Progress In Electromagnetics Research Letters, Vol. 3, 179–186, 2008

SCALAR EQUATIONS FOR SCATTERING BY ROTATIONALLY SYMMETRIC RADIALLY INHOMOGENEOUS ANISOTROPIC SPHERE

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Abstract—A set of scalar differential equations for treating scattering by rotationally symmetric radially inhomogeneous anisotropic sphere is presented. These equations may easily be treated by applying the integral equation method that has been developed. Besides a possible different expansion in a Neumann series is also discussed.

1. INTRODUCTION

The consideration of electromagnetic waves in anisotropic media occurs in a variety of citations [1–20]. Usually the treatment presented in the literature is limited to the homogeneous anisotropic case [1], which is already complex. However the evolution of the technology makes necessary the handling of the more difficult case of the inhomogeneous anisotropic scatterer.

When the medium is isotropic but inhomogeneous a handy method based on a volume integral equation has been developed [21, 22]. In this method the unknown function is expanded in a vector Dini series, which allows direct calculation of the resulting integrals containing the Green function. For the case of the homogeneous but gyrotropic or biaxial spherical bodies a similar approach has been developed [18, 19]. This method consists in constructing a basis of functions that satisfy the Maxwell equations in the whole space. Next this set of functions should be used in conjunction with the appropriate boundary conditions to form the actual solution.

It will be shown that the method developed for the inhomogeneous spheres can be applied to the case of scattering by inhomogeneous anisotropic spheres. Although the procedure seems straightforward the expansions may be slowly convergent, since, as it has been known from the solution of anisotropic spheres with only three parameters (ε_r, ε_θ, μ), the solution is expressed in terms of spherical Bessel functions of non-integral order [23].

In case of inhomogeneity dependent only on r the solution is easier to obtain, since the field can be expressed in terms of scalar potentials as has been done in the case of radially inhomogeneous cylindrical fibers [24]. The treatment of the resulting equations is similar to the ones for the scalar (acoustic) problem, which has been developed by the author and his collaborators [25, 26].

The corresponding equations are given below. Next a direct verification in a special case is given as well as the possibility of expanding the unknown function in terms of Neumann series is discussed.

2. APPLICATION OF THE VECTOR DINI SERIES METHOD

Here we consider a rotationally symmetric radially inhomogeneous anisotropic dielectric sphere of radius α described in spherical coordinates by the constitutional parameters

$$\bar{\bar{\varepsilon}} = \begin{pmatrix} \varepsilon_{rr} & 0 & 0 \\ 0 & \tau & \sigma \\ 0 & -\sigma & \tau \end{pmatrix}, \quad \mu = \mu_0 \tag{1}$$

The sphere is inside a homogeneous isotropic space described by $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$, while $\exp(i\omega t)$ is the assumed time dependence. An EM field of arbitrary direction is imposed. The method that has been developed for inhomogeneous dielectric spheres [21], can be easily extended to the present case. Using the expansions for the field inside the sphere ($r \leq \alpha$)

$$\vec{D}(\vec{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sum_{\ell=1}^{\infty} \left[A_{mn\ell} \vec{M}_{mn\ell} \left(\frac{\gamma^M_{mn\ell}}{a}, \vec{r} \right) + B_{mn\ell} \vec{N}_{mn\ell} \left(\frac{\gamma^N_{mn\ell}}{a}, \vec{r} \right) \right] \tag{2}$$

$$\begin{aligned} \vec{E}(\vec{r}) \; = \; & \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{\ell=1}^{\infty} \left[\Gamma_{mn\ell} \vec{M}_{mn\ell} \left(\frac{\gamma^M_{mn\ell}}{a}, \vec{r} \right) + \Delta_{mn\ell} \vec{N}_{mn\ell} \left(\frac{\gamma^N_{mn\ell}}{a}, \vec{r} \right) \right. \\ & \left. + Z_{mn\ell} \vec{L}_{mn\ell} \left(\frac{\gamma^L_{mn\ell}}{a}, \vec{r} \right) \right] \end{aligned} \tag{3}$$

where the various symbols have been defined in [21], we derive again the following equations, as in [21], which are repeated here for convenience

$$\Gamma_{mn\ell} = A_{mn}^{\text{inc}} T_{n\ell}^{M}(k_0) + k_0^2 \frac{A_{mn\ell} - \Gamma_{mn\ell}}{\left(\frac{\gamma_{mn\ell}^{M}}{a}\right)^2 - k_0^2} - i k_0^3 a^2 T_{n\ell}^{M}(k_0)$$

$$\sum_p \frac{\left[\begin{array}{c} -\frac{\gamma_{mnp}^{M}}{a} j_n'(\gamma_{mnp}^{M}) h_n(k_0 a) + \\ k_0 h_n'(k_0 a) j_n(\gamma_{mnp}^{M}) \end{array}\right]}{\left(\frac{\gamma_{mnp}^{M}}{a}\right)^2 - k_0^2} (A_{mnp} - \Gamma_{mnp}) \tag{4}$$

$$\Delta_{mn\ell} = B_{mn}^{\text{inc}} T_{n\ell}^{N}(k_0) + k_0^2 \frac{B_{mn\ell} - \Delta_{mn\ell}}{\left(\frac{\gamma_{mn\ell}^{N}}{a}\right)^2 - k_0^2} - i k_0^3 a^2 T_{n\ell}^{N}(k_0) \cdot$$

$$\sum_p \frac{\left[\begin{array}{c} \frac{\gamma_{mnp}^{N}}{a} j_n\left(\gamma_{mnp}^{N}\right) \frac{1}{k_0 a} \left[x h_n(x)\right]_{x=k_0 a}' - \\ k_0 h_n(k_0 a) \frac{1}{\gamma_{mnp}^{N}} \left[x j_n(x)\right]_{x=ka}' \end{array}\right]}{\left(\frac{\gamma_{mnp}^{N}}{a}\right)^2 - k_0^2} (B_{mnp} - \Delta_{mnp}) \tag{5}$$

Next we should eliminate one of the two groups of unknowns $\{A, B\}$ or $\{\Gamma, \Delta, Z\}$ of the expansions (12) and (13). Since Z are not present in the final expressions it is better to eliminate Γ, Δ so we write

$$\begin{aligned} \vec{E} &= \sum_{m,n,\ell} \left[A_{mn\ell} \bar{\bar{\varepsilon}}^{-1}(\vec{r}) \cdot \vec{M}_{mn\ell} + B_{mn\ell} \bar{\bar{\varepsilon}}^{-1}(\vec{r}) \cdot \vec{N}_{mn\ell}\right] \\ &= \sum_{m,n,\ell} \left[\Gamma_{mn\ell} \vec{M}_{mn\ell} + \Delta_{mn\ell} \vec{N}_{mn\ell} + Z_{mn\ell} \vec{L}_{mn\ell}\right] \end{aligned} \tag{6}$$

Selecting now $\bar{\bar{\varepsilon}}$ as in (1) we find that

$$\bar{\bar{\varepsilon}}^{-1} = \begin{pmatrix} \frac{1}{\varepsilon_{rr}} & 0 & 0 \\ 0 & \frac{\tau}{\tau^2 + \sigma^2} & -\frac{\sigma}{\tau^2 + \sigma^2} \\ 0 & \frac{\sigma}{\tau^2 + \sigma^2} & \frac{\tau}{\tau^2 + \sigma^2} \end{pmatrix} \tag{7}$$

and after some manipulation we derive that

$$\overline{\overline{\varepsilon}}^{-1}\cdot\vec{M}_{mn}=\frac{\tau}{\tau^2+\sigma^2}\vec{M}_{mn}(\vec{r})+\frac{\sigma}{\tau^2+\sigma^2}\sqrt{n(n+1)}\vec{B}_{mn}(\theta,\varphi)j_n(kr) \quad (8)$$

$$\overline{\overline{\varepsilon}}^{-1}\cdot\vec{N}_{mn} = \frac{1}{\varepsilon_{rr}}n(n+1)\vec{P}_{mn}(\theta,\varphi)\frac{j_n(kr)}{kr}+\sqrt{n(n+1)} \left(\frac{\tau}{\tau^2+\sigma^2}\vec{B}_{mn}(\vec{r})-\frac{\sigma}{\tau^2+\sigma^2}\vec{C}_{mn}(\theta,\varphi)\right)\frac{1}{kr}\frac{d}{dr}[rj_n(kr)] \quad (9)$$

Here we may apply the orthogonality process to produce a system of equations for the unknown coefficients. However a difficulty may occur since the terms cannot be easily expanded in a small number of terms. So the convergence of the series in (2), (3) may be very poor. For this reason we are going to develop in the following section an alternative approach. This is restricted to radially inhomogeneous anisotropic spheres only, however, it is also applied to the case of spheres containing both electric and magnetic anisotropies.

3. THE SET OF EQUATIONS

Following Monzon [1] we define the six-parameter material in (r,θ,φ)

$$\overline{\overline{\varepsilon}}=\begin{pmatrix}\varepsilon_{rr} & 0 & 0\\ 0 & \tau & \sigma\\ 0 & -\sigma & \tau\end{pmatrix},\quad \overline{\overline{\mu}}=\begin{pmatrix}\mu_{rr} & 0 & 0\\ 0 & \gamma & \zeta\\ 0 & -\zeta & \gamma\end{pmatrix} \quad (10)$$

where all parameters are dependent on r only. Using the procedure outlined in that paper we find after very lengthy manipulations the following set of equations

$$\frac{\partial^2 S_r}{\partial r^2}-\frac{\frac{\partial\tau}{\partial r}}{\tau}\frac{\partial S_r}{\partial r}+\left[\omega^2\tau\gamma-\frac{n(n+1)}{r^2}\frac{\tau}{\varepsilon_{rr}}+\omega^2\frac{\tau}{\gamma}\zeta^2\right]S_r+ +j\omega T_r\left(\frac{\sigma}{\tau}\frac{\partial\tau}{\partial r}-\frac{\partial\sigma}{\partial r}\right)-j\omega\frac{\partial T_r}{\partial r}\left(\frac{\tau}{\gamma}\zeta+\sigma\right)=0 \quad (11)$$

$$\frac{\partial^2 T_r}{\partial r^2}-\frac{\frac{\partial\gamma}{\partial r}}{\gamma}\frac{\partial T_r}{\partial r}+\left[\omega^2\tau\gamma-\frac{n(n+1)}{r^2}\frac{\gamma}{\mu_{rr}}+\omega^2\frac{\gamma}{\tau}\sigma^2\right]T_r+ -j\omega S_r\left(\frac{\zeta}{\gamma}\frac{\partial\gamma}{\partial r}-\frac{\partial\zeta}{\partial r}\right)+j\omega\frac{\partial S_r}{\partial r}\left(\frac{\gamma}{\tau}\sigma+\zeta\right)=0 \quad (12)$$

where we have defined $S_r = r^2\varepsilon_{rr}E_r$, $T_r = r^2\mu_{rr}H_r$. Obviously, when the parameters in $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ are independent of r we derive the corresponding equations of [1]. Once the equations for the radial components have been solved the expressions for the transverse components may be found. However, they are not necessary for the integral approach discussed here.

4. DISCUSSION

In general with r-dependent parameters the solution of the above system of equations can be formed using the method developed in the corresponding scalar case for the inhomogeneous density [26]. However, particular attention should be given to the discontinuities of the various parameters since their differentiation gives rise to delta functions. We may overcome this difficulty by stating the problem in a different way: by considering as initial problem the scattered field by a homogeneous anisotropic sphere with values of the parameters equal to those at the interface. Then no delta functions appear in the treatment of the integral equation.

To illustrate the application of the method we consider the simple case of the three-parameter material (with ($\varepsilon_r = \varepsilon_{rr}$, $\varepsilon_\theta = \tau$, μ)). Then we take the following integral equation for S_r, following the method of Chew [27]:

$$S = S^o - \int_0^a \frac{1}{\varepsilon_\theta} \frac{\partial \varepsilon_\theta}{\partial r'} \frac{\partial S}{\partial r'} G(r, r')\, dr' + \int_0^a \omega^2 \left(\varepsilon_\theta \mu - \varepsilon_\theta^0 \mu^0\right) S \cdot G(r, r')\, dr' - $$

$$-n(n+1) \int_0^a \frac{1}{r^2} \left(\frac{\varepsilon_\theta}{\varepsilon_r} - 1\right) S \cdot G(r, r') dr' \tag{13}$$

when the Green function G fulfils the equation [28]

$$\frac{\partial^2 G}{\partial r^2} + \left[\omega^2 \varepsilon_\theta^0 \mu^0 - \frac{n(n+1)}{r^2}\right] G = -\delta(r - r') \tag{14}$$

A direct substitution $S = Arj_v(kr)$ (A a constant), with $k^2 = \omega^2 \varepsilon_\theta \mu$ reveals after long manipulations that the equation is fulfilled only if $v = -\frac{1}{2} + \sqrt{n(n+1)\frac{\varepsilon_\theta}{\varepsilon_r} + \frac{1}{4}}$ [23] as expected from the alternative solution based on separation of variables.

The general solution of the problem can also be formed (especially when $\varepsilon_\theta = \varepsilon_\theta^0$) via a Neumann series [29]

$$S = r \sum_{n=0}^{\infty} \alpha_n j_n(kr) \tag{15}$$

However some difficult integrals appear for the $j_n^2(kr)$ term that should be expressed in terms of $Si(r)$ and $Ci(r)$, rendering the calculation very cumbersome.

5. CONCLUSION

A set of two scalar equations for treating scattering by rotationally symmetric radially inhomogeneous anisotropic sphere is presented. Using the scalar integral equation formulation [26], and expanding the unknown function in Dini or Neumann series the problem can be transformed to an algebraic one. Alternatively the vector Dini series method [21, 22] can, also, be applied.

ACKNOWLEDGMENT

The present paper has been supported by the program of basic research PEBE 2007 of NTUA.

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