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PLANE WAVE DIFFRACTION BY A STRONGLY ELONGATED OBJECT ILLUMINATED IN THE PARAXIAL DIRECTION

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Abstract—After a short presentation of the boundary layer method extended to strongly elongated objects by Andronov and Bouche [1], the author develops some techniques for deriving explicit formulas for the asymptotic currents on a strongly elongated object of revolution excited by an electromagnetic plane wave propagating in the paraxial direction. The performance of the different techniques are demonstrated by comparing numerical results obtained for the asymptotic currents on an elongated prolate ellipsoid with those obtained by solving the EFIE.

1. INTRODUCTION

In the classical formulation of creeping waves, the radius of transverse curvature to the geodesics is supposed large or at least of the same order than the radius of curvature of the geodesics. As a consequence, one finds the transverse radius of curvature only in smaller order corrections. Hong [2] and Voltmer [3] have obtained correction terms to the attenuation constants and diffraction coefficients of order $(2/ka)^{2/3}$. The same results have been obtained with the boundary layer method for $\rho_t \gg k^{-1/3}$ by Andronov and Bouche [4]. In the case of a moderately elongated body: $k^{-2/3} \ll \rho_t \ll k^{-1/3}$ the transverse curvature appears in the second order of the asymptotic expansion and modifies the amplitude factor in the asymptotics of creeping waves. We consider here the case of a strongly elongated body: $\rho_t = O(k^{-2/3})$. By extending the boundary layer method to this case, Andronov and Bouche [1] have shown that the transverse curvature appears in the principal order and modifies the differential equation verified by the

Fourier transform of the amplitude of the creeping waves which is no longer an Airy equation but a biconfluent Heun equation. Whereas the theory of creeping waves on strongly elongated bodies is now well established, no attempt has been made so far to control the accuracy of the formulas obtained which, as shown by Andronov [5], predict an enhancement of the magnetic creeping wave. Comparison of the asymptotic currents on the surface with the results obtained by a strictly numerical solution encounters several problems. One of the difficulties is that no analytical solution of the Heun equation expressed in terms of tabulated functions is available. Another difficulty is the derivation of a closed form expression of the incident field in the semi-geodesic co-ordinate system used in the boundary layer theory [6]. We will present some techniques permitting to overcome both difficulties and test their accuracy on an elongated prolate ellipsoid, by comparing the asymptotic currents to the currents obtained by solving the Electric Field Integral Equation (EFIE). The remainder of the paper is organized as follows. In Section 2 we recall the main steps of the boundary layer theory extended to strongly elongated bodies. In section 3, we present a first approach for deriving explicit formulas for the asymptotic currents and in section 4 an alternative approach is given. In both Sections 3 and 4 numerical results are shown and compared with a reference solution. Finally, some concluding remarks are made in Section 5.

2. CREEPING WAVES ON STRONGLY ELONGATED BODIES

The behavior of acoustic or electromagnetic creeping waves propagating on the surface of a strongly elongated convex body has been first studied by Andronov and Bouche [1] using the boundary layer method. We recall here the main steps of their approach applied to the electromagnetic creeping waves. We consider a family of geodesics followed by a creeping wave on a convex object and the curves orthogonal to these geodesics. In order to locate a point M in space, we define a system of semi-geodesic co-ordinates (s, a, n) where s denotes the curvilinear abscissa along any geodesic of the family, a denotes the curvilinear abscissa along a reference curve orthogonal to the geodesics and n denotes the distance of M to the surface along the normal.

The quadratic tensor g_{ij} of these co-ordinates defined by:

$$\left|d\overrightarrow{OM}\right|^2 = g_{ij}dx^i dx^j, \quad x^i = (s, a, n) \tag{1}$$

will be needed for writing Maxwell's equations in these co-ordinates. The derivation of the components of this tensor by differential geometry is straightforward and is given in [6]. The result is:

$$g_{ij} = \begin{pmatrix} \left(1+\dfrac{n}{\rho}\right)^2 + \tau^2 n^2 & -h\tau\left[2n + n^2\left(\dfrac{1}{\rho}+\dfrac{1}{\rho_t}\right)\right] & 0 \\ -h\tau\left[2n + n^2\left(\dfrac{1}{\rho}+\dfrac{1}{\rho_t}\right)\right] & h^2\left[\left(1+\dfrac{n}{\rho_t}\right)^2 + \tau^2 n^2\right] & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where ρ is the radius of curvature of the geodesic passing trough the point P corresponding to the normal projection of M on the surface, ρ_t the radius of curvature of the curve orthogonal to the geodesic passing though P which is also the radius of curvature of the wave front at P, h is the divergence of the geodesics along s and τ is the torsion of the geodesic.

With the notations:

$$\sqrt{\varepsilon}\,\vec{e} = \vec{E}, \qquad \sqrt{\mu}\,\vec{h} = \vec{H}$$

Maxwell's equations in vacuum, with the $e^{-i\omega t}$ time convention, read:

$$rot\vec{E} = ik\vec{H}, \qquad rot\vec{H} = -ik\vec{E} \tag{3}$$

In the co-ordinate system (s, a, n) the first equation (3) becomes:

$$\begin{aligned} ikH^s &= \frac{1}{\sqrt{g}}(\partial_a E_n - \partial_n E_a) \\ ikH^a &= \frac{1}{\sqrt{g}}(\partial_n E_s - \partial_s E_n) \\ ikH^n &= \frac{1}{\sqrt{g}}(\partial_s E_a - \partial_a E_s) \end{aligned} \tag{4}$$

where g is the determinant of the quadratic tensor (2). The second equation (3) gives three equations which may be deduced from (4) by the substitution $\vec{E} \to \vec{H},\ \vec{H} \to -\vec{E}$.

The covariant components in (4) are related to the contravariant components by:

$$V_i = g_{ij}V^j \quad i, j = s, a, n \tag{5}$$

In the boundary layer n is small, of order $k^{-2/3}$. However since $\rho_t = O(k^{-2/3})$ for a strongly elongated object, we must retain the terms $\frac{n^2}{\rho_t}$ and $\frac{n^2}{\rho_t^2}$ which are respectively of order $k^{-2/3}$ and 1. Then, according to (4), Maxwell's equations (3) expressed with only contravariant components take the form:

$$\begin{aligned}
ikH^s &= \frac{1}{\sqrt{g}}\left\{\partial_a E^n - \partial_n\left[-\tau hn\left(2+\frac{n}{\rho_t}\right)E^s + h^2\left(1+\frac{n}{\rho_t}\right)^2 E^a\right]\right\}\\
ikH^a &= \frac{1}{\sqrt{g}}\left\{\partial_n\left[\left(1+\frac{2n}{\rho}\right)E^s - \tau hn\left(2+\frac{n}{\rho_t}\right)^2 E^a\right] - \partial_s E^n\right\}\\
ikH^n &= \frac{1}{\sqrt{g}}\left\{\partial_s\left[-\tau hn\left(2+\frac{n}{\rho_t}\right)E^s + h^2\left(1+\frac{n}{\rho_t}\right)^2 E^a\right]\right.\\
&\quad\left. - \partial_a\left[\left(1+\frac{2n}{\rho}\right)E^s - \tau hn\left(2+\frac{n}{\rho_t}\right)^2 E^a\right]\right\} \qquad (6)
\end{aligned}$$

where g has also been approximated by neglecting the terms of order smaller than $k^{-2/3}$.

$$g \cong \left(\frac{1+2n}{\rho}\right)\left(\frac{\rho_t+n}{\rho_t}\right)^2$$

To these equations we have to add three other equations obtained by the substitution $\vec{E} \to \vec{H},\ \vec{H} \to -\vec{E}$.

In order to solve these equations, the general procedure consists in dividing the shadowed part of the surface boundary layer of the convex body into two regions: the penumbra region in the vicinity of the surface shadow boundary and the deep shadow region. In each region, the solution is stated in the form of an asymptotic expansion with unknown coefficients but with a term describing explicitly the dominant behaviour of the phase. This general analytic form of the solution is usually called an "Ansatz". The coefficients of the asymptotic expansions are determined recursively by substituting the stated form of the solution into the Maxwell equations and boundary conditions and equating terms of similar order in the large parameter k.

In the penumbra region, we start with the following Ansatz:

$$\vec{E} = \exp(iks)\sum_{j=0}^{N}\vec{E}_j(s,a,n)k^{-j/3}, \quad \vec{H} = \exp(iks)\sum_{j=0}^{N}\vec{H}_j(s,a,n)k^{-j/3} \qquad (7)$$

However since the thickness of the boundary layer and the radius of the transverse curvature on strongly elongated bodies are of the same order, the asymptotics loses its locality with respect to the a coordinate. Consequently, in order to construct the asymptotic expansion one needs to know precisely the shape of the cross-section of the body.

In the case of a plane wave incident along the axis of a body of revolution, the incident field behaves like $\cos\varphi$ where $\varphi = a[\rho_t(s = 0)]^{-1}$ and the diffracted field has the same behaviour. It is therefore possible to incorporate explicitly the φ behaviour in the phase term of the Ansatz (7) by writing:

$$\vec{E} = \vec{E}_+ - \vec{E}_-, \quad \vec{H} = \vec{H}_+ + \vec{H}_-$$

where:

$$\begin{aligned} \vec{E}_\pm &= \exp(iks \pm i\varphi) \sum_{j=0}^{N} \vec{E}_j(s, a, n) k^{-j/3} \\ \vec{H}_\pm &= \exp(iks \pm i\varphi) \sum_{j=0}^{N} \vec{H}_j(s, a, n) k^{-j/3} \end{aligned} \tag{8}$$

For an arbitrary convex cross-section, a similar Ansatz can be stated but φ is then a more complicated function of the variable a.

Subsequently, we assume that we deal with a body of revolution. Then by substituting (8) into (6) and in the homologous equations derived from the second Maxwell equation (3) and equate the terms of order $k, k^{2/3}$ and $k^{1/3}$ we obtain a hierarchy of twelve equations relating the components of the first three coefficients $\vec{E}_j, \vec{H}_j (j = 0, 1, 2)$ of the asymptotic expansion (8). A part of these equations can be used to express the components E^n, E^s and H^n, H^s via E^a, H^a. The other equations form a degenerated system whose compatibility conditions give a system of differential equations for the components E^a and H^a. The details concerning the derivation of the expressions for the two principal order terms are given in [1] and [7]. By introducing the reduced variables:

$$\sigma = \frac{ms}{\rho}, \quad v = \frac{kn}{m}, \quad \kappa = \frac{k\rho_t}{m}, \quad m = \left(\frac{k\rho}{2}\right)^{1/3} \tag{9}$$

and Fourier transforming with respect to σ the system of differential equations verified by E^a and H^a, we obtain:

$$\frac{\partial^2 U_o}{\partial v^2} + \frac{3}{v+\kappa}\frac{\partial}{\partial v} U_o + \left(v - \xi - \left(\frac{1}{v+\kappa}\right)^2\right) U_o = 0 \tag{10}$$

where U_o is the Fourier transform of either $E_o^a = (E_o^a)^+ - (E_o^a)^-$ or $H_o^a = (H_o^a)^+ + (H_o^a)^-$ and ξ the spectral variable.

In the derivation of (10) we have neglected the variation of $\rho(s)$ and $\rho_t(s)$ with respect to s, in the penumbra region close to the shadow boundary. The differential equation:

$$\frac{d^2V}{dv^2} + \frac{3}{v+\kappa}\frac{dV}{dv} + \left(v - \xi - \left(\frac{1}{v+\kappa}\right)^2\right)V = 0. \tag{11}$$

is a biconfluent Heun equation [8].

An exact solution of that equation is not available. However, some particular solutions can be obtained when tends to zero. One class of particular solutions satisfying (11) is given by the solutions of the equation:

$$\frac{d^2V}{dv^2} + \frac{3}{\kappa}\frac{dV}{dv} + \left(v - \xi - \frac{1}{\kappa^2}\right)V = 0 \tag{12}$$

which transforms in an Airy equation by the transformation $W = \exp\left(\frac{3v}{\kappa}\right)V$ and gives:

$$V = e^{-\frac{3v}{2\kappa}} W\left(\xi + \frac{13}{4\kappa^2} - v\right) \tag{13}$$

where $W(x)$ is the general solution of the following Airy equation:

$$\frac{d^2W}{dx^2} - xW = 0 \tag{14}$$

By applying the transformation

$$Y = (v+\kappa)^{3/2}V \tag{15}$$

equation (11) reduces to:

$$\frac{d^2Y}{dv^2} + \left(v - \xi - \frac{7}{4(v+\kappa)^2}\right)Y = 0 \tag{16}$$

From this equation we can derive another class of particular solutions of (11) for $v \to 0$ by neglecting the term v in the denominator of (16) which transforms this equation in an Airy equation the general solution of which is $Y = W\left(\xi - v + \frac{7}{4\kappa^2}\right)$.

Moreover, since for $v \to 0$ we have $(v+\kappa)^{-2} \cong \kappa^{-2}\left(1 - \frac{2v}{\kappa}\right)$ and by keeping the term $v\kappa^{-1}$, we obtain again an Airy equation defining a third class of particular solutions of (11) for $v \to 0$.

All these particular solutions satisfy equation (11) only at the limit $v \to 0$, but they do not satisfy it in a given neighbourhood of $v = 0$.

Now, we are not only interested in a solution valid for $v \to 0$ giving the field at an observation point located on the surface of the elongated body, but also in the correct value of the derivative of the field at that point in order to substitute it in the boundary conditions. Moreover we observe that for $\xi = -\frac{7}{4\kappa^2}$, the differential equation (16) has a turning point or transition point at $v = 0$. This property has to be taken into account in the research of an approximate solution in the neighbourhood of this point.

Before going on in our investigation of solutions of the Heun equation (11), we come back to the physical problem and define the conditions which must be verified by the diffracted field.

3. MAGNETIC FIELD ON THE SURFACE — ASYMPTOTIC CURRENT

Equation (10) is verified in vacuum by any solution of Maxwell's equations expressed in the co-ordinates (s, a, n) satisfying the special conditions verified on a strongly elongated object. It is important to note that the formulation presented until now is only a mathematical representation, the object being absent. Equation (10) is therefore verified by the incident field, the diffracted field and the total field.

When $v \to \infty$, the transformed equation (16) reduces to the Airy equation:

$$\frac{d^2Y}{dv^2} + (v - \xi)Y = 0$$

which has two independent solutions given by the Miller type Airy functions W_1 and W_2 having the properties [9, 10]:

$$\begin{aligned} W_1(\xi - v) &\to 0 \quad \text{when} \quad v \to \infty \\ W_2(\xi - v) &\to \infty \quad \text{when} \quad v \to \infty \end{aligned}$$

Since the diffracted field must satisfy the radiation condition when $v \to \infty$, the solution of (16) must behave like $W_1(\xi - v)$. Hence, if we denote by $Y_\xi^{(1)}(v, \kappa)$ the solution of (16) satisfying:

$$\lim_{v \to \infty} Y_\xi^{(1)}(v, \kappa) = W_1(v - \kappa) \tag{17}$$

the Fourier transform of the diffracted field is given by:

$$U_o^d(\xi, v, \kappa) = A(\xi, \kappa)(v + \kappa)^{-3/2} Y_\xi^{(1)}(v, \kappa) \tag{18}$$

where U_o^d is the Fourier transform of either $(E_o^a)^d$ or $(H_o^a)^d$.

According to (16), we have:

$$\lim_{\kappa\to\infty} Y_\xi^{(1)}(v,\kappa) = W_1(v-\xi) \tag{19}$$

and consequently we must have:

$$\lim_{\kappa\to\infty} \kappa^{-3/2} A(\xi,\kappa) = A(\xi) \tag{20}$$

in order to recover the classical solution corresponding to a non elongated object.

The solution $Y_\xi^{(1)}(v,\kappa)$ can be calculated by solving (16) numerically using the method of Runge-Kutta. Starting with the function $W_1(\xi - v)$ and its first derivative with respect to v for v large, one obtains by this method the values on the surface ($v = 0$) of:

$$Y_\xi^{(1)}(o,\kappa) \quad \text{and} \quad \left[\frac{dY_\xi^{(1)}}{dv}(v,\kappa)\right]_{v=o} \tag{21}$$

All particular solutions mentioned so far must be Airy functions of the type W_1 in order to satisfy the necessary conditions (17) and (19). Let $W_1(\xi,v,\kappa)$ be a particular solution satisfying (16) for $v \to 0$. In order that this solution satisfy also the radiation condition, we should have:

$$\frac{W_1(\xi,o,\kappa)}{\left[\dfrac{d}{dv}W_1(\xi,v,\kappa)\right]_{v=o}} = \frac{Y_\xi^{(1)}(o,\kappa)}{\left[\dfrac{d}{dv}Y_\xi^{(1)}(o,\kappa)\right]_{v=o}} \tag{22}$$

where the right hand side of (22) is determined numerically.

According to (17) and (19) this condition is satisfied for all values of ξ when κ is large compared to 1 or for all values of κ when ξ is large compared to 1. However, (22) is not satisfied for ξ close to the value $-7(4\kappa^2)^{-1}$ for which the Heun equation (11) has a turning point at $v = 0$. The deviation from the correct value of the first quotient in (22) depends on κ. It augments when κ decreases and may be large for $\kappa = 1$. An example is shown on Fig. 1. The curves show the variation with respect to ξ for different values of κ of the quotient of the second member of (22) over the first member of this equation computed with the particular solution (13). We see that for κ equal to 1.6 the modulus of this quotient differs from unity in a limited interval extending approximately from $\xi = -5$ to $\xi = 5$ with a maximum deviation at the turning point. When the values of are augmented, the deviations from unity appear in the same interval but with an amplitude which declines rapidly.

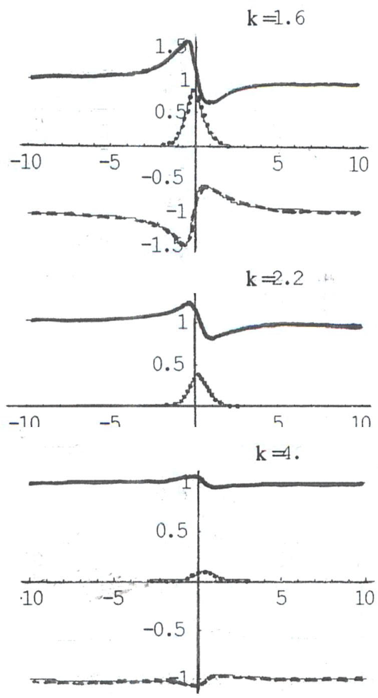

Figure 1. Variation of $R(\xi,\kappa)$ with ξ for different values of κ Real ----- Im ...... Modulus ——.

$$R(\xi,\kappa)=\frac{Y_\xi^{(1)}(o,\kappa)}{\left[\frac{d}{dv}Y_\xi^{(1)}(v,\kappa)\right]_{v=0}}\left\{\frac{W_1\left(\xi+\frac{13}{4\kappa^2}\right)}{W_1'\left(\xi+\frac{13}{4\kappa^2}\right)}\right\}^{-1}$$

Coming back to equation (18) we see that in order to determine completely the diffracted field we have to determine $A(\xi,\kappa)$. This coefficient is obtained by applying the boundary conditions on the surface which involve the incident field and its derivative with respect to v, on the surface.

Using the method described in [11] consisting in expanding the incident field in a Luneburg-Kline series and expressing the eikonal equation in the (s,a,n) co-ordinates, Andronov [12] obtained the following equation where S is the eikonal:

$$\left(1-\frac{2n}{\rho}\right)\left(\frac{\partial S}{\partial s}\right)^2+\left(\frac{\partial S}{\partial n}\right)^2+\frac{3}{ik\rho_t}\frac{1}{1+\dfrac{n}{\rho_t}}\frac{\partial S}{\partial n}=1 \tag{23}$$

The solution of (23) can be searched in the form of the series:

$$S=s+a_{20}s^2+a_{11}sn+a_{o2}n^2+a_{30}s^3+a_{21}s^2n+\cdots$$

the coefficients of which are determined by substituting this series in (23) and equating the terms of similar power of s and n. Following this procedure we find that the asymptotics of the eikonal $S(s,a,n)$ takes the form:

$$S(s,a,n)=s+\frac{3is^2}{4k\rho\rho_t}+\frac{sn}{\rho}-\frac{s^3}{6\rho^2}\left(1-\frac{9}{4}\frac{\rho}{\rho_t}\frac{1}{k^2\rho_t^2}\right)-\frac{3ins^2}{4k\rho\rho_t}+0(n^2) \tag{24}$$

or, in reduced variables:

$$S(s,a,n)=s+\frac{1}{\kappa}\left[\sigma v-\frac{\sigma^3}{3}\left(1-\frac{9}{2\kappa^3}\right)+\frac{3i\sigma^2}{2\kappa^2}\left(1-\frac{v}{\kappa}\right)\right] \tag{25}$$

If we neglect the terms of order κ^{-3}, this procedure leads to an explicit expression for the Fourier transform of the incident field:

$$U_o^i(\xi,v,\kappa)=\frac{1}{\sqrt{\pi}}\exp\left(\frac{3\xi}{2\kappa}+\frac{9}{4\kappa^3}\right)e^{-\frac{3}{2}\frac{v}{\kappa}}V\left(\xi+\frac{9}{4\kappa^2}-v\right) \tag{26}$$

where V is the Miller-type Airy function defined by:

$$V=\frac{W_1-W_2}{2i}$$

It is important to observe that (26) does not verify the Heun equation (10) for $v\to 0$. Moreover, in order to apply the boundary conditions, an explicit expression for the derivative of the incident magnetic field is also needed. Indeed, if we substitute (8) into the boundary conditions for a perfectly conducting surface $E^s=0,\ E^a=0$, we obtain for the dominant term of the components on the binormal:

$$\begin{aligned} E_o^a &= 0 \\ \frac{\partial H_o^a}{\partial v}+\frac{2}{\kappa}H_o^a &= 0 \end{aligned} \tag{27}$$

where $H_o^a=(H_o^a)^i+(H_o^a)^d$.

The derivative of the incident field involves higher order terms in the series expansion of the eikonal in powers of s and n. Especially it can be shown that the term s^2n is important. Unfortunately, by keeping this term in the expression of the incident field, its Fourier transform can no longer be obtained by analytical integration.

By substituting (26) into the Fourier transform of the boundary conditions we derive the expression of $A(\xi,\kappa)$ and by an inverse Fourier transform of the sum of (18) and (26) we obtain finally the expression of the total magnetic field and of the derivative of the total electric field on the surface:

$$H_o^a(o,\sigma,\kappa) = \frac{1}{\sqrt{\pi}} e^{\frac{9}{4\kappa^3}} \int_{-\infty}^{+\infty} e^{i\left(\sigma - i\frac{3}{2\kappa}\right)\xi} F(\xi,\kappa)d\xi \tag{28}$$

$$F(\xi,\kappa) = \frac{V\left(\xi + \frac{9}{4\kappa^2}, o\right)\frac{\partial Y_\xi^{(1)}}{\partial v}(o,\kappa) + V'\left(\xi + \frac{9}{4\kappa^2}, o\right)Y_\xi^{(1)}(o,\kappa)}{\frac{\partial Y_\xi^{(1)}}{\partial v}(o,\kappa) + \frac{1}{2\kappa}Y_\xi^{(1)}(o,\kappa)}$$

$$\underline{\partial E_o^a}(o,\sigma,\kappa) = \frac{1}{\sqrt{\pi}} e^{\frac{9}{4\kappa^3}} \int_{-\infty}^{+\infty} e^{i\left(\sigma - i\frac{3}{2\kappa}\right)\xi} G(\xi,\kappa)d\xi \tag{29}$$

$$G(\xi,\kappa) = \frac{V\left(\xi + \frac{9}{4\kappa^2}, o\right)\frac{\partial Y_\xi^{(1)}}{\partial v}(o,\kappa) + V'\left(\xi + \frac{9}{4\kappa^2}, o\right)Y_\xi^{(1)}(o,\kappa)}{Y_\xi^{(1)}(o,\kappa)}$$

where we have used the notation:

$$\frac{\partial}{\partial v}V\left(\xi + \frac{9}{4\kappa^2} - v\right)\Big|_{v=0} = -V'\left(\xi + \frac{9}{4\kappa^2}\right) \tag{30}$$

The current on a perfectly conducting surface is given by:

$$\vec{J} = \hat{n} \times \vec{H} \tag{31}$$

where $\vec{H}$ is the total magnetic field on the surface.

With the magnetic field expressed in the co-ordinate system (s,a,n), (31) takes the form:

$$\vec{J} = \hat{n} \times \left(hH_o^s\hat{s} + hH^a\hat{a}\right)$$

where $\hat{s}$ and $\hat{a}$ are unit vectors along the co-ordinates s and a and where h is the divergence of the geodesics which for axial incidence can be approximated by 1 in the penumbra region. Coming back to the system of equations verified by the coefficients of the Ansatz (8) it can be shown that on the surface:

$$H_o^s = 0, \qquad \frac{H_1^s}{k^{1/3}} = \frac{i}{m}\frac{\partial E_o^a}{\partial v}$$

Hence the asymptotic approximation of H^s is given by:

$$H^s = \frac{i}{m} \quad \frac{\partial E_o^a}{\partial v} \tag{32}$$

and by neglecting the higher order terms of H^a, the asymptotic form of the current on the surface is given by:

$$\vec{J} = -\left(\hat{a}\frac{1}{m}\frac{\partial E_o^a}{\partial v} + \hat{s}H_o^a\right) \tag{33}$$

We have evaluated the asymptotic currents numerically using the formulas (28) and (29) and compared the results with those given by the Method of Moments. The elongated object that we have chosen is a perfectly conducting prolate ellipsoid defined by $\rho(o) = 6.25\,\mathrm{m}$ and $\rho_t(o) = 0.3125\,\mathrm{m}$, the abscissa $s = 0$ corresponding to the shadow boundary. The incident plane wave propagates along the axis of revolution of the ellipsod, with its magnetic field of amplitude unity perpendicular to the plane of the geodesic along which the

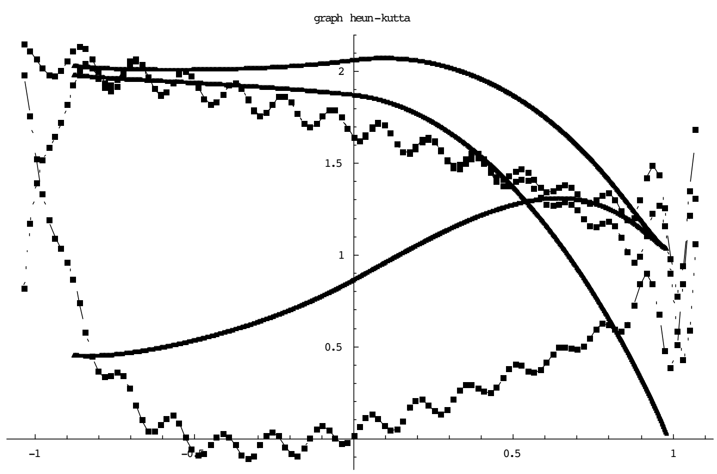

Figure 2. Asymptotic solution (solid lines) compared to MoM solution (dotted lines).

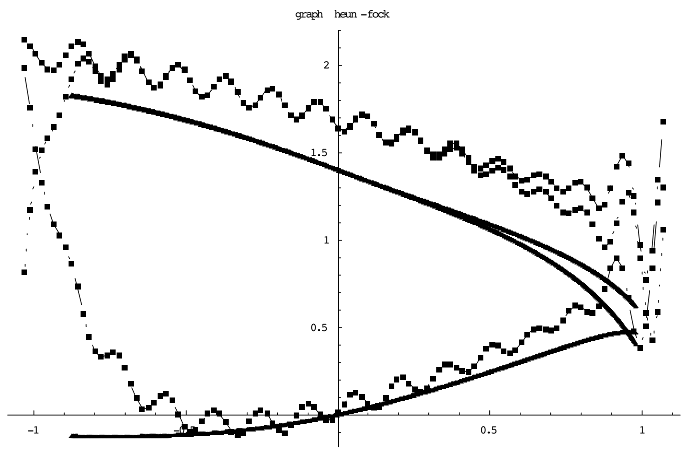

Figure 3. Classical Fock solution (solid lines) compared to MoM solution (dotted lines).

current is calculated. In that configuration we have $H^s = 0$. The results for a frequency of 1 GHz are given on Fig. 2 where the curves representing the real part, the imaginary part and the modulus of the current are drawn as a function of the abscissa s. The curves with the ripples represent the Method of Moments (MoM) results. Those corresponding to the asymptotic currents have been obtained by solving numerically the Heun equation via the Runge-Kutta procedure. They have no ripples since the contribution of the creeping wave following the geodesic surrounding the ellipsod which is weaker due to a longer path, has not been added to that corresponding to the direct way. The comparison of the curves show that our asymptotic formulas overestimate by about 20% the modulus of the MoM current at the shadow boundary. On Fig. 3 the MoM results are compared to the Fock solution. From these results it is evident that classical creeping wave theory does not predict the weaker attenuation of the amplitude of these waves on an elongated object as confirmed by the MoM results. On the other hand our asymptotic solution predicts the correct behaviour.

In order to improve our asymptotic formulas we describe in the next chapter another method for calculating the derivative of the incident field which seems to be the weak point of our approach.

4. INDIRECT DETERMINATION OF THE DERIVATIVE OF THE INCIDENT FIELD

Another procedure for calculating the derivative of the incident field consists in specifying the behaviour of the incident field when v tends to infinity.

Let $V_\xi(v,\kappa)$ be the solution of (16) which behaves like $V(\xi - v)$ when $v \to \infty$.

Since the incident field is a solution of (10) we have:

$$\left(\tilde{H}_o^a\right)^i (\xi, v, \kappa) = a(\kappa,\xi)(v+\kappa)^{-3/2} V_\xi(v,\kappa) \tag{34}$$

When $\kappa \to \infty$, we must recover the expression of the incident field for a classical non elongated object. Hence:

$$\lim_{\kappa\to\infty} \left(\tilde{H}_o^a\right)^i (\xi, v, \kappa) = \frac{1}{\sqrt{\pi}} V(\xi - v) \tag{35}$$

and by stating: $a(\kappa,\xi) = \frac{\kappa^{3/2}}{\sqrt{\pi}}\alpha(\kappa,\xi)$ we get:

$$\left(\tilde{H}_o^a\right)^i (\xi, v, \kappa) = \frac{\kappa^{3/2}}{\sqrt{\pi}}(v+\kappa)^{-3/2}\alpha(\xi,\kappa) V_\xi(v,\kappa) \tag{36}$$

with:

$$\lim_{\kappa\to\infty} = \alpha(\kappa,\xi) = 1 \tag{37}$$

In (36), only the coefficient α remains unknown. We can determine it by identifying (36) to (26) for $v = 0$ which gives:

$$\alpha = \exp\left(\frac{3\xi}{2\kappa} + \frac{9}{\kappa^3}\right) \frac{V\left(\xi + \dfrac{9}{4\kappa^2}\right)}{V_\xi(o,\kappa)} \tag{38}$$

Since $V_\xi(o,\kappa)$ tends to $V(\xi)$ when κ tends to infinity, we see that α satisfies (37).

Knowing the incident field we determine the coefficient $A(\xi,\kappa)$ of the diffracted field by applying the boundary conditions (27). The final result for the component H_o^a of the total magnetic field and the derivative of the component E_o^a of the total electric field on the surface is:

$$H_o^a(v,\sigma,\kappa) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{i\sigma\xi} \alpha(\xi,\kappa) \frac{d\xi}{\dfrac{\partial}{\partial v} Y_\xi^{(1)}(v,\kappa)|_{v=0} + \dfrac{1}{2\kappa} Y_\xi^{(1)}(o,\kappa)} \tag{39}$$

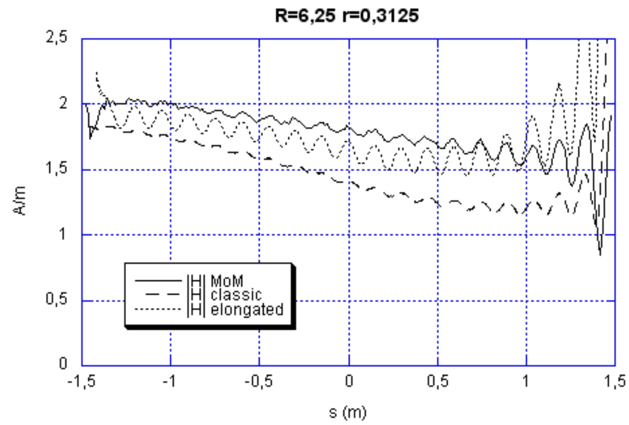

Figure 4. Amplitude of the current along a generatrix of an elongated prolate ellipsoid illuminated by a plane wave propagating in the direction of its axis.

$$\left[\frac{\partial}{\partial v}E_o^a(v,\sigma,\kappa)\right]_{v=0} = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty} e^{i\sigma\xi}\alpha(\xi,\kappa)\frac{d\xi}{Y_\xi^{(1)}(o,\kappa)} \tag{40}$$

In order to obtain (39), (40), we have established, using (16), that the derivative with respect to v of the Wronskian between $Y_\xi^{(1)}(v,\kappa)$ and $V_\xi(v,\kappa)$ is equal to zero for all values of v. This Wronskian can therefore be calculated for $v \to \infty$. It is therefore equal to the Wronskian between $W_1(\xi - v)$ and $V(\xi - v)$, the value of which is (-1).

Different approaches have been tried for the computation of the integral (39). When ξ takes positive and negative values on the real axis, we had no difficulties for calculating $Y_\xi^{(1)}(v,\kappa)$ by the method of Runge-Kutta, starting with its value $W_1(\xi - v)$ for v large. However, until now, we did not succeed, using the same method, for the computation of $V_\xi(v,\kappa)$ in (38). If we replace in (38) the ratio of the Airy functions by 1 and keep only the exponential term, then the results show that (39) overestimates H_o^a at the shadow boundary. If we state $\alpha = 1$, then the integral (39) converges poorly on the negative ξ axis. A deformation of the integration contour on the negative axis into the contour $(\infty\exp(2i\pi/3), 0)$ would be adequate, but then the Runge-Kutta algorithm must be adapted to complex values of ξ. The results shown on Figure 4 correspond to $\alpha = 1$, $Y_\xi^{(1)}(v,\kappa) = W_1\left(\xi + \frac{7}{4\kappa^2} - v\right)$ which is a particular solution of (16) when $v \to 0$. The curves give the variation at 1 GHz of the modulus of the current on the same ellipsod as in Section 2, for an incident magnetic field of amplitude unity, as a function of the abscissa s along the geodesic located in the plane defined by the incident electric field and the axis of the ellipsod. Now the interaction between the direct creeping wave and the creeping wave propagating around the axis of the ellipsoid has been taken into account in the asymptotic evaluation. The Method of Moments (MoM) results give the highest value at the shadow boundary ($s = 0$), compared to the two other curves. The results obtained with the formula for a non elongated (classical) body, underestimates the value of the field, whereas the actual solution based on formula (39) gives intermediate values close to the MoM results.

5. CONCLUSION

Explicit formulas for the asymptotic currents on a strongly elongated body have been derived and tested on an elongated prolate ellipsod by comparing the results with those obtained by the Method of Moments (MoM).

The main results are

1) the confirmation by the MoM that the magnetic creeping wave is less attenuated on an elongated object than on a non elongated object
2) The results given by the asymptotics of elongated objects are close to those given by the MoM.

Some problems remain in the numerical computation of the Fock integral for elongated objects and in the determination of the normal derivative of the incident field on the surface.

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