

$$\Phi^T(t)\Phi(t) = 1, \quad (11)$$

$$\Phi(t)\Phi^T(t)V = \tilde{V}\Phi(t), \quad (12)$$

where, V is an m -vector and $\tilde{V} = \text{diag}(V)$. Moreover, it can be clearly concluded that for every $m \times m$ matrix B :

$$\Phi^T(t)B\Phi(t) = \hat{B}^T\Phi(t), \quad (13)$$

where, \hat{B} is an m -vector with elements equal to the diagonal entries of matrix B .

2.3. BPFs Expansion

The expansion of a function $f(t)$ over $[0, 1)$, with respect to $\phi_i(t)$, $i = 0, 1, \dots, m-1$ may be compactly written as

$$f(t) \simeq \sum_{i=0}^{m-1} f_i \phi_i(t) = F^T\Phi(t) = \Phi^T(t)F, \quad (14)$$

where, $F = [f_0, f_1, \dots, f_{m-1}]^T$ and f_i is defined by (7).

Now, assume $k(t, s)$ is a function of two variables in $\mathcal{L}^2([0, 1) \times [0, 1))$. It can be similarly expanded with respect to BPFs as

$$k(t, s) \simeq \Phi^T(t)K\Psi(s), \quad (15)$$

where, $\Phi(t)$ and $\Psi(s)$ are m_1 and m_2 dimensional BPF vectors respectively, and K is the $m_1 \times m_2$ block-pulse coefficient matrix with k_{ij} , $i = 0, 1, \dots, m_1 - 1$, $j = 0, 1, \dots, m_2 - 1$ as follows:

$$k_{ij} = m_1 m_2 \int_0^1 \int_0^1 k(t, s) \phi_i(t) \psi_j(s) dt ds. \quad (16)$$

For convenience, we put $m_1 = m_2$.

2.4. Operational Matrix

Computing $\int_0^t \phi_i(\tau) d\tau$ follows:

$$\int_0^t \phi_i(\tau) d\tau = \begin{cases} 0, & t < ih, \\ t - ih, & ih \leq t < (i+1)h, \\ h, & (i+1)h \leq t < 1. \end{cases} \quad (17)$$

Note that $t - ih$, equals to $h/2$, at mid-point of $[ih, (i+1)h]$. So, we can approximate $t - ih$, for $ih \leq t < (i+1)h$, by $h/2$.

Now, expressing $\int_0^t \phi_i(\tau) d\tau$, in terms of the BPFs follows:

$$\int_0^t \phi_i(\tau) d\tau \simeq \left[0, \dots, 0, \frac{h}{2}, h, \dots, h \right] \Phi(t), \quad (18)$$

in which $h/2$, is i th component. Therefore,

$$\int_0^t \Phi(\tau) d\tau \simeq P\Phi(t), \quad (19)$$

where, $P_{m \times m}$ is called operational matrix of integration and can be represented as

$$P = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (20)$$

So, the integral of every function f can be approximated as follows:

$$\int_0^t f(\tau) d\tau \simeq \int_0^t F^T \Phi(\tau) d\tau \simeq F^T P \Phi(t). \quad (21)$$

3. DIRECT METHOD

In this section, by using results obtained in previous section about BPFs, an effective and accurate direct method for solving nonlinear Volterra-Fredholm integral and integro-differential equations is presented.

First, we require to prove the following lemma.

Lemma 1. *Let m -vectors X and X_n be BPFs coefficients of $x(s)$ and $[x(s)]^n$ respectively. If*

$$X = (x_0, x_1, \dots, x_{m-1})^T, \quad (22)$$

then

$$X_n = (x_0^n, x_1^n, \dots, x_{m-1}^n)^T, \quad (23)$$

where, $n \geq 1$ is a positive integer.

Proof. When $n = 1$, (23) follows at once from $[x(s)]^n = x(s)$. Suppose that (23) holds for n , we shall deduce it for $n + 1$. Since $[x(s)]^{n+1} = x(s)[x(s)]^n$, from assumption, Eqs. (14) and (12) follows:

$$\begin{aligned} [x(s)]^{n+1} &\simeq (X^T \Phi(s)) \cdot (X_n^T \Phi(s)) \\ &= X^T \Phi(s) \Phi^T(s) X_n \\ &\simeq X^T \tilde{X}_n \Phi(s). \end{aligned} \quad (24)$$

Now, using (23) we obtain

$$X^T \tilde{X}_n = (x_0^{n+1}, x_1^{n+1}, \dots, x_{m-1}^{n+1})^T. \quad (25)$$

Therefore, (23) holds for $n + 1$, and the lemma is established. \square

So, the components of X_n can be computed in terms of components of vector X .

3.1. Volterra-Fredholm Integral Equation

Consider the following nonlinear Volterra-Fredholm integral equation of the second kind:

$$x(s) + \lambda_1 \int_0^s k_1(s, t) [x(t)]^{n_1} dt + \lambda_2 \int_0^1 k_2(s, t) [x(t)]^{n_2} dt = y(s), \quad (26)$$

$$0 \leq s < 1,$$

where the parameters λ_1 and λ_2 , and the functions $y(s)$, $k_1(s, t)$ and $k_2(s, t)$ are known but $x(s)$ is not. Moreover, $k_1(s, t), k_2(s, t) \in \mathcal{L}^2([0, 1] \times [0, 1])$ and $y(s) \in \mathcal{L}^2([0, 1])$.

Approximating functions $x(s)$, $[x(s)]^{n_1}$, $[x(s)]^{n_2}$, $y(s)$, $k_1(s, t)$ and $k_2(s, t)$ with respect to BPFs, using (14) and (15) gives

$$\begin{aligned}
 x(s) &\simeq X^T \Phi(s) = \Phi^T(s) X \\
 [x(s)]^{n_1} &\simeq X_{n_1}^T \Phi(s) = \Phi^T(s) X_{n_1} \\
 [x(s)]^{n_2} &\simeq X_{n_2}^T \Phi(s) = \Phi^T(s) X_{n_2} \\
 y(s) &\simeq Y^T \Phi(s) = \Phi^T(s) Y \\
 k_1(s, t) &\simeq \Phi^T(s) K_1 \Phi(t) \\
 k_2(s, t) &\simeq \Phi^T(s) K_2 \Phi(t),
 \end{aligned} \tag{27}$$

where m -vectors X , X_{n_1} , X_{n_2} , Y , and $m \times m$ matrices K_1 and K_2 are BPFs coefficients of $x(s)$, $[x(s)]^{n_1}$, $[x(s)]^{n_2}$, $y(s)$, and $k_1(s, t)$, $k_2(s, t)$, respectively.

For solving Eq. (26), we substitute (27) into (26), therefore,

$$\begin{aligned}
 Y^T \Phi(s) &\simeq X^T \Phi(s) \\
 &+ \lambda_1 \Phi^T(s) K_1 \int_0^s \Phi(t) \Phi^T(t) X_{n_1} dt \\
 &+ \lambda_2 \Phi^T(s) K_2 \int_0^1 \Phi(t) \Phi^T(t) X_{n_2} dt.
 \end{aligned} \tag{28}$$

Using Eq. (10) and Eq. (12) follows:

$$\begin{aligned}
 Y^T \Phi(s) &\simeq X^T \Phi(s) \\
 &+ \lambda_1 \Phi^T(s) K_1 \tilde{X}_{n_1} \int_0^s \Phi(t) dt \\
 &+ \lambda_2 \Phi^T(s) K_2 D X_{n_2}.
 \end{aligned} \tag{29}$$

Using operational matrix P , in Eq. (20), gives

$$\begin{aligned}
 Y^T \Phi(s) &\simeq X^T \Phi(s) \\
 &+ \lambda_1 \Phi^T(s) K_1 \tilde{X}_{n_1} P \Phi(s) \\
 &+ \lambda_2 (K_2 D X_{n_2})^T \Phi(s),
 \end{aligned} \tag{30}$$

in which, $\lambda_1 K_1 \tilde{X}_{n_1} P$ is an $m \times m$ matrix. Eq. (13) follows:

$$\Phi^T(s) \lambda_1 K_1 \tilde{X}_{n_1} P \Phi(s) \simeq \hat{X}_{n_1}^T \Phi(s), \tag{31}$$

where, \hat{X}_{n_1} is an m -vector with components equal to the diagonal entries of the matrix $\lambda_1 K_1 \tilde{X}_{n_1} P$.

Now, Combining (30) and (31), and replacing \simeq with $=$ gives

$$Y^T \Phi(s) = X^T \Phi(s) + \hat{X}_{n_1}^T \Phi(s) + \lambda_2 (K_2 D X_{n_2})^T \Phi(s), \quad (32)$$

or

$$X + \hat{X}_{n_1} + \lambda_2 K_2 D X_{n_2} = Y. \quad (33)$$

Equation (33) is a nonlinear system of m algebraic equations for the m unknowns x_0, x_1, \dots, x_{m-1} , components of X can be obtained by an iterative method. Hence, an approximate solution $x(s) \simeq X^T \Phi(s)$ can be computed for Eq. (26) without using any projection method.

3.2. Volterra-Fredholm Integro-Differential Equation

Consider the following Volterra-Fredholm nonlinear integro-differential equation:

$$\begin{cases} x'(s) + q(s)x(s) + \lambda_1 \int_0^s k_1(s,t)[x(t)]^{n_1} dt \\ \quad + \lambda_2 \int_0^1 k_2(s,t)[x(t)]^{n_2} dt = y(s), \\ x(0) = x_0, \quad 0 \leq s < 1, \quad n_1, n_2 \geq 1, \end{cases} \quad (34)$$

where the parameters λ_1 and λ_2 and \mathcal{L}^2 functions $q(s)$, $y(s)$, $k_1(s,t)$ and $k_2(s,t)$ are known but $x(s)$ is not. Note the appearance of initial condition equation in Eq. (34). This is necessary to ensure existence of a solution.

Approximating functions $x'(s)$ and $q(s)$ with respect to BPFs, from (14) and (15) gives

$$\begin{aligned} x'(s) &\simeq X'^T \Phi(s) = \Phi^T(s) X' \\ q(s) &\simeq Q^T \Phi(s) = \Phi^T(s) Q, \end{aligned} \quad (35)$$

where m -vectors X' and Q are BPFs coefficients of $x'(s)$ and $q(s)$ respectively.

For solving Eq. (34), we substitute (27) and (35) into (34), therefore,

$$\begin{aligned} Y^T \Phi(s) &\simeq X'^T \Phi(s) + Q^T \Phi(s) \Phi^T(s) X \\ &\quad + \lambda_1 \Phi^T(s) K_1 \int_0^s \Phi(t) \Phi^T(t) X_{n_1} dt \\ &\quad + \lambda_2 \Phi^T(s) K_2 \int_0^1 \Phi(t) \Phi^T(t) X_{n_2} dt. \end{aligned} \quad (36)$$

Using Eq. (10) and Eq. (12) follows:

$$\begin{aligned}
 Y^T \Phi(s) &\simeq X'^T \Phi(s) + (\tilde{Q} \Phi(s))^T X \\
 &\quad + \lambda_1 \Phi^T(s) K_1 \tilde{X}_{n_1} \int_0^s \Phi(t) dt \\
 &\quad + \lambda_2 \Phi^T(s) K_2 D X_{n_2}.
 \end{aligned} \tag{37}$$

Using operational matrix P , in Eq. (20), gives

$$\begin{aligned}
 Y^T \Phi(s) &\simeq X'^T \Phi(s) + X^T \tilde{Q} \Phi(s) \\
 &\quad + \lambda_1 \Phi^T(s) K_1 \tilde{X}_{n_1} P \Phi(s) \\
 &\quad + \lambda_2 (K_2 D X_{n_2})^T \Phi(s),
 \end{aligned} \tag{38}$$

in which, $\lambda_1 K_1 \tilde{X}_{n_1} P$ is an $m \times m$ matrix. Eq. (13) follows:

$$\Phi^T(s) \lambda_1 K_1 \tilde{X}_{n_1} P \Phi(s) \simeq \hat{X}_{n_1}^T \Phi(s), \tag{39}$$

where, \hat{X}_{n_1} is an m -vector with components equal to the diagonal entries of the matrix $\lambda_1 K_1 \tilde{X}_{n_1} P$. Combining (38) and (39) gives

$$Y^T \Phi(s) \simeq X'^T \Phi(s) + X^T \tilde{Q} \Phi(s) + \hat{X}_{n_1}^T \Phi(s) + \lambda_2 (K_2 D X_{n_2})^T \Phi(s), \tag{40}$$

or

$$X' + \tilde{Q} X + \hat{X}_{n_1} + \lambda_2 K_2 D X_{n_2} \simeq Y. \tag{41}$$

Note that \tilde{Q} is a diagonal matrix, so $\tilde{Q}^T = \tilde{Q}$.

Now, X' must be computed in terms of X . Note that

$$\begin{aligned}
 x(s) - x(0) &= \int_0^s x'(\tau) d\tau \\
 &\simeq \int_0^s X'^T \Phi(\tau) d\tau \\
 &\simeq X'^T P \Phi(s).
 \end{aligned} \tag{42}$$

Therefore,

$$x(s) \simeq X'^T P \Phi(s) + X_0^T \Phi(s), \tag{43}$$

where, X_0 is the m -vector of the form $X_0 = [x_0, x_0, \dots, x_0]^T$, consequently, using (27)

$$X \simeq P^T X' + X_0. \tag{44}$$

Now, combining (41) and (44), and replacing \simeq with $=$ follows:

$$(I + P^T \tilde{Q}) X + P^T \hat{X}_{n_1} + \lambda_2 P^T K_2 D X_{n_2} = P^T Y + X_0. \quad (45)$$

Equation (45) is a nonlinear system of m algebraic equations for the m unknowns x_0, x_1, \dots, x_{m-1} , components of X can be obtained by an iterative method. Hence, an approximate solution $x(s) \simeq X^T \Phi(s)$ can be computed for Eq. (34) without using any projection method.

4. NUMERICAL EXAMPLES

The direct method, presented in this article, is applied to four examples. These examples are selected from different references, so the numerical results obtained here can be compared with both the exact solution and other numerical results.

The computations associated with the examples were performed using Matlab 7 on a Personal Computer.

Example 1. Consider the following nonlinear Volterra integral equation [23]:

$$x(s) - \frac{1}{2} \int_0^s x^2(t) dt = \sin s + \frac{1}{8} \sin 2s - \frac{1}{4} s, \quad (46)$$

with the exact solution $x(s) = \sin s$. The numerical results are shown in Table 1.

Table 1. Numerical results for example 1.

s	Exact solution	Approximate solution, m=32	Approximate solution, m=64
0	0	0.015624	0.007812
0.1	0.099833	0.109157	0.101388
0.2	0.198669	0.201731	0.194073
0.3	0.295520	0.292533	0.299995
0.4	0.389418	0.380765	0.387978
0.5	0.479426	0.493076	0.486266
0.6	0.564642	0.572351	0.565930
0.7	0.644218	0.646598	0.640624
0.8	0.717356	0.715165	0.720611
0.9	0.783327	0.777451	0.782351

Example 2. For the following nonlinear Fredholm integral equation [23]:

$$x(s) - \frac{1}{2} \int_0^1 t x^2(t) dt = s^2 - \frac{1}{12}, \quad (47)$$

with the exact solution $x(s) = s^2$, Table 2 shows the numerical results.

Example 3. For the following nonlinear integro-differential equation [29]:

$$x'(s) + 2sx(s) - \int_0^s (s+t)x^3(t)dt - \int_0^1 (s-t)x(t)dt = y(s), \quad (48)$$

where, $y(s) = (-\frac{2}{3}s + \frac{1}{9})e^{3s} + (2s+1)e^s + (\frac{4}{3} - e)s + \frac{8}{9}$, with the initial condition $x(0) = 1$, and the exact solution $x(s) = e^s$, Table 3 gives the numerical results.

Example 4. For the following nonlinear integro-differential equation [22]:

$$x'(s) + \int_0^s 3 \cos(s-t) x^2(t) dt = y(s), \quad (49)$$

where, $y(s) = 2 \sin s \cos s$, with the initial condition $x(0) = 1$, and the exact solution $x(s) = \cos s$, Table 4 shows the numerical results.

Example 5. As the final example, we solve an electromagnetic scattering problem via presented method. The scattering problems have been surveyed by many authors [32–42]. Here, we consider a thin wire scatterer and obtain the current induced on it. We have modeled this problem in detail in [13, 15]. However, for a wire of length L and radius a coincided with z -axis, the final form of current integral equation is [13, 15]

$$\int_{-L/2}^{L/2} I_z(z') G(z, z') dz' = A \cos kz + j \frac{4\pi\omega\epsilon_0}{k^2 \sin \alpha} e^{jkz \cos \alpha}, \quad (50)$$

in which,

$$G(z, z') = \int_0^{2\pi} \frac{e^{-jkR}}{R} d\phi',$$

$$R = \sqrt{(z - z')^2 + \left(2a \sin \frac{\phi'}{2}\right)^2},$$

α , is the incident angle,
 k , is free space wave number.

Also, A is an unknown coefficient that should be determined.

However, this is a Fredholm integral equation of the first kind and applying the presented method to this equation gives the approximate solution of $I_z(z)$. Figure 1 shows the current magnitude for $\alpha = \frac{\pi}{2}$, $a = 0.001L$, and $L = \lambda, 1.5\lambda, 2\lambda$.

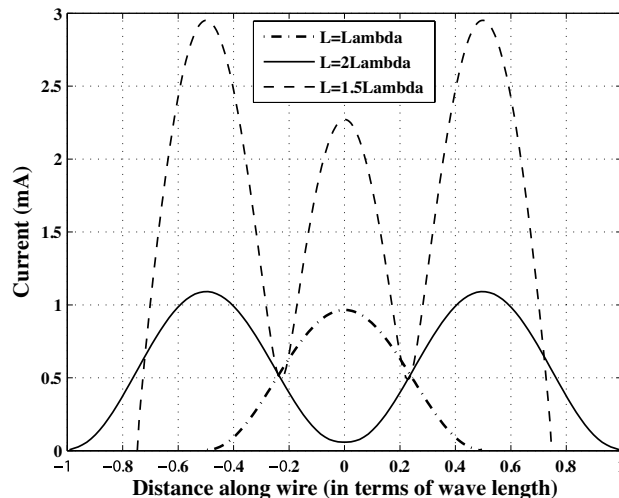


Figure 1. Current magnitude along the thin wire of length $\lambda, 1.5\lambda$ and 2λ , for $\alpha = \frac{\pi}{2}$ and $a = 0.001L$.

5. ERROR EVALUATION, COMMENT ON THE RESULTS, CONCLUSION

A new direct method based on BPFs and its operational matrix was proposed. This approach, without applying any projection method, transforms a nonlinear Volterra-Fredholm integral or integro-differential equation to a set of algebraic equations. applicability and accuracy was checked on some examples. In these examples the approximate solution was briefly compared with exact solution only at specific points. But, it should be noted that at mid-point of every subinterval $[ih, (i + 1)h]$, for $i = 0, 1, \dots, m - 1$, the approximate solution is more accurate and this accuracy will increase as m increases. On the other hand, error at some points different from mid-points may get worse as m increases. Of course these oscillations are negligible

Table 2. Numerical results for example 2.

s	Exact solution	Approximate solution, $m = 32$	Approximate solution, $m = 64$
0	0	0.000217	0.000054
0.1	0.010000	0.011936	0.010308
0.2	0.040000	0.041233	0.038140
0.3	0.090000	0.088108	0.092828
0.4	0.160000	0.152561	0.158746
0.5	0.250000	0.265842	0.257867
0.6	0.360000	0.371311	0.361871
0.7	0.490000	0.494358	0.483453
0.8	0.640000	0.634983	0.647515
0.9	0.810000	0.793186	0.807183

Table 3. Numerical results for example 3.

s	Exact solution	Approximate solution, $m = 64$	Approximate solution, $m = 128$
0.1	1.105171	1.106931	1.102592
0.2	1.221403	1.215726	1.220458
0.3	1.349859	1.356245	1.350925
0.4	1.491825	1.489553	1.495340
0.5	1.648721	1.661734	1.655194
0.6	1.822119	1.825082	1.817881
0.7	2.013753	2.004501	2.012222
0.8	2.225541	2.236265	2.227346
0.9	2.459603	2.456184	2.465482

and thus the mean-absolute error generally reduces as m increases. This can be clearly caused by definition of operational matrix P . As illustrated in Eqs. (17) and Eq. (18), the diagonal elements $\frac{h}{2}$ of operational matrix P are approximate values of $t - ih$. Note that, at mid-point of every subinterval $[ih, (i + 1)h]$ the diagonal elements of operational matrix are exactly $\frac{h}{2}$. In general, the results illustrate efficiency and accuracy of the method. For showing this matter and comparing the results obtained from current method with other methods, the mean-absolute errors of the numerical examples at mid-points and the points s in Tables 1–4 for different values of m are computed.

Table 4. Numerical results for example 4.

s	Exact solution	Approximate solution, $m = 32$	Approximate solution, $m = 64$
0.1	0.995004	0.993909	0.994818
0.2	0.980067	0.979341	0.980962
0.3	0.955336	0.956179	0.953922
0.4	0.921061	0.924624	0.921657
0.5	0.877583	0.869984	0.873809
0.6	0.825336	0.820041	0.824460
0.7	0.764842	0.762895	0.767870
0.8	0.696707	0.699047	0.693363
0.9	0.621610	0.629055	0.622866

Firstly, consider the mean-absolute error as follows:

$$E_m = \frac{1}{m} \sum_{i=1}^m |x(s_i) - x_m(s_i)|, \quad (51)$$

where, $x(s)$ is the exact solution and $x_m(s)$ is the approximate solution.

For example 1, the mean-absolute errors at mid-points from Eq. (51) are 4.9E-6 and 1.2E-6, for $m = 32$ and $m = 64$, respectively. But for ten points s in Table 1, these errors are 7.1E-3, for $m = 32$ and 3.6E-3, for $m = 64$. These errors for example 2 at mid-points are 2.7E-5, for $m = 32$ and 6.8E-6, for $m = 64$, and at ten points s in Table 2 are 6.6E-3 and 3.3E-3, for $m = 32$ and $m = 64$, respectively. [23] proposes the decomposition method to solve these problems. It seems that the direct method is more accurate and practical than the decomposition method. The current method can be run with increasing m until the computed results have appropriate accuracy.

The mean-absolute errors of example 3 at mid-points are 1.6E-4, for $m = 64$ and 3.9E-5, for $m = 128$. These errors at nine points s in Table 3 are 6.2E-3 and 3.1E-3, for $m = 64$ and $m = 128$, respectively. [29] has solved this problem by Taylor polynomial method. Comparing the direct method with the method proposed in [29] shows that the accuracy of Taylor polynomial method is slightly better. But, it seems that the number of calculations of the direct method is lower.

In example 4, the mean-absolute errors at mid-points are 8.1E-5 and 2E-5, for $m = 32$ and $m = 64$, respectively, but for nine points s in Table 4, these errors are 3.4E-3, for $m = 32$ and 1.7E-3, for $m = 64$. For this problem, the direct method achieves a high accuracy, higher than that obtained by the decomposition algorithm presented in [22].

The benefits of this method are low cost of setting up the equations without applying any projection method such as Galerkin, collocation, Also, the nonlinear system of algebraic equations is sparse.

Finally, this method can be easily extended and applied to systems of nonlinear Volterra-Fredholm integral equations and integro-differential equations of the forms Eqs. (1) and (2). Also, this method can be applied to nonlinear Volterra-Fredholm integro-differential equations of any order with suitable initial conditions.

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