Progress In Electromagnetics Research C, Vol. 1, 229–240, 2008

DYNAMICAL PROBLEM OF A ROTATIONAL CHARGED DIELECTRIC RIGID BODY IN A UNIFORM MAGNETIC FIELD

G.-Q. Zhou and X. Xiao

Department of Physics
Wuhan University
Wuhan 430072, China

W.-J. Zhou

School of Electrical Engineering
Wuhan University
Wuhan, 430072, China

Abstract—Based on the theory about charge moment tensor and the magnetic moment of a rotational charged body, and by means of Eulor's equation for a rotational rigid body in classic mechanics, the Lagrangian and the dynamic equation of a charged dielectric rigid body under a uniform magnetic field has been derived; Also two symmetric simple cases have been solved or analyzed under condition of slow rotation and no gravitation, the corresponding invariants have also been found.

1. INTRODUCTION

A charged body rotating around a fixed axis has definite magnetic moment, and an object with magnetic moment will sustain moment of force imposed by an external magnetic field [1–6]. References [1–4] has deduced a series of rules and given some examples about calculating the magnetic moment of a rotational charged body based on a strict and delicate analogue relation. Especially the interesting and useful concepts such as charge moment tensor $\tilde{T}$, scalar charge moment, principal axis and principal-axis scalar charge moment introduced in references [1, 2] make it very easy for us to compute the magnetic moment of an arbitrary rotational charged body with respect to an arbitrary given axis. Meanwhile, the relationship between the quadric distributive law of scalar charge moment and the parameters of tensor $\tilde{T}$ has also been discussed in detail, and conditions of zero magnetic moment for an arbitrary rotational charged body have been formulated explicitly in references [2].

On the other hand, to compute $\tilde{T}$ and magnetic moment $\vec{P}_m$ of a rotational charged body is only the first step to study its dynamic and kinetic behaviors in electromagnetic fields. In view of electromagnetism, cosmology and space technology, it is of special meaning and extraordinary importance to derive a dynamic equation for a rotational charged body under a given electromagnetic field.

Generally speaking, movement of a rigid body can be viewed as superposition of the movement of its center of mass and the rotation around the center, the latter is just the concerned theme of the present paper. As to a rigid body with an arbitrary mass distribution rotating around a fixed point, there are rarely several circumstances which can be strictly solved [7, 8], such as Eulor-Poinsat case under zero moment of force, Lagrange-Poisson case for a rigid body with a symmetry of degree 4 (that is, $I_x = I_y \neq I_z$), and a more special example — the so-called Kovaleveskaya case. The rotation of a charged body under vanishing magnetic moment or zero moment of force belongs to Eulor-Poinsat case. Its solution can be expressed with the complicated Jacobi function (a function related to elliptic integral) [9]. Then a question is brought forth subsequently — how about a rotational charged body in an electromagnetic field?

In the present paper, by means of Lagrange dynamical theory and Eulor's equation for a rotational rigid body, the Lagrangian and dynamic equations of a rotational charged rigid body under a uniform magnetic field has been successfully derived; Also two symmetric simple cases have been solved and analyzed under conditions of slow rotation and no gravitation. We limit our research object to the dielectric rigid body under conditions of slow rotation and no gravitation so that the damping effect of electromagnetic radiation and the relativistic effects caused by rotation can all be ignorable.

2. FUNDAMENTAL CONCEPTS OF CHARGE MOMENT TENSOR FOR A ROTATIONAL CHARGED BODY

In an arbitrary body-coordinate system O-xyz which is rigidly linked with a charged body of definite volume (area, line or discrete) charge distribution, the so-called **charge moment tensor** with respect to the given origin O is defined as:

$$\tilde{T}(O) = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \tag{1}$$

Under the case of disperse charge distribution, the tensor element is

$$T_{\alpha\beta(O)} = \sum_i Q_i \left[r_i^2 \delta_{\alpha\beta} - x_{i\alpha} x_{i\beta} \right] \quad (\alpha \text{ and } \beta = 1,\, 2,\, 3) \tag{2}$$

here $\vec{r}_i$ is the position vector of point charge Q_i, $\vec{r}_i = (x_{i1}, x_{i2}, x_{i3}) = (x_i,\, y_i,\, z_i)$, and for the case of continual charge distribution such as a charged body with a volume charge density of $\rho_e(x_1,\, x_2,\, x_3)$,

$$T_{\alpha\beta}(O) = \int_v \rho_e\,(x_1,\, x_2,\, x_3) \left[r^2 \delta_{\alpha\beta} - x_\alpha x_\beta \right] dv \tag{3}$$

here $dv = dx_1 dx_2 dx_3$, $r^2 = x_1^2 + x_2^2 + x_3^2$.

Then the so-called **scalar charge moment** T_l with respect to the same point O and arbitrary direction (provided its direction cosine is $\vec{l} = (\cos\theta_1,\, \cos\theta_2,\, \cos\theta_3)$) is

$$\begin{aligned} T_l(O, \vec{l}\,) &= \vec{l} \cdot \tilde{T} \cdot \vec{l} = (\cos\theta_1 \cos\theta_2 \cos\theta_3) \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \end{pmatrix} \\ &= T_{11}\cos^2\theta_1 + T_{22}\cos^2\theta_2 + T_{33}\cos^2\theta_3 \\ &\quad + 2T_{12}\cos\theta_1\cos\theta_2 + 2T_{23}\cos\theta_2\cos\theta_3 + 2T_{31}\cos\theta_3\cos\theta_1 \end{aligned} \tag{4}$$

Then it can be immediately deduced that a rotational charged body with an angular velocity of $\vec{\omega}$ with respect the same axis $(O,\, \vec{l}\,)$, must has a **scalar charge moment** (given $\vec{\omega} = \omega \vec{l}\,$):

$$T_l = \frac{\vec{\omega}}{\omega} \cdot \tilde{T} \cdot \frac{\vec{\omega}}{\omega} \tag{5}$$

And **magnetic moment**

$$\vec{P}_m \left(O, \vec{l}\, \right) = \frac{1}{2} T_l \vec{\omega} = \frac{1}{2} \left(\frac{\vec{\omega}}{\omega} \cdot \tilde{T} \cdot \frac{\vec{\omega}}{\omega} \right) \vec{\omega} = \frac{1}{2} l_i \tilde{T}_{ij} l_j \vec{\omega} \tag{6}$$

here the repeated indices represent summation from 1 to 3 according to Einstein's convention.

Concepts of **principal axes** and corresponding **principal scalar charge moments** have also been proposed in reference [1].

In the new Cartesian coordinate body system O-xyz spanned with above three **principal axes**, the charge moment tensor can be expressed as a diagonal form

$$\tilde{T}(O) = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix} \tag{7}$$

Based on the concepts of principal axes and the three principal-axis scalar charge moments, scalar charge moment T_l with respect to an arbitrary direction $\vec{l} = (\cos\theta_1, \cos\theta_2, \cos\theta_3)$ and the given point O is

$$T_l = \vec{l} \cdot \tilde{T} \cdot \vec{l} = T_1 \cos^2\theta_1 + T_2 \cos^2\theta_2 + T_3 \cos^2\theta_3 \tag{8}$$

Then according to Equation (6),

$$\vec{P}_m\left(O, \vec{l}\right) = \frac{1}{2}\left(T_1 \cos^2\theta_1 + T_2 \cos^2\theta_2 + T_3 \cos^2\theta_3\right)\vec{\omega} \tag{9}$$

3. DYNAMIC EQUATION OF ROTATIONAL CHARGED DIELECTRIC RIGID BODY IN A UNIFORM MAGNETIC FIELD

We select Cartesian coordinate system O-xyz in the body reference with the center-of-mass point O as its origin and the inertia principal axes as its three axes. Therefore, according to formula (6), the moment of force generated by the uniform magnetic field is

$$\begin{aligned}\vec{M} &= \vec{P}_m \times \vec{B} = M_x\vec{i} + M_y\vec{j} + M_z\vec{k} \\ &= \frac{1}{2}T_l\left[(\omega_y B_z - \omega_z B_y)\vec{i} + (\omega_z B_x - \omega_x B_z)\vec{j} + (\omega_x B_y - \omega_y B_x)\vec{k}\right]\end{aligned} \tag{10}$$

Thus the dynamic rotation equation of the rigid body is Euler's Equation

$$\begin{cases} I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = M_x \\ I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = M_y \\ I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = M_z \end{cases} \tag{11}$$

Or concretely expressed as

$$\begin{cases} I_x\dot{\omega}_x - \left[\left(I_y\omega_z + \dfrac{1}{2}T_lB_z\right)\omega_y - \left(I_z\omega_y + \dfrac{1}{2}T_lB_y\right)\omega_z\right] = 0 \\ I_y\dot{\omega}_y - \left[\left(I_z\omega_x + \dfrac{1}{2}T_lB_x\right)\omega_z - \left(I_x\omega_z + \dfrac{1}{2}T_lB_z\right)\omega_x\right] = 0 \\ I_z\dot{\omega}_z - \left[\left(I_x\omega_y + \dfrac{1}{2}T_lB_y\right)\omega_x - \left(I_y\omega_x + \dfrac{1}{2}T_lB_x\right)\omega_y\right] = 0 \end{cases} \tag{12}$$

here

$$\tilde{I}(O) = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \tag{13}$$

is the principal-axes moment of inertia of the rigid body with respect to point O.

Equation (12) is effective for an arbitrary rotational charged dielectric rigid body in a gravitationless environment and a uniform magnetic field.

4. THE LAGRANGIAN OF ROTATIONAL CHARGED DIELECTRIC RIGID BODY IN AN UNIFORM MAGNETIC FIELD

We can also make use of Lagrange's dynamic theory to depict the same rotation system. Suppose ω_x, ω_y, ω_z are respectively the three components of the angular velocity in the body reference system and Cartesian coordinate frame O-xyz, also suppose the direction of external uniform magnetic field is along the axis $O\zeta$ of coordinate frame O-$\xi\eta\zeta$ in a fixed reference system, and φ, θ, ψ respectively stand for the angle of precession, nutation, and rotation, shown as Fig. 1.

Then the Eulor's kinetic equation is

$$\begin{cases} \omega_x = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_y = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_z = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases} \tag{14a}$$

It can be rewritten as a matrix form

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \equiv \tilde{M} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \tag{14b}$$

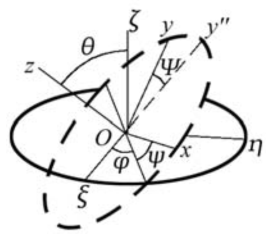

Figure 1. Eulor's angles describing the movement of a rigid body around its center of mass.

Here

$$\tilde{M} = \begin{pmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \tag{15}$$

We select the three Eulor's angles φ, θ, ψ as the generalized coordinates of the system, then the rotation kinetic energy of this system is

$$\begin{aligned} E_k(\varphi,\theta,\psi;\dot{\varphi},\dot{\theta},\dot{\psi}) &= \frac{1}{2}\left(\omega_x \quad \omega_y \quad \omega_z\right)\tilde{I}\left(\omega_x \quad \omega_y \quad \omega_z\right)^T \\ &= \frac{1}{2}\left(\dot{\varphi} \quad \dot{\theta} \quad \dot{\psi}\right)M^T\tilde{I}M\left(\dot{\varphi} \quad \dot{\theta} \quad \dot{\psi}\right)^T \end{aligned} \tag{16}$$

Here the superscript "T" represents transposition of a matrix.

On the other hand,

$$\begin{aligned} \vec{P}_m &= \frac{1}{2}\left(\frac{\vec{\omega}}{\omega}\cdot T_l\cdot\frac{\vec{\omega}}{\omega}\right)\vec{\omega} \\ &= \frac{1}{2}\left[\left(\dot{\varphi} \quad \dot{\theta} \quad \dot{\psi}\right)M^T\tilde{T}M\left(\dot{\varphi} \quad \dot{\theta} \quad \dot{\psi}\right)^T\right]\left(\dot{\varphi} \quad \dot{\theta} \quad \dot{\psi}\right)M^T \end{aligned} \tag{17}$$

And the uniform magnetic field $\vec{B}$ along with direction $O\zeta$ can be expressed in terms of Eulor's angles φ, θ, ψ and the value of $\vec{B}$

$$\begin{aligned} \vec{B} &= B\sin\theta\cos\psi\vec{i} + B\sin\theta\sin\psi\vec{j} + B\cos\theta\vec{k} \\ &= B\left(\sin\theta\cos\psi \quad \sin\theta\sin\psi \quad \cos\theta\right)^T \end{aligned} \tag{18}$$

Then the potential energy of rotational charged dielectric rigid body in a uniform magnetic field is

$$\begin{aligned}V_m &= -\vec{P}_m \cdot \vec{B}\\ &= -\frac{B}{2\omega^2}\left[\begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix} M^T \tilde{T} M \begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix}^T\right] \cdot\\ &\qquad \begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix} M^T \begin{pmatrix}\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta\end{pmatrix}^T \end{aligned} \tag{19}$$

here

$$\omega^2 = \dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta \tag{20}$$

Thus the Lagrangian of this system is

$$\begin{aligned}L(\varphi, \theta, \psi; \dot{\varphi}, \dot{\theta}, \dot{\psi}) &= E_k - V_m\\ &= \frac{1}{2}\begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix} M^T \tilde{I} M \begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix}^T\\ &\quad + \frac{B}{2\omega^2}\left[\begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix} M^T \tilde{T} M \begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix}^T\right] \cdot\\ &\quad \begin{pmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{pmatrix} M^T (\sin\theta\cos\psi \;\; \sin\theta\sin\psi \;\; \cos\theta)^T\end{aligned} \tag{21}$$

Note that although $\tilde{I}$ is diagonal in the principal-axes coordinate system, generally the charge moment tensor $\tilde{T}$ isn't definitely of a diagonal form at the same time, but still expressed as formula (1).

Then the Lagrange dynamic equations of the conservative system are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \tag{22}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{23}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = 0 \tag{24}$$

The above equation determines the dynamic behaviors of this rotation system. It is easy for us to recognize immediately that φ is the ignorable variable (i.e., the cyclic coordinate) of the system and the corresponding generalized momentum P_φ is a invariant of this system with

$$P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} \tag{25}$$

$$\dot{P}_\varphi = \frac{\partial L}{\partial \varphi} = 0 \tag{26}$$

Equations (20)–(24) is complete and actually equivalent to Equation (12), but just as that instructed in first section, there are rarely several cases permit a strict solution. In the subsequent chapters, we will give some conclusions and discussion about two simply or symmetric circumstances.

5. SIMPLE CONCLUSION ABOUT A SYMMETRIC CASE

When the concerned dielectric rigid body is of a 4-degree symmetric axis about the distribution of its mass and charge, that is, the center of mass and center of charge is just the same point-origin O of body reference system. The principal axes of inertia moment tensor $\tilde{I}$ and those of the charge moment tensor $\tilde{T}$ are therefore also collinear and construct the above body Cartesian coordinate system, in which both the inertia moment tensor $\tilde{I}$ and the charge moment tensor $\tilde{T}$ are diagonal, i.e., respectively expressed as formula (13) and (7). At the same time, we have

$$I_x = I_y \neq I_z, \quad T_1 = T_2 \neq T_3 \tag{27}$$

According to formula (6) and (7), the magnetic moment corresponding to angular velocity $\vec{\omega}$ is

$$\begin{aligned}\vec{P}_m &= \frac{1}{2}\left(\frac{\vec{\omega}}{\omega}\cdot\tilde{T}\cdot\frac{\vec{\omega}}{\omega}\right)\vec{\omega} \\ &= \frac{1}{2\omega^2}\left(T_1\omega_x^2 + T_2\omega_y^2 + T_3\omega_z^2\right)\vec{\omega} \\ &= \frac{1}{2\omega^2}\left[T_1\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + T_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2\right]\vec{\omega}\end{aligned} \tag{28}$$

then

$$E_k = \frac{I_x}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{I_z}{2}\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2 \tag{29}$$

and

$$\begin{aligned}V_m &= -\vec{P}_m\cdot\vec{B} \\ &= -\frac{B}{2\omega^2}\left[T_1\left(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta\right)+T_3\left(\dot{\varphi}\cos\theta+\dot{\psi}\right)^2\right]\left(\dot{\psi}\cos\theta+\dot{\varphi}\right)\end{aligned} \tag{30}$$

At last we attain the Lagrangian of above dielectric rigid body with a 4-degree symmetric axis, that is

$$
\begin{aligned}
& L\left(\varphi,\theta,\psi;\dot{\varphi},\dot{\theta},\dot{\psi}\right) = E_k - V_m \\
= & \frac{I_x}{2}\left(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta\right)+\frac{I_z}{2}\left(\dot{\psi}+\dot{\varphi}\cos\theta\right)^2 \\
& +\frac{B}{2\omega^2}\left[T_1\left(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta\right)+T_3\left(\dot{\varphi}\cos\theta+\dot{\psi}\right)^2\right]\left(\dot{\psi}\cos\theta+\dot{\varphi}\right) \quad (31)
\end{aligned}
$$

Here ω^2 is also given by formula (20). We can immediately recognize that φ and ψ are the cyclic coordinates (i.e., the so-called ignorable variables) of this system. According to Lagrange dynamic Equations (22)–(24), it is easy to find that corresponding generalized momentum $P_\varphi = \frac{\partial L}{\partial \dot{\varphi}}$ and $P_\psi = \frac{\partial L}{\partial \dot{\psi}}$ are all the invariants of the system.

6. AN EXAMPLE OF STRICT SOLUTION

If the rotational charged dielectric is made of the same particles with an unique charge-mass ratio of q_i/m_i, $(i = 1, 2, \dots, n)$, for the discrete case, and we define that

$$\gamma \equiv q_i/2m_i, \quad (i = 1, 2, \dots, n) \quad (32a)$$

or under the continual case,

$$\gamma \equiv \rho_e(\vec{r})/2\rho_m(\vec{r}) \quad (32b)$$

Take the case of continual distribution as reference, in a fixed inertial reference system, the magnetic moment can be expressed as

$$\vec{P}_m = \frac{1}{2}\int \rho_e(\vec{r})(\vec{r}\times\vec{v})d^3\vec{r} \quad (33)$$

And the total angular momentum can be formulized as

$$\vec{J} = \frac{1}{2}\int \rho_m(\vec{r})(\vec{r}\times\vec{v})d^3\vec{r} \quad (34)$$

where ρ_m is the mass density of the rigid body. From (32a) or (32b) and compare (33) with (34), we have

$$\vec{P}_m = \gamma\vec{J} \quad (35)$$

Then the moment of the force generated by the magnetic field is

$$\begin{aligned}\vec{M} &= \vec{P}_m \times \vec{B} = \gamma \vec{J} \times \vec{B} \\ &= -\gamma \vec{B} \times \vec{J}\end{aligned} \tag{36}$$

According to angular momentum theorem, in a fixed inertial reference system,

$$\begin{aligned}\vec{M} &= \frac{d\vec{J}}{dt} = -\gamma \vec{B} \times \vec{J} \\ &= \vec{\omega} \times \vec{J}\end{aligned} \tag{37}$$

here

$$\vec{\omega} = -\gamma \vec{B} \tag{38}$$

This is just accordance with the sufficient and necessary condition for a vector to make a regular precession movement attained in reference [10], therefore the angular momentum $\vec{J}$ will precess with a precession angular velocity of $\vec{\omega}$. This is in fact equivalent to Larmor precession which depicts the movement of a particle with magnetic moment in a magnetic field. The different point is the rotational charged dielectric rigid body is of an arbitrary shape and distribution of charge and mass but the same charge-mass ratio.

7. CONCLUDING REMARKS

Research of a charged dielectric rigid body in a uniform magnetic field is an important and valuable content involved in space technology, cosmology, celestial physics, and electrical engineering. It is worthy of mention that, even for a rotational rigid body under the background without electromagnetic field, there are only several examples such as Eulor-Poinsat case, Lagrange-Poisson case, Kovaleveskaya case that can be strictly solved. In order to study the rotation problem of a charged dielectric rigid body in a uniform field, we ignore the existence of gravitation and induction electricity (for slow rotation and dielectric medium).

Making use of existent theory about the relationship between the charge moment tensor and the magnetic moment of a rotational charged body, Eulor's equation and Lagrange dynamic theory in analytic mechanics, we successfully derived the Lagrangian of a rotational charged dielectric rigid body in a uniform magnetic field, give concrete and simple conclusion under two cases: i, a system with symmetric distribution of charge and mass. *ii*, a system consisting of the same particle with an unique charge-mass ratio ($q_i/m_i \equiv 2\gamma$), and recognize the ignorable variable and corresponding invariant. In a subsequent paper, we will discuss in detail the dynamics of a rotational charged dielectric rigid body in another kind of electromagnetic field.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China under Grant No. 10775105.

REFERENCES

1. Zhou, G.-Q., "Charge moment tensor and the magnetic moment of rotational charged bodies," *Progress In Electromagnetics Research*, PIER 68, 156–160, New York, 2007.
2. Zhou, G.-Q., "The magnetic-moment quadric and conditions of vanishing magnetic moment for a rotational charged body," *Progress In Electromagnetics Research*, PIER 70, 211–223, New York, 2007.
3. Zhou, G.-Q., "Several rules about the magnetic moment of rotational charged bodies," *Progress In Electromagnetics Research Symposium (PIERS 2007 Beijing Proceedings)*, 89–93, The Electromagnetics Academy, Cambridge, 2007.
4. Xu, D. F. and Z.-S. Yu, "The calculation of magnetic moment for a electrified body rotating about a fixed axis," *College Physics*, Vol. 16, No. 4, 3–4, 1997.
5. Liu, J.-P., *Electrodynamics*, 153–165, Higher Education Press, Beijing, 2004.
6. Chen, J.-F., *Classical Mechanics*, 161–169, Press of China University of Sci. & Tech., Hefei, 1990.
7. Zhou, Y.-B., *Theoretical Mechanics*, 202–212, People's Education Press, Beijing, 1974.
8. Zhang, Q.-R., *Classical Mechanics*, 94–112, Science Press, Beijing, 2002.
9. Gradsbteyn, I. S. and L. M. Ryzbik, *Table of Integrals, Series, and Productions*, 1059–1064, Elsevier (Singapore) Pte. Ltd., 2004.
10. Zhou, G.-Q., "Some properties and rules about a regular precession vector," *Physics and Engineering*, Vol. 11, No. 1, 22–25, 2001.
11. Tchernyi, V. V., et al., "Possible electromagnetic nature of

the Saturn's rings: Superconductivity and magnetic levitation," *Progress In Electromagnetics Research*, PIER 52, 277–299, New York, 2005.

12. Capozzoli. A., et al., "Global optimization and antennas synthesis and diagnosis, Part one: Concepts, tools, strategies and performances," *Progress In Electromagnetics Research*, Vol. 56, 195–232, 2006.
13. Bellett, P. T., et al, "An investigation of magnetic antennas for ground penetrating radar," *Progress In Electromagnetics Research*, PIER 43, 257–271, 2003.
14. Dmtriev, V., "Tables of the second rank constitutive tensors for linear homogeneous media described by point magnetic groups of symmetry," *Progress In Electromagnetics Research*, PIER 28, 43–95, 2000.
15. Yin, W.-Y., et al., "The Mueller matrix of a two-layer eccentrically bianisotropic cylinder linear array with double helical conductances of the surfaces: Classification of the magnetic symmetry groups," *Progress In Electromagnetics Research*, PIER 30, 105–130, 2001.