Progress In Electromagnetics Research B, Vol. 6, 295–306, 2008

PROPAGATION OF WAVES IN A BIFURCATED CYLINDRICAL WAVEGUIDE WITH WALL IMPEDANCE DISCONTINUITY

A. Büyükaksoy, A. Demir, and F. Hacıvelioğlu

Department of Mathematics
Gebze Institute of Technology
P. O. Box 141, Gebze, Kocaeli 41400, Turkey

Abstract—In the present work the radiation of sound from a bifurcated circular waveguide formed by a semi-infinite rigid duct inserted axially into a larger infinite tube with discontinuous wall impedance is reconsidered through an alternative approach which consists of using the mode matching technique in conjunction with the Wiener-Hopf method. By expressing the total field in the appropriate waveguide region in terms of normal modes and using the Fourier transform technique elsewhere, we end up with a single modified Wiener-Hopf equation whose solution involves an infinite system of algebraic equations. This system is solved numerically and the influence of some parameters on the radiation phenomenon is shown graphically. The equivalence of the direct method described in [1] and the present mixed method are shown numerically.

1. INTRODUCTION

In a previous work by the authors [1], the radiation of sound from a bifurcated circular waveguide formed by a semi-infinite rigid duct inserted axially into a larger infinite tube with discontinuous wall impedance (see Fig. 1) has been analyzed by using the Fourier transform technique. The related boundary value problem is then formulated as a matrix Wiener-Hopf equation and solved rigorously through the "weak factorization" [2], or "pole removal" [3, 4] method. When the Neumann boundary condition satisfied on the inner semi-infinite cylinder is replaced by a more general one, such as the impedance type boundary condition, the resulting matrix Wiener-Hopf equation becomes very complicated. Therefore, a hybrid method of formulation consisting of expressing the field in the region $a < \rho < b$,

$z < 0$ in terms of normal waveguide modes and using the Fourier transform technique elsewhere may be adopted. The use of this hybrid method results in a single modified Wiener-Hopf equation involving infinitely many unknown expansion coefficients satisfying an infinite system of linear algebraic equations [7]. An alternative approach would be the very well known mode matching technique. The advantage of the mixed method used in this paper over the pure mode-matching technique is that the edge conditions are incorporated rigorously in the analysis through Wiener-Hopf procedure, while the mode matching requires a proper application of edge conditions by checking the convergence rate of the modal amplitudes. Indeed, by using the hybrid method we end-up with a single set of infinitely many expansion coefficients satisfying an infinite system of linear algebraic equations which can be solved by numerical methods.

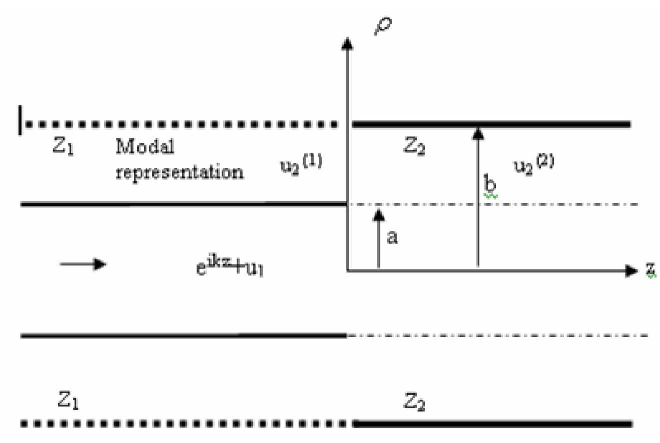

Figure 1. Geometry of the problem for mixed formulation.

2. MIXED METHOD OF FORMULATION

We now express the total field as

$$u_T(\rho, z) = \begin{cases} e^{ikz} + u_1(\rho, z), & 0 < \rho < a \\ u_2^{(1)}(\rho, z)H(-z) + u_2^{(2)}(\rho, z)H(z), & a < \rho < b \end{cases} \tag{1}$$

where $H(z)$ denotes the unit step function.

$u_2^{(1)}(\rho, z)$ which satisfies the Helmholtz equation and the following boundary conditions

$$\frac{\partial}{\partial\rho}u_2^{(1)}(a,z) = 0, \tag{2a}$$

$$\eta_1 u_2^{(1)}(b,z) - \frac{1}{ik}\frac{\partial}{\partial\rho}u_2^{(1)}(b,z) = 0 \tag{2b}$$

can be expressed in terms of normal waveguide modes as:

$$u_2^{(1)}(\rho,z)=\sum_{m=0}^{\infty}A_m[J_0(\gamma_m\rho)Y_1(\gamma_m a)-J_1(\gamma_m a)Y_0(\gamma_m\rho)]\,e^{-i\alpha_m z} \tag{3a}$$

with

$$\gamma_m = \sqrt{k^2-\alpha_m^2}, \quad \mathrm{Im}\alpha_m > \mathrm{Im}k \tag{3b}$$

and $\pm\alpha_m$ being the symmetrical zeros of $K(\alpha)M_1(\eta_1,\alpha)$. Here $M_p(\eta_j,\alpha)$ stands for

$$M_p(\eta_j,\alpha) = [J_p(Ka)\,\mathcal{Y}(\eta_j,\alpha) - Y_p(Ka)\,\mathcal{J}(\eta_j,\alpha)], \ p=0,1;\ j=1,2. \tag{4a}$$

with

$$\mathcal{J}(\eta_j,\alpha) = \eta_j J_0(Kb) + \frac{K(\alpha)}{ik}J_1(Kb) \tag{4b}$$

and

$$\mathcal{Y}(\eta_j,\alpha) = \eta_j Y_0(Kb) + \frac{K(\alpha)}{ik}Y_1(Kb) \tag{4c}$$

Here $K(\alpha)$ stands for

$$K(\alpha) = \sqrt{k^2-\alpha^2}. \tag{4d}$$

The square root function is defined in the complex α-plane cut along $\alpha = k$ to $\alpha = k + i\infty$ to $\alpha = -k - i\infty$, such that $K(0) = k$. The boundary and continuity relations are

$$\frac{\partial}{\partial\rho}u_1(a,z) = 0 \tag{5a}$$

$$\frac{\partial}{\partial\rho}u_1(a,z) - \frac{\partial}{\partial\rho}u_2^{(2)}(a,z) = 0, \qquad z>0 \tag{5b}$$

$$e^{ikz} + u_1(a,z) = u_2^{(2)}(a,z), \qquad z>0 \tag{5c}$$

$$\eta_2 u_2^{(2)}(b,z) - \frac{1}{ik}\frac{\partial}{\partial\rho}u_2^{(2)}(b,z) = 0, \qquad z>0 \tag{5d}$$

$$u_2^{(1)}(\rho,0) = u_2^{(2)}(\rho,0), \qquad a<\rho<b \tag{5e}$$

and

$$\frac{\partial}{\partial z} u_2^{(1)}(\rho, 0) = \frac{\partial}{\partial z} u_2^{(2)}(\rho, 0), \quad a < \rho < b. \tag{5f}$$

To obtain a unique solution the following edge conditions should also be taken into account

$$u_1(a, z) = \mathcal{O}(1), \quad z \to 0 \tag{5g}$$

$$\frac{\partial}{\partial \rho} u_1(a, z) = \mathcal{O}\left(|z|^{-1/2}\right), \quad z \to 0 \tag{5h}$$

Consider first the region $0 < \rho < a$. By applying full range Fourier transform to the Helmholtz equation satisfied by $u_1(\rho, z)$, we get

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + K(\alpha)\right]\left[F_-(\rho, \alpha) + F_+(\rho, \alpha)\right] = 0 \tag{6a}$$

with

$$F_\pm(\rho, \alpha) = \pm\frac{1}{2\pi}\int_0^{\pm\infty} u_1(\rho, z)\, e^{i\alpha z} dz \tag{6b}$$

The solution of (6a) is

$$F_-(\rho, \alpha) + F_+(\rho, \alpha) = A(\alpha) J_0(K\rho). \tag{7}$$

Taking the derivative of (7) and using the Fourier transform of the boundary condition in (5a), namely

$$\frac{\partial}{\partial \rho} F_-(\rho, \alpha)\Big|_{\rho=a} = \dot{F}_-(a, \alpha) = 0, \tag{8a}$$

we obtain

$$F_+(a, \alpha) = -\frac{J_0(Ka)}{K J_1(Ka)}\dot{F}_+(a, \alpha) - F_-(a, \alpha) \tag{8b}$$

and

$$A(\alpha) = -\frac{\dot{F}_+(a, \alpha)}{K(\alpha) J_1(Ka)}. \tag{9a}$$

By using the edge condition in (5h) and taking into account that $K(\alpha) \sim i|\alpha|$, $J_1(Ka) = \mathcal{O}(|\alpha|^{-1/2} e^{a|\alpha|})$ we can show that $\dot{F}_+(a, \alpha) = \mathcal{O}(|\alpha|^{-1/2})$ and

$$A(\alpha) = \mathcal{O}\left(|\alpha|^{-1} e^{-a|\alpha|}\right), \tag{9b}$$

for $|\alpha| \to \infty$.

Now, consider the region $a < \rho < b$, $z > 0$. By taking the half-range Fourier transform of the Helmholtz equation satisfied by $u_2(\rho, z)$ we get

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + K^2(\alpha)\right] G_+(\rho,\alpha) = f(\rho) - i\alpha g(\rho). \tag{10a}$$

Here $G_+(\rho,\alpha)$ is a function regular in the upper $\alpha-$ plane $\mathrm{Im}(\alpha) > \mathrm{Im}(-k)$, defined by

$$G_+(\rho,\alpha) = \frac{1}{2\pi}\int_0^\infty u_2^{(2)}(\rho,z)e^{i\alpha z}dz, \tag{10b}$$

while $f(\rho)$ and $g(\rho)$ stand for

$$f(\rho) = \frac{\partial}{\partial z}u_2^{(2)}(\rho,0), \tag{10c}$$

$$g(\rho) = u_2^{(2)}(\rho,0). \tag{10d}$$

The solution of the non homogeneous equation in (10a) can be obtained by using the Green's function technique. The result is

$$\begin{aligned} G_+(\rho,\alpha) = & -\frac{\dot{F}_+(a,\alpha)}{K(\alpha)\,M_1(\eta_2,\alpha)}\left[J_0(K\rho)\,\mathcal{Y}(\eta_2,\alpha) - Y_0(K\rho)\,\mathcal{J}(\eta_2,\alpha)\right] \\ & + \frac{1}{K(\alpha)\,M_1(\eta_2,\alpha)}\int_a^b \{f(t) - i\alpha g(t)\}\,Q(t,\rho,\alpha)t\,dt \end{aligned} \tag{11a}$$

with

$$Q(\rho,t,\alpha) = \frac{\pi}{2}\begin{cases} \begin{array}{l}[J_0(Kt)\,\mathcal{Y}(\eta_2,\alpha) - Y_0(Kt)\,\mathcal{J}(\eta_2,\alpha)] \\ \quad\times[J_0(K\rho)Y_1(Ka) - J_1(Ka)Y_0(K\rho)]\end{array}, & a \le \rho \le t \\ \begin{array}{l}[J_0(K\rho)\,\mathcal{Y}(\eta_2,\alpha) - Y_0(K\rho)\,\mathcal{J}(\eta_2,\alpha)] \\ \quad\times[J_0(Kt)Y_1(Ka) - J_1(Ka)Y_0(Kt)]\end{array}, & t \le \rho \le b \end{cases} \tag{11b}$$

The regularity of the right hand-side of (11a) is ensured if

$$\dot{F}_+(a,\beta_n) = \frac{K_n}{\pi}\frac{J_1(K_n a)}{\mathcal{J}(\eta_2,\beta_n)}\theta_n\left[f_n - i\beta_n g_n\right], \tag{12a}$$

with

$$\begin{bmatrix} f_n \\ g_n \end{bmatrix} = \frac{\pi^2}{2\theta_n} \int_a^b \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \left[J_0(K_n t)\, \mathcal{Y}(\eta_2, \beta_n) - Y_0(K_n t)\, \mathcal{J}(\eta_2, \beta_n) \right] t dt, \tag{12b}$$

$$K_n = K(\beta_n) \tag{12c}$$

and

$$\theta_n = -\frac{1}{k^2} + \left(\frac{\eta_2}{K_n} \right)^2 - \left(\frac{\mathcal{J}(\eta_2, \beta_n)}{K_n J_1(K_n a)} \right)^2 \tag{12d}$$

In (12a)–(12d), β_n denotes the zeros of $K(\alpha) M_1(\eta_2, \alpha)$:

$$K(\pm\beta_n) M_1(\eta_2, \pm\beta_n) = 0, \qquad \operatorname{Im}\beta_n > \operatorname{Im} k.$$

By using the Fourier transform of the continuity relation in (5c)

$$F_+(a, \alpha) = G_+(a, \alpha) + \frac{1}{2\pi i} \frac{1}{\alpha + k} \tag{13}$$

and the following eigenfunction expansion

$$\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} f_m \\ g_m \end{bmatrix} \left[J_0(K_m t)\, \mathcal{Y}(\eta_2, \beta_m) - Y_0(K_m t)\, \mathcal{J}(\eta_2, \beta_m) \right] \tag{14}$$

we end up with the following modified Wiener-Hopf equation

$$\begin{aligned} &\frac{2}{\pi a} \frac{L(\alpha)}{k^2 - \alpha^2} \dot{F}_+(a, \alpha) + F_-(a, \alpha) = \\ &\frac{2}{\pi a} \sum_{m=1}^{\infty} \frac{\mathcal{J}(\eta_2, \beta_m)}{K_m J_1(K_m a)} \frac{f_m - i\alpha g_m}{\alpha^2 - \beta_m^2} - \frac{1}{2\pi i} \frac{1}{\alpha + k}, \end{aligned} \tag{15}$$

whose solution reads

$$\begin{aligned} &\dot{F}_+(a, \alpha) = -\frac{ak}{2i} \frac{1}{L_+(\alpha) L_+(k)} + \\ &\frac{1}{2k} \frac{k + \alpha}{L_+(\alpha)} \sum_{m=1}^{\infty} \frac{(k + \beta_m)\, \mathcal{J}(\eta_2, \beta_m)}{K_m L_+(\beta_m)\, J_1(K_m a)} \frac{f_m + i\beta_m g_m}{\beta_m(\alpha + \beta_m)}. \end{aligned} \tag{16}$$

Here $L^+(\alpha)$ is a function regular and free of zeros in the upper half plane $\operatorname{Im}\alpha > -\operatorname{Im}k$ resulting from the Wiener-Hopf factorization of

$$L(\alpha) = \frac{\mathcal{J}(\eta_2, \alpha)}{J_1(Ka)\, M_1(\eta_2, \alpha)}. \tag{17a}$$

as

$$L(\alpha) = L^+(\alpha)\, L^-(\alpha) \tag{17b}$$

with

$$L^-(\alpha) = L^+(-\alpha) \tag{17c}$$

The explicit expression of $L^+(\alpha)$ is [1]:

$$\begin{aligned} L^+(\alpha) \;=\; & \left\{ \frac{\mathcal{J}(\eta_2, 0)}{J_1(ka)\left[J_1(ka)\,\mathcal{Y}(\eta_2, 0) - Y_1(ka)\,\mathcal{J}(\eta_2, 0)\right]} \right\}^{1/2} e^{-\alpha T} \\ & \times \prod_{m=1}^{\infty} \frac{(1+\alpha/\xi_m)}{(1+\alpha/\chi_m)(1+\alpha/\beta_m)} \end{aligned} \tag{18a}$$

where T stands for

$$T = \frac{i}{\pi}\left[b \ln b - a \ln a - (b-a)\ln(b-a)\right] \tag{18b}$$

and ξ_m and χ_m are the roots of the following equations:

$$\sqrt{k^2 - (\chi_m)^2}\, J_1\left(a\sqrt{k^2 - (\chi_m)^2}\right) \;=\; 0 \tag{18c}$$

$$ik\eta_2 J_0\left(b\sqrt{k^2-(\xi_m)^2}\right) + \sqrt{k^2-(\xi_m)^2}\, J_1\left(b\sqrt{k^2-(\xi_m)^2}\right) \;=\; 0. \tag{18d}$$

The unknown coefficients A_m and f_m, g_m appearing in (3a) and (12a), respectively are to be determined trough the regularity condition (12a) and the following matching conditions derived from (5e), (5f):

$$\begin{aligned} & -i \sum_{m=1}^{\infty} A_m \alpha_m \left[J_0(\gamma_m \rho)\, Y_1(\gamma_m a) - Y_0(\gamma_m \rho)\, J_1(\gamma_m a)\right] = \\ & \sum_{m=1}^{\infty} f_m \left[J_0(K_m \rho)\, \mathcal{Y}(\eta_2, \beta_m) - Y_0(K_m \rho)\, \mathcal{J}(\eta_2, \beta_m)\right] \end{aligned} \tag{19a}$$

$$\begin{aligned} & \sum_{m=1}^{\infty} A_m \left[J_0(\gamma_m \rho)\, Y_1(\gamma_m a) - Y_0(\gamma_m \rho)\, J_1(\gamma_m a)\right] = \\ & \sum_{m=1}^{\infty} g_m \left[J_0(K_m \rho)\, \mathcal{Y}(\eta_2, \beta_m) - Y_0(K_m \rho)\, \mathcal{J}(\eta_2, \beta_m)\right] \end{aligned} \tag{19b}$$

Multiplying both sides of (19a,b) by

$$\rho\left[J_0\left(K_m\rho\right)\mathcal{Y}\left(\eta_2,\beta_m\right)-Y_0\left(K_m\rho\right)\mathcal{J}\left(\eta_2,\beta_m\right)\right] \tag{20}$$

and integrating from a to b, we get

$$f_m-i\alpha g_m=-\frac{\pi^2 i}{2\theta_m}\sum_{j=1}^{\infty}\left(\alpha+\alpha_j\right)\Delta_{jm}A_j \tag{21}$$

where Δ_{jm} stands for:

$$\begin{aligned}\Delta_{jm} \;=\; & \frac{2}{\pi}\frac{1}{\alpha_j^2-\beta_m^2}\left\{\eta_2\left[Y_1(\gamma_j a)J_0\left(\gamma_j b\right)-J_1(\gamma_j a)Y_0\left(\gamma_j b\right)\right]+\right.\\ & \left.\frac{\gamma_j}{ik}\left[Y_1(\gamma_j a)J_1\left(\gamma_j b\right)-J_1(\gamma_j a)Y_1\left(\gamma_j b\right)\right]\right\}\end{aligned} \tag{22}$$

Substituting (21) in (12a), and using (16) we get a single infinite system of linear algebraic equations for the modal expansion coefficients A_j which are solved numerically.

$$\begin{aligned}&\frac{K_r J_1\left(K_r a\right)}{\mathcal{J}\left(\eta_2,\beta_r\right)}\sum_{j=1}^{\infty}\left(\beta_r+\alpha_j\right)\Delta_{jr}A_j+\\ &\frac{\pi}{2k}\frac{k+\beta_r}{L_+\left(\beta_r\right)}\sum_{m=1}^{\infty}\frac{\left(k+\beta_m\right)\mathcal{J}\left(\eta_2,\beta_m\right)}{K_m L_+\left(\beta_m\right)J_1(K_m a)}\frac{1}{\beta_m\left(\beta_r+\beta_m\right)\theta_m}\\ &\sum_{j=1}^{\infty}\left(\beta_m-\alpha_j\right)\Delta_{jm}A_j=-\frac{ak}{\pi}\frac{1}{L_+\left(\beta_r\right)L_+\left(k\right)}.\end{aligned} \tag{23}$$

To solve this system we assume that the convergence of the infinite series involved is rapid enough to allow truncation at $n=N$. The value of N is increased until the transmission coefficient does not change in a given number of decimal place. It was seen that the convergence is very fast and the truncation number may be chosen as $N=3$ [1].

The radiated field is obtained by evaluating the following integral

$$u_1(\rho,z)=-\int_{\mathcal{L}}\dot{F}_+\left(a,\alpha\right)\frac{J_0(K\rho)}{K(\alpha)J_1(Ka)}e^{-i\alpha z}d\alpha \tag{24}$$

The transmission coefficient $\mathcal{T}$ of the fundamental mode is defined as the complex coefficient multiplying the travelling wave term

$\exp(i\xi_1 z)$ and is computed from the contribution of the first pole at $\alpha = -\xi_1$. The result is

$$\mathcal{T} = \pi\left[\frac{i}{k}(k-\xi_1)\sum_{m=1}^{\infty}\frac{(k+\beta_m)\,\mathcal{J}(\eta_2,\beta_m)}{K_m L_+(\beta_m)\,J_1(K_m a)}\frac{f_m+i\beta_m g_m}{\beta_m(-\xi_1+\beta_m)}-\frac{ka}{L_+(k)}\right]\times\frac{L_+(\xi_1)\,M_1(\eta_2,-\xi_1)}{K(\xi_1)\mathcal{J}'(\eta_2,-\xi_1)} \tag{25}$$

where the dash ($'$) denotes the derivative with respect to α.

Similarly, the reflection coefficient $\mathcal{R}$ of the fundamental mode which is defined as to be the complex coefficient of $\exp(-ikz)$ is computed as contribution from the pole at $\alpha = k$, which is

$$\mathcal{R} = \pi\left[\frac{2i}{kaL_+(k)}\sum_{m=1}^{\infty}\frac{\mathcal{J}(\eta_2,\beta_m)\,[f_m+i\beta_m g_m]}{\beta_m K_m L_+(\beta_m)\,J_1(K_m a)}-\left(\frac{1}{L_+(k)}\right)^2\right] \tag{26}$$

3. COMPUTATIONAL RESULTS

In this section some graphical results showing the effects of the geometrical and physical parameters on reflection coefficient given in (26) are presented.

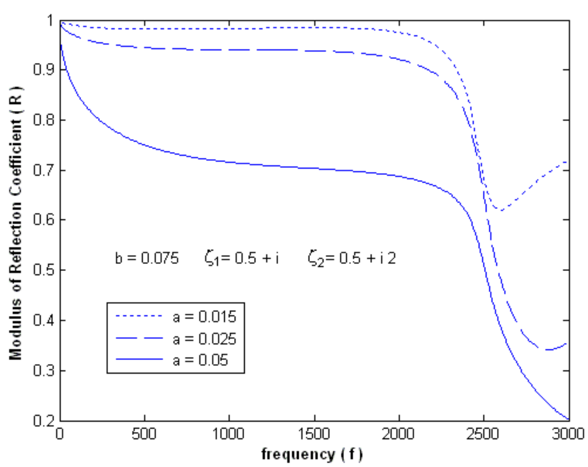

Figure 2. Modulus of the reflection coefficient $|\mathcal{R}|$ versus frequency f for different values of inner duct radius a.

Figure 2 shows the variation of the reflection coefficient versus the frequency for different values of the inner cylinder radius. It is seen that the amplitude of the reflection coefficient decreases with increasing values of the inner cylinder radius.

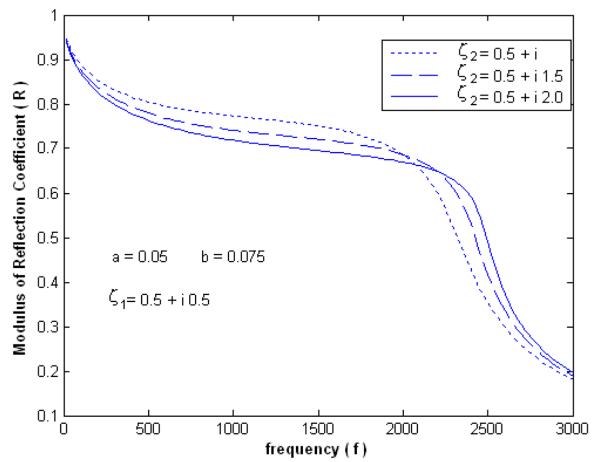

Figure 3. Modulus of the Reflection Coefficient $|\mathcal{R}|$ versus frequency f for different values of ζ_2.

Figure 3 shows the variation of the reflection coefficient versus the frequency for different values of the surface impedance $\zeta_2 = 1/\eta_2$ of the part $z > 0$ while the impedance $\zeta_1 = 1/\eta_1$ of the part $z < 0$ is kept constant. It is observed that the amplitude of the reflected field decreases up to a certain frequency range, as the contrast $|\zeta_2 - \zeta_1|$ increases. Beyond this frequency range a reversed behavior is observed.

4. CONCLUSION

The direct method of formulation described in [1] is simpler, but requires the Wiener-Hopf factorization of a kernel matrix. In some special cases, as in the present problem, the factorization can be accomplished rather easily

The mixed method of formulation described in this work requires more complicated calculations but results into a single modified Wiener-Hopf equation. The advantage of the hybrid method is that it can be applied to more general cases.

For the present geometry, the direct method is effective only when the inner semi-infinite duct is completely rigid or completely soft. But it becomes very complicated when the wall of the duct is characterized by a more general boundary condition, such as the impedance boundary condition.

The table below shows the amplitude of the transmission coefficient $|\mathcal{T}|$ for different values of the radius of the outer cylinder, i.e., kb which is calculated by using the direct and the mixed methods of formulation.

$ka = 1, \quad \zeta_1 = 0.5 + i, \quad \zeta_2 = 0.5 + 2i$

kb	direct method	hybrid method
1.1	0.5133849698328619	0.5133845112922173
1.2	0.4209788115694715	0.4209775205339662
1.3	0.35291199842048	0.3529098482749924
1.4	0.2997297433023099	0.2997268632262918
1.5	0.2570210560706732	0.2570176495735579
1.6	0.222129699067328	0.2221260006399043
1.7	0.1932691928701588	0.1932654441155111
1.8	0.1691646981412051	0.1691611349790878
1.9	0.1488724971776491	0.1488693436665429
2	0.1316761672364317	0.1316736334944602
2.1	0.1170216347123282	0.1170199174828079
2.2	0.1044742038425272	0.1044734888820861
2.3	0.09368898884613652	0.09368945471940107
2.4	0.08438999241100219	0.0843918155695917
2.5	0.07635501295749399	0.07635837423485263

It is seen that the solutions obtained by these two different methods coincide exactly.

REFERENCES

1. Büyükaksoy, A. and A. Demir, "Radiation of sound from a semi-infinite rigid duct inserted axially into a larger infinite tube with wall impedance discontinuity," *ZAMM*, Vol. 86, No. 7, 563–571, 2006.
2. Idemen, M., "A new method to obtain exact solutions of vector Wiener-Hopf equations," *ZAMM*, Vol. 59, 656–658, 1979.
3. Abrahams, I. D., "Scattering of sound by two parallel semi-infinite screens," *Wave Motion*, Vol. 9, 289–300, 1987.
4. Abrahams, I. D. and G. R. Wickham, "The scattering of water waves by two semi-infinite vertical walls," *Wave Motion*, Vol. 14, 145–168, 1991.

5. Rawlins, A. D., "A bifurcated circular waveguide problem," *IMA J. Appl. Math.*, Vol. 54, 59–81, 1995.
6. Nilsson, B. and O. Brander, "The propagation of sound in cylindrical ducts with mean flow and bulk reacting lining. I. Modes in an infinite duct," *IMA J. Appl. Math.*, Vol. 26, 269–298, 1980.
7. Demir, A. and A. Büyükaksoy, "Radiation of plane sound waves by a rigid circular cylindrical pipe with a partial internal impedance loading," *Acustica*, Vol. 89, 578–585, 2003.