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SCATTERING OF DIPOLE FIELD BY A FINITE CONDUCTING AND A FINITE IMPEDANCE CYLINDER

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Abstract—Scattering of a spherical wave by a finite PEC and a finite impedance cylinder is studied theoretically. Source of excitation is spherical wave is produced by a dipole with arbitrary orientation. First we derive surface fields induced on an infinite cylinder when it is illuminated by an arbitrary oriented dipole. By the use of the currents on the infinite cylinder excited by an arbitrarily oriented dipole, approximate expressions for two components of the far-zone scattered electric field of a finite PEC and a finite impedance cylinder are derived. The validity of the approximation depends on the conditions that the cylinder must be long compared to the wavelength. The theoretical results are compared to the available results in the literature.

1. INTRODUCTION

Scattering of electromagnetic wave by a circular cylinder is a classical problem and it has been studied by many workers. Much of the information on the theoretical and numerical results is available [1–49]. The problem of scattering of a dipole field by a perfectly conducting infinite cylinder has been of great importance in the past and as such has received considerable attention. It was recognized, for instance, that the directive properties of linear antennas could be improved considerably by situating them in the vicinity of a conducting cylinder of the proper radius. Even more directive patterns were obtained by using arrays of such antennas fed with proper phase. Such patterns are given in the work of Carter [43]. His method consists of obtaining the far field directly by means of reciprocity between the radiating element and a passive receiving dipole situated in the far field of the radiator. Luke [44] obtained essentially the same results using the Green's function method. Moullin [45] has related results in his book and LePage, Harrington and Schlecht [46, 47] and Walsh [48] consider similar problems. A radial dipole in a cylindrical wedge region is considered by Wait [49] and theoretical and experimental radiation patterns in the principal plane are compared. However, studies on the scattering of a dipole wave by an impedance cylinder seems to be very few, especially for the cylinder with finite length. This problem is a simple model of radio hazard due to the higher building with circular cross section and is important practically and theoretically.

The configuration of interest here is an asymmetrical structure; the exciting dipole is not necessarily situated in the plane equidistant from the two ends of the cylinder. Therefore, the radiation pattern is not necessarily symmetrical about the mid plane of the system. We divide our paper in six sections. In Section 2, we consider the scattering of dipole field from infinite PEC cylinder. In Section 3, scattering of dipole field from finite PEC cylinder is considered utilizing the results of infinite PEC cylinder. In Section 4, we find the scattering of dipole field from infinite impedance cylinder. In Section 5, the scattering of dipole field from finite impedance cylinder is obtained. In Section 6, the results and discussions are presented.

2. INFINITE PEC CIRCULAR CYLINDER

First we obtain the surface fields of an infinitely long PEC circular cylinder which is excited by an arbitrarily oriented dipole. Later these surface fields have been utilized to study the scattering from a finite PEC and impedance cylinder. The source of excitation is small electric dipole. Field produced by an isolated dipole has been termed as primary field. Field produced due to the presence of cylinder is termed as a secondary field.

The primary field created by the dipole is derived in Appendix A. Our interest is to determine the unknown secondary field. Enforcement of the boundary conditions on the surface of the cylinder yield the secondary field contribution. As our interest is to consider the orientation of dipole in all cylindrical coordinates directions, so first consider the dipole is ρ-directed.

(I) ρ-Directed Dipole Field

The Fourier transform of the vector potential has been denoted by $\widetilde{A}_\rho$ and $\widetilde{A}_\phi$. We decompose each transform into two parts

$$\widetilde{A}_{\rho n} = \widetilde{A}^0_{\rho n} + \widetilde{A}^1_{\rho n}, \qquad \widetilde{A}_{\phi n} = \widetilde{A}^0_{\phi n} + \widetilde{A}^1_{\phi n} \tag{1a}$$

where $\widetilde{A}^0_{\rho n}$ and $\widetilde{A}^0_{\phi n}$ are the functions associated with situation in the absence of the cylinder. It may be noted that $\widetilde{A}_{\ell n}$ are the coefficients of Fourier series of the transformed function $\widetilde{A}_\ell$, where $\ell = \rho, \phi$ etc. The expressions are given by (A7e) and (A7f). On the other hand $\widetilde{A}^1_{\rho n}$ and $\widetilde{A}^1_{\phi n}$ are associated with the secondary field contribution due to the presence of circular cylinder. We may assume secondary field contribution in the form of unknowns

$$\widetilde{A}^1_{\rho n} = \frac{\mu J_\rho}{4j}\left\{a_n H_n^{(2)\prime}(\chi\rho) + b_n \frac{n}{\chi\rho} H_n^{(2)}(\chi\rho)\right\} \tag{1b}$$

$$\widetilde{A}^1_{\phi n} = \frac{\mu J_\rho}{4}\left\{c_n H_n^{(2)\prime}(\chi\rho) + d_n \frac{n}{\chi\rho} H_n^{(2)}(\chi\rho)\right\} \tag{1c}$$

By enforcing the boundary conditions, on the surface of PEC circular cylinder, given by

$$E_\phi = 0, \quad E_z = 0 \qquad \text{at} \quad \rho = a \tag{1d}$$

This condition is found to be satisfied if we set $\widetilde{A}_{\phi n} = 0$ and $\frac{\partial}{\partial\rho}(\rho\widetilde{A}_{\rho n}) = 0$. Hence we have

$$a_n = d_n = -H_n^{(2)\prime}(\chi\rho_0)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}$$

$$b_n = c_n = -\frac{n}{\chi\rho_0} H_n^{(2)}(\chi\rho_0)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}$$

The vector potential is obtained as follows.

(i) $\rho > \rho_0$:

$$\begin{aligned} A_\rho = {}& \frac{\mu J_\rho}{8\pi j}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\ & \times\int_{-\infty}^{\infty}\left[\left\{\frac{n}{\chi\rho_0}J_n(\chi\rho_0) - \frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}\right\}\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right. \\ & \left. + \left\{J_n{}'(\chi\rho_0) - H_n^{(2)\prime}(\chi\rho_0)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho)\right]\exp[-jh(z-z_0)]dh \end{aligned} \tag{2a}$$

$$
\begin{aligned}
A_\phi &= \frac{\mu J_\rho}{8\pi}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\
&\times\int_{-\infty}^{\infty}\left[\left\{\frac{n}{\chi\rho_0}J_n(\chi\rho_0)-\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho)\right. \\
&+\left.\left\{J_n{}'(\chi\rho_0)-H_n^{(2)\prime}(\chi\rho_0)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right]\exp[-jh(z-z_0)]dh
\end{aligned}
\tag{2b}
$$

(ii) $\rho < \rho_0$:

$$
\begin{aligned}
A_\rho &= \frac{\mu J_\rho}{8\pi j}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\
&\times\int_{-\infty}^{\infty}\left[\left\{\frac{n}{\chi\rho}J_n(\chi\rho)-\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}\right\}\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\right. \\
&+\left.\left\{J_n{}'(\chi\rho)-H_n^{(2)\prime}(\chi\rho)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho_0)\right]\exp[-jh(z-z_0)]dh
\end{aligned}
\tag{3a}
$$

$$
\begin{aligned}
A_\phi &= \frac{\mu J_\rho}{8\pi}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\
&\times\int_{-\infty}^{\infty}\left[\left\{J_n'(\chi\rho)-H_n^{(2)\prime}(\chi\rho)\frac{J_n'(\chi a)}{H_n^{(2)}(\chi a)}\right\}\frac{n}{\chi\rho_0}H_n^{(2)\prime}(\chi\rho_0)\right. \\
&+\left.\frac{n}{\chi\rho}\left\{J_n(\chi\rho)-H_n^{(2)}(\chi\rho)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho_0)\right]\exp[-jh(z-z_0)]dh
\end{aligned}
\tag{3b}
$$

(II) ϕ-Directed Dipole Field

We can obtain the results quite similarly to the case of ρ-directed dipole. These are given by

(i) $\rho > \rho_0$:

$$
\begin{aligned}
A_\rho &= -\frac{\mu J_\phi}{8\pi}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\
&\times\int_{-\infty}^{\infty}\left[\left\{\frac{n}{\chi\rho_0}J_n(\chi\rho_0)-\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho)\right.
\end{aligned}
$$

$$+\left\{J_n{}'(\chi\rho_0)-H_n^{(2)\prime}(\chi\rho_0)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}\right\}\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\Bigg]\exp[-jh(z-z_0)]dh \tag{4a}$$

$$\begin{aligned}A_\phi \;=\; & -\frac{j\mu J_\phi}{8\pi}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\ & \times\int_{-\infty}^{\infty}\left[\left\{\frac{n}{\chi\rho_0}J_n(\chi\rho_0)-\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right. \\ & \left.+\left\{J_n{}'(\chi\rho_0)-H_n^{(2)\prime}(\chi\rho_0)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho)\right]\exp[-jh(z-z_0)]dh\end{aligned} \tag{4b}$$

(ii) $\rho < \rho_0$:

$$\begin{aligned}A_\rho \;=\; & -\frac{\mu J_\phi}{8\pi}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\ & \times\int_{-\infty}^{\infty}\left[\left\{J_n'(\chi\rho)-H_n^{(2)\prime}(\chi\rho)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\right. \\ & \left.+\frac{n}{\chi\rho}\left\{J_n(\chi\rho)-H_n^{(2)}(\chi\rho)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho_0)\right]\exp[-jh(z-z_0)]dh\end{aligned} \tag{5a}$$

$$\begin{aligned}A_\phi \;=\; & \frac{\mu J_\phi}{8\pi j}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\ & \times\int_{-\infty}^{\infty}\left[\left\{\frac{n}{\chi\rho}J_n(\chi\rho)-\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\}\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\right. \\ & \left.+\left\{J_n'(\chi\rho)-H_n^{(2)\prime}(\chi\rho)\frac{J_n'(\chi a)}{H_n^{(2)\prime}(\chi a)}\right\}H_n^{(2)\prime}(\chi\rho_0)\right]\exp[-jh(z-z_0)]dh\end{aligned} \tag{5b}$$

(III) z-Directed Dipole Field

(i) $\rho > \rho_0$

$$\begin{aligned}A_z \;=\; & \frac{\mu J_z}{8\pi j}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)]\int_{-\infty}^{\infty}\left\{J_n(\chi\rho_0)-H_n^{(2)}(\chi\rho_0)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\right\} \\ & \times H_n^{(2)}(\chi\rho)\exp[-jh(z-z_0)]dh\end{aligned} \tag{6a}$$

(ii) $\rho < \rho_0$

$$A_z = \frac{\mu J_z}{8\pi j} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \int_{-\infty}^{\infty} \left\{ J_n(\chi\rho) - H_n^{(2)}(\chi\rho) \frac{J_n(\chi a)}{H_n^{(2)}(\chi a)} \right\} \times H_n^{(2)}(\chi\rho_0) \exp[-jh(z - z_0)] dh \tag{6b}$$

2.1. Magnetic Field on the Surface of Perfectly Conducting Cylinder

The magnetic field on the surface is readily obtained from the vector potential given in (3a), (3b), (5a), (5b) and (6b). The result is given as follows.

(I) ρ**- Directed Dipole Field**

$$H_\phi = \frac{jJ_\rho}{4\pi^2 a} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \times \int_{-\infty}^{\infty} \frac{h}{\chi} \left\{ \frac{n^2}{\chi a \chi\rho_0} \frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)\prime}(\chi a)} - \frac{H_n^{(2)\prime}(\chi\rho_0)}{H_n^{(2)}(\chi a)} \right\} \exp[-jh(z - z_0)] dh \tag{7a}$$

$$H_z = \frac{jJ_\rho}{4\pi^2 a} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \times \int_{-\infty}^{\infty} \frac{n}{\chi\rho_0} \frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)\prime}(\chi a)} \exp[-jh(z - z_0)] dh \tag{7b}$$

(II) ϕ**- Directed Dipole Field**

$$H_\phi = \frac{J_\phi}{4\pi^2 a} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \times \int_{-\infty}^{\infty} \frac{h}{\chi} \left\{ \frac{n}{\chi a} \frac{H_n^{(2)\prime}(\chi\rho_0)}{H_n^{(2)\prime}(\chi a)} - \frac{n}{\chi\rho_0} \frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)}(\chi a)} \right\} \exp[-jh(z - z_0)] dh \tag{8a}$$

$$H_z = \frac{J_\phi}{4\pi^2 a} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \int_{-\infty}^{\infty} \frac{H_n^{(2)\prime}(\chi\rho_0)}{H_n^{(2)\prime}(\chi a)} \exp[-jh(z - z_0)] dh \tag{8b}$$

(III) z- Directed Dipole Field

$$H_\phi = -\frac{J_z}{4\pi^2 a}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)]\int_{-\infty}^{\infty}\frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)}(\chi a)}\exp[-jh(z-z_0)]dh \tag{9a}$$

$$H_z = 0 \tag{9b}$$

2.2. Far Field Scattered by an Infinitely Long Conducting Cylinder

Interest is to evaluate the integral representation of the vector potential when the observation point is far from the origin of the cylindrical coordinate system. Consider a useful and simple integral of the form given by

$$I = \int_{-\infty}^{\infty} H_n^{(2)}(\chi\rho)H_n^{(2)}(\chi\rho_0)\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}\exp[-jh(z-z_0)]dh \tag{10a}$$

Hankel function $H_n^{(2)}(\chi\rho)$ is replaced by its asymptotic approximation

$$H_n^{(2)}(\chi\rho) \simeq \sqrt{\frac{2}{\pi\chi\rho}}\exp\left[-j\left(\chi\rho-\frac{2n+1}{4}\pi\right)\right] \tag{10b}$$

Then integral I becomes

$$\begin{aligned} I &= \sqrt{\frac{2}{\pi\rho}}\exp\left[j\left(\frac{2n+1}{4}\pi\right)\right] \\ &\quad\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{\chi}}\frac{J_n(\chi a)}{H_n^{(2)}(\chi a)}H_n^{(2)}(\chi\rho_0)\exp[-j\chi\rho-jh(z-z_0)]dh \end{aligned} \tag{10c}$$

Now we transform the variable by $\rho = R\sin\theta$, $z-z_0 = R\cos\theta$; $h = k\cos\alpha$ and $\chi = k\sin\alpha$, hence I becomes

$$\begin{aligned} I &= \sqrt{\frac{2}{\pi\rho}}\exp\left[j\left(\frac{2n+1}{4}\pi\right)\right] \\ &\quad\times\int_C\frac{J_n(ka\sin\alpha)}{H_n^{(2)}(ka\sin\alpha)}H_n^{(2)}(k\rho_0\sin\alpha)\sqrt{k\sin\alpha}\exp\Big[-jkR\cos(\theta-\alpha)\Big]d\alpha \end{aligned} \tag{10d}$$

where C is the contour given by $[-j\infty, \pi+j\infty]$. Application of the steepest descent method yields

$$I \simeq \frac{2}{R}\exp\left[-jkR+j\frac{(n+1)\pi}{2}\right]\frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}H_n^{(2)}(k\rho_0\sin\theta) \tag{11a}$$

When $\rho_0 \gg a$, I is simplified to

$$I \simeq \frac{2\sqrt{2}}{\sqrt{\pi k R \rho \rho_0}} \exp\left[-jkR - jk\rho_0 \sin\theta + jn\pi + j\frac{3\pi}{4}\right] \frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)} \tag{11b}$$

Integral form considered in (10a) and its corresponding far-fields are very useful in evaluation of the far-fields corresponding to (2), (4) and (5).

Far field expressions corresponding to (2), (4) and (5) can be obtained by comparing our expressions with (11a) and by substituting

$$\chi = k\sin\theta, \quad h = k\cos\theta; \quad H_n^{(2)\prime}(\chi\rho) \simeq -jH_n^{(2)}(k\rho\sin\theta) \tag{11c}$$

The results are given as follows.

(I) ρ-Directed Dipole Field

$$A_\rho^0 = \mu J_\rho \frac{\exp(-jkR)}{4\pi R}\cos(\phi-\phi_0), \quad A_\phi^0 = \mu J_\rho \frac{\exp(-jkR)}{4\pi R}\sin(\phi-\phi_0) \tag{12a}$$

$$\begin{aligned} A_\rho^1 \simeq& -\frac{\mu J_\rho}{4\pi R}\exp[-jkR]\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0+\frac{\pi}{2}\right)\right] \\ &\times\left\{\frac{n}{k\rho_0\sin\theta}\frac{n}{k\rho\sin\theta}\frac{J_n'(ka\sin\theta)}{H_n^{(2)\prime}(ka\sin\theta)}H_n^{(2)}(k\rho_0\sin\theta)\right. \\ &\left.-j\frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}H_n^{(2)\prime}(k\rho_0\sin\theta)\right\} \end{aligned} \tag{12b}$$

$$\begin{aligned} A_\phi^1 \simeq& -\frac{\mu J_\rho}{4\pi R}\exp[-jkR]\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0+\frac{\pi}{2}\right)\right] \\ &\times\left\{\frac{n}{k\rho_0\sin\theta}\frac{J_n'(ka\sin\theta)}{H_n^{(2)\prime}(ka\sin\theta)}H_n^{(2)}(k\rho_0\sin\theta)\right. \\ &\left.+\frac{jn}{k\rho\sin\theta}\frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}H_n^{(2)\prime}(k\rho_0\sin\theta)\right\} \end{aligned} \tag{12c}$$

(II) ϕ-Directed Dipole Field

$$A_\rho^0 = \mu J_\phi \frac{\exp(-jkR)}{4\pi R}\sin(\phi-\phi_0), \quad A_\phi^0 = \mu J_\phi \frac{\exp(-jkR)}{4\pi R}\cos(\phi-\phi_0) \tag{13a}$$

$$A_\rho^1 \simeq \frac{\mu J_\phi}{4\pi R}\exp\left[-jkR\right]\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0+\frac{\pi}{2}\right)\right]$$
$$\times\left\{\frac{n}{k\rho_0\sin\theta}\frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}H_n^{(2)}(k\rho_0\sin\theta)\right.$$
$$\left.+\frac{jn}{k\rho\sin\theta}\frac{J_n'(ka\sin\theta)}{H_n^{(2)\prime}(ka\sin\theta)}H_n^{(2)\prime}(k\rho_0\sin\theta)\right\} \tag{13b}$$

$$A_\phi^1 \simeq \frac{\mu J_\phi}{4\pi R}\exp\left[-jkR\right]\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0+\frac{\pi}{2}\right)\right]$$
$$\times\left\{-\frac{n}{k\rho_0\sin\theta}\frac{n}{k\rho\sin\theta}\frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}H_n^{(2)}(k\rho_0\sin\theta)\right.$$
$$\left.+j\frac{J_n'(ka\sin\theta)}{H_n^{(2)\prime}(ka\sin\theta)}H_n^{(2)\prime}(k\rho_0\sin\theta)\right\} \tag{13c}$$

(III) z-Directed Dipole Field

$$A_z^0 = \mu J_z\frac{\exp(-jkR)}{4\pi R} \tag{14a}$$

$$A_z^1 \simeq -\frac{\mu J_z}{4\pi R}\exp\left[-jkR\right]$$
$$\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0+\frac{\pi}{2}\right)\right]\frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}H_n^{(2)}(k\rho_0\sin\theta) \tag{14b}$$

3. FINITE PEC CIRCULAR CYLINDER

The determination of the far-zone radiation from an arbitrarily oriented dipole near a finite perfectly conducting cylinder is of interest from both a theoretical and a practical point of view. From a theoretical aspect, the problem presents a distinct mathematical difficulty; the surface of the perfectly conducting cylinder cannot be described by assigning a constant value to one of the coordinates. A rigorous solution to the problem is, therefore, very difficult since the boundary condition of zero tangential electric field must be imposed over parts of two different coordinate surfaces. However, if the cylinder is long compared to a radius and wavelength an approximate solution to the problem is obtained [50].

In solving such a problem, the main issue is to determine the current distribution induced on the cylinder, therefore, it is necessary in the approximate treatment of the long finite cylinder to have available the expressions for the currents on the infinite cylinder. Using a cylindrical coordinate system with the cylinder axis and the z-axis coincident, and the dipole at $\rho = \rho_0$, $\phi = \phi_0$, the magnetic field at the surface of the cylinder ($\rho = a$) has been shown in Eqs. (7)–(9). The surface current densities on the cylinder are then

$$J_z = H_\phi(\rho = a) \qquad J_\phi = -H_z(\rho = a) \tag{15}$$

and the currents on the cylinder are completely determined.

3.1. The Finite Cylinder Excited by an Arbitrarily Oriented Dipole

The determination of the far-zone field of a finite cylinder, i.e., a cylinder not of infinite extent in the $\pm z$ direction, when excited by an arbitrarily electric dipole presents greater difficulty than the case of the infinite cylinder. The case of the finite cylinder involves complicated boundary conditions in that the tangential electric field must vanish over parts of two different coordinate surfaces. As shown in Fig. 1, the tangential field must vanish at $\rho = a$ for $-l_1 < z < l_2$ and also at $z = l_2, z = -l_1$ for $\rho < a$. These boundary conditions introduce considerable complications and usually the solutions to problems of this sort involve some method of approximation.

Frequently the far-zone field expres-sions for the infinite cylinder are used as an approximation to the case of a finite cylinder. However, this procedure yields poor results, as has been pointed out by [50], especially near the axis of the cylinder where the expression for the θ-component of the electric field of the infinite cylinder diverges. Also the minor lobes which are found to exist for the finite cylinder of small radius are not predicted in this approximation. In present work our procedure does not yield these problems.

In order to carry out a more accurate approximate solution to this problem the current distribution on the finite cylinder will be taken identical to the distribution on the infinite cylinder. The procedure is as follows: First the current distribution on the infinite cylinder is determined. Then the portions of the infinite cylinder above $z = l_2$ and below $z = -l_1$ are removed with the assumption that the current on the remaining portion of the cylinder is unchanged. The far zone field of this unchanged portion of the current between $-l_1$ and l_2 is then computed and added to the field of the dipole to obtain an expression for the total far-zone field of the configuration. A similar approximation has been used by Meixner [51] in treating the radiation from a slit in a finite plane.

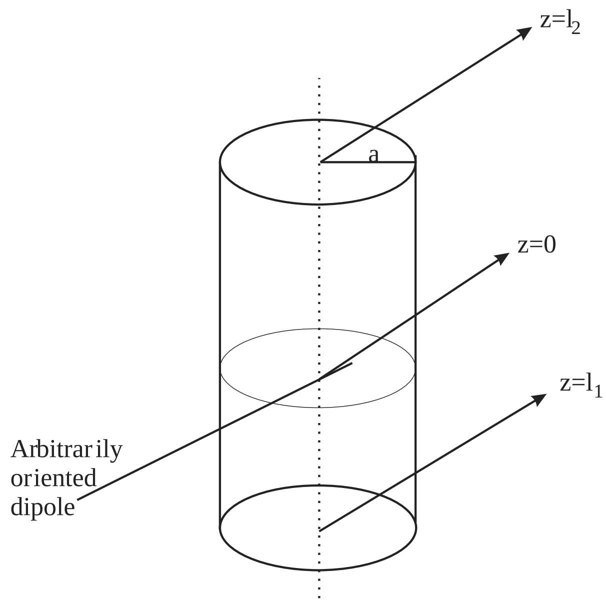

Figure 1. Finite cylinder with arbitrarily oriented dipole.

Following two proportionalities hold for the surface current density [50]:

$$J_z \alpha \frac{e^{-jkz}}{\ln\left(\dfrac{kz}{k^2a^2}\right)}$$

$$J_\phi \alpha \frac{\sin\phi e^{-jkz}}{(kz)^2}$$

where J_z and J_ϕ are the z- and ϕ-components of the surface current densities on the cylinder. From above expressions, it is evident that at large value of kz, contribution of J_z and J_ϕ goes to zero. So the major portion of the radiation is then be produced by the central region of the infinite cylinder i.e., $-l_1 \leq z \leq l_2$ and the approximation used is justified for the conditions $kl_2 \gg 1$ and $kl_1 \gg 1$.

(I) ρ-Directed Dipole Field

The far-zone radiation field of a ρ-directed dipole near a finite cylinder will now be derived using the approximation discussed above. The physical configuration and coordinate system are shown in Fig. 2.

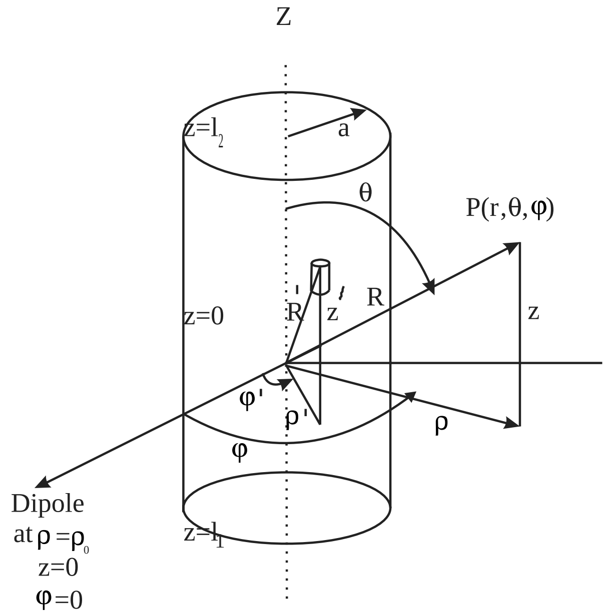

Figure 2. Finite cylinder with arbitrarily oriented dipole and coordinate system.

The field is to be calculated at an arbitrary point P. The field due to current on the cylinder will first be derived and the dipole field will be added to give the total field. The far-zone ϕ-component of electric field due to a surface current density $\mathbf{J}$ flowing over an area ds is [55]

$$A_\phi = \frac{\mu_0}{4\pi r} \int \int_S \mathbf{J}(\rho', \phi', z').\mathbf{i}_\phi e^{-jk\mathbf{r}'.\mathbf{i}_r} \tag{16}$$

where $\mathbf{i}_r$ and $\mathbf{i}_\phi$ are unit vectors in the r- and ϕ-direction respectively. For the cylinder of Fig. 1, (15) becomes

$$\begin{aligned} E_\phi &= \frac{j\omega^2 \mu_0 a e^{-jkr}}{4\pi r} \int_{-l_1}^{l_2} \\ &\quad \times \int_0^{2\pi} H_z(\rho' = a) \cos(\phi' - \phi) e^{jka\cos(\phi'-\phi)\sin\theta + z'\cos\theta} d\phi' dz' \end{aligned} \tag{17}$$

where M is the dipole moment. Substituting H_z into (16), one obtains

$$E_\phi = \frac{M\omega^2 \mu_0 a e^{-jkr}}{8\pi^3 r \rho_0} \sum_{n=1}^{\infty} n \int_0^{2\pi} \sin n\phi' \cos(\phi' - \phi) e^{jka\cos(\phi'-\phi)\sin\theta} d\phi'$$

$$\times \int_{-l_1}^{l_2} \int_C \frac{H_n^{(2)}(\chi\rho_0)}{\chi a H_n^{(2)\prime}(\chi a)} e^{-jkhz'\cos\theta} e^{jkz'\cos\theta} dh dz' \tag{18}$$

It can be shown that first integral in (17) is given by

$$\int_0^{2\pi} \sin n\phi' \cos(\phi'-\phi) e^{jka\cos(\phi'-\phi)\sin\theta} d\phi' = 2\pi j^{n-1} \sin n\phi J_n'(ka\sin\theta) \tag{19}$$

Therefore (17) becomes

$$E_\phi = \frac{M\omega^2\mu_0 a e^{-jkr}}{4\pi^2 r\rho_0} \sum_{n=1}^{\infty} n j^{n-1} \sin n\phi J_n'(ka\sin\theta)$$
$$\times \int_{-l_1}^{l_2} \int_C \frac{H_n^{(2)}(\chi\rho_0)}{\chi a H_n^{(2)\prime}(\chi a)} e^{-jkhz'\cos\theta} e^{jkz'\cos\theta} dh dz' \tag{20}$$

Performing the integration over z' in (19) and making the substitution $h = kh'$, one obtains

$$E_\phi = \frac{Mke^{-jkr}}{4\pi^2\epsilon_0 r\rho_0} \sum_{n=1}^{\infty} n j^n \sin n\phi J_n'(ka\sin\theta)$$
$$\times \int_{C'} \frac{H_n^{(2)}(k\rho_0\sqrt{1-h^2})[e^{ikl_2(h'-\cos\theta)} - e^{-ikl_1(h'-\cos\theta)}]}{H_n^{(2)\prime}(ka\sqrt{1-h'^2})\sqrt{1-h'^2}(h-\cos\theta)} dh' \tag{21}$$

in adding the contribution to E_ϕ from the dipole alone, the total ϕ-component of the far-zone field becomes

$$E_{\phi_{total}} = -\frac{Mke^{-jkr}}{4\pi\epsilon_0 r}$$
$$\times \left[k\sin\phi e^{jk\rho_0\sin\theta\cos\phi} - \frac{1}{k\rho_0}\sum_{n=1}^{\infty} n j^n \sin n\phi J_n'(ka\sin\theta)\Psi_n\right] \tag{22}$$

where

$$\Psi_n = \int_{C'} \frac{H_n^{(2)}(k\rho_0\sqrt{1-h'^2})[e^{jkl_2(h'-\cos\theta)} - e^{-jkl_1(h'-\cos\theta)}]}{H_n^{(2)\prime}(ka\sqrt{1-h'^2})\sqrt{1-h'^2}(h'-\cos\theta)} dh' \tag{23}$$

The far-zone θ-component of electric field due to surface current density $\mathbf{J}$ is

$$E_\theta = -\frac{j\omega\mu_0 e^{-jkr}}{4\pi r} \int_S \mathbf{J}.\mathbf{i}_\theta e^{jk\mathbf{r}'.\mathbf{i}_r} dS' \tag{24}$$

where $\mathbf{i}_\theta$ and $\mathbf{i}_r$ are unit vectors in the θ- and r-directions, respectively. For the cylinder of Fig. 2, Eq. (23) becomes

$$E_\theta = \frac{j\omega\mu_0 a e^{jkr}}{4\pi r}\int_{-l_1}^{l_2}\int_0^{2\pi} H_\phi(\rho' = a)\sin\theta e^{jk[a\cos(\phi'-\phi)\sin\theta + z'\cos\theta]}d\phi' dz' \tag{25}$$

Substituting H_ϕ into (24), one obtains

$$\begin{aligned} E_\theta = & \frac{jM\omega^2\mu_0 a\sin\theta e^{-jkr}}{16\pi^3 r}\sum_{n=0}^{\infty}\epsilon_n\int_0^{2\pi}\cos n\phi' e^{jka\cos(\phi'-\phi)\sin\theta}d\phi' \\ & \times\int_{-l_1}^{l_2}\int_C\frac{h}{\chi a}\left[\frac{H_n^{(2)\prime}(\chi\rho_0)}{H_n^{(2)}(\chi a)} - \frac{n^2}{\chi^2 a\rho_0}\frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)\prime}(\chi a)}\right]e^{-jhz'}e^{jkz'\cos\theta}dhdz' \end{aligned} \tag{26}$$

It can be shown that the first integral in (25) is given by

$$\int_0^{2\pi}\cos n\phi' e^{-jka\cos(\phi'-\phi)\sin\theta}d\phi' = 2\pi j^n\cos n\phi J_n(ka\sin\theta)$$

Thus (25) becomes

$$\begin{aligned} E_\theta \ = \ & \frac{jM\omega^2\mu_0 a\sin\theta e^{-jkr}}{8\pi^2 r}\sum_{n=0}^{\infty}\epsilon_n j^n\cos n\phi J_n(ka\sin\theta) \\ & \times\int_{-l_1}^{l_2}\int_C\frac{h}{\chi a}\left[\frac{H_n^{(2)\prime}(\chi\rho_0)}{H_n^{(2)}(\chi a)} - \frac{n^2}{\chi^2 a\rho_0}\frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)\prime}(\chi a)}\right]e^{-jhz'}e^{jkz'\cos\theta}dhdz' \end{aligned} \tag{27}$$

Performing the integration over z' and letting $h = kh'$, one obtains

$$E_\theta \ = \ \frac{Mk^2\sin\theta e^{jkr}}{8\pi^2\epsilon_0 r}\sum_{n=0}^{\infty}\epsilon_n j^n\cos n\phi J_n(ka\sin\theta)\gamma_n \tag{28}$$

where

$$\begin{aligned} \gamma_n \ = \ & \int_{C'}\frac{h'}{\sqrt{1-h'^2}} \\ & \times\left[\frac{H_n^{(2)\prime}(k\rho_0\sqrt{1-h'^2})}{H_n^{(2)}(ka\sqrt{1-h'^2})} - \frac{n^2}{k^2 a\rho_0(1-h'^2)}\frac{H_n^{(2)}(k\rho_0\sqrt{1-h'^2})}{H_n^{(2)\prime}(ka\sqrt{1-h'^2})}\right] \\ & \times\frac{[e^{jkl_2(h'-\cos\theta)} - e^{-jkl_1(h'-\cos\theta)}]}{(h'-\cos\theta)}dh' \end{aligned} \tag{29}$$

Adding the contribution to E_θ due to the dipole alone, the total θ-component of the far-zone electric field becomes

$$E_{\theta_{total}} = \frac{Mk^2 e^{-jkr}}{4\pi\epsilon_0 r} \times \left[\cos\theta\cos\phi e^{jk\rho_0\sin\theta\cos\phi} - \frac{\sin\theta}{2\pi}\sum_{n=0}^{\infty}\epsilon_n j^n \cos n\phi J_n(ka\sin\theta)\gamma_n\right] \tag{30}$$

(II) ϕ-Directed Dipole Field

We write only results.

$$E_{\phi_{total}} = -\frac{Mke^{-jkr}}{4\pi\epsilon_0 r} \times \left[k\cos\phi e^{jk\rho_0\sin\theta\cos\phi} - \frac{1}{2\pi}\sum_{n=0}^{\infty}\epsilon_n j^n \cos n\phi J_n'(ka\sin\theta)\Psi_n\right] \tag{31}$$

where

$$\Psi_n = \int_{C'} \frac{H_n^{(2)\prime}(k\rho_0\sqrt{1-h'^2})[e^{ikl_2(h'-\cos\theta)} - e^{-ikl_1(h'-\cos\theta)}]}{H_n^{(2)\prime}(ka\sqrt{1-h'^2})(h'-\cos\theta)} dh' \tag{32}$$

and

$$E_{\theta_{total}} = \frac{Mk^2 e^{-jkr}}{4\pi\epsilon_0 r} \times \left[\cos\theta\sin\phi e^{jk\rho_0\sin\theta\cos\phi} - \frac{\sin\theta}{\pi}\sum_{n=1}^{\infty} j^n \sin n\phi J_n(ka\sin\theta)\gamma_n\right] \tag{33}$$

where

$$\gamma_n = \int_{C'} \frac{h'}{\sqrt{1-h'^2}} \times \left[\frac{n}{ka\sqrt{1-h^2}}\frac{H_n^{(2)\prime}(k\rho_0\sqrt{1-h'^2})}{H_n^{(2)\prime}(ka\sqrt{1-h'^2})} - \frac{n}{k\rho_0(1-h'^2)}\frac{H_n^{(2)}(k\rho_0\sqrt{1-h'^2})}{H_n^{(2)}(ka\sqrt{1-h'^2})}\right] \times \frac{[e^{jkl_2(h'-\cos\theta)} - e^{-jkl_1(h'-\cos\theta)}]}{(h'-\cos\theta)} dh' \tag{34}$$

(iii) z-Directed Dipole Field

$$E_{\theta_{total}} = \frac{Mk^2 e^{-jkr}}{4\pi\epsilon_0 r}\left[\sin\theta + \frac{1}{2\pi}\sum_{n=0}^{\infty}\epsilon_n j^n \cos n\phi J_n(ka\sin\theta)\gamma_n\right] \quad (35)$$

where

$$\gamma_n = \int_{C'} \frac{H_n^{(2)}(k\rho_0\sqrt{1-h'^2})}{H_n^{(2)}(ka\sqrt{1-h'^2})}\frac{\left[e^{jkl_2(h'-\cos\theta)} - e^{-jkl_1(h'-\cos\theta)}\right]}{(h'-\cos\theta)}dh' \quad (36)$$

4. INFINITE IMPEDANCE CIRCULAR CYLINDER

An arbitrarily oriented dipole current is placed in the vicinity of an infinitely long circular cylinder with surface impedance. The boundary conditions are given by

$$E_z = Z_1 H_\phi, \qquad E_\phi = -Z_2 H_z \quad (37)$$

It is assumed that surface impedance is anisotropic. The analytical procedure is quite similar to the case of perfectly conducting cylinder. In this section we derive the expression of the scattered field and the surface field of the cylinder.

4.1. Expressions for the Scattered Field

(I) ρ-Directed Dipole Field

To derive the field produced by ρ-directed dipole current, we need two components of vector potential. In the presence of an impedance cylinder, this situation does not change. The Fourier transform of the vector potential is written as

$$\widetilde{A}_{\rho n} = \widetilde{A}^0_{\rho n} + \widetilde{A}^1_{\rho n}, \qquad \widetilde{A}_{\phi n} = \widetilde{A}^0_{\phi n} + \widetilde{A}^1_{\phi n} \quad (38a)$$

where the superscripts "0" and "1" represent the primary and scattered waves, respectively. These are given by

$$\widetilde{A}^0_{\rho n} = \frac{\mu J_\rho}{4j}\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho) + H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi\rho)\right], \quad (38b)$$

$$\widetilde{A}^1_{\rho n} = \frac{\mu J_\rho}{4j}\left[a_n H_n^{(2)\prime}(\chi\rho) + b_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right], \quad (38c)$$

$$\widetilde{A}^0_{\phi n} = \frac{\mu J_\rho}{4}\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi\rho) + H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho)\right], \quad (38d)$$

$$\widetilde{A}^1_{\phi n} = \frac{\mu J_\rho}{4}\left[b_n H_n^{(2)\prime}(\chi\rho) + a_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right] \quad (38e)$$

where a_n and b_n are the expansion coefficients which are to be determined from the boundary conditions (36). The Fourier transform of the tangential components of the electromagnetic field is obtained from (37) [52, 53].

$$\widetilde{E}_{zn} = \frac{Z_0 J_\rho}{4j}\frac{h\chi}{k}\Big[H_n^{(2)\prime}(\chi\rho_0)J_n(\chi\rho) + a_n H_n^{(2)}(\chi\rho)\Big] \tag{39a}$$

$$\begin{aligned}\widetilde{H}_{\phi n} = -\frac{J_\rho}{4}h\Bigg\{&\Bigg[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho) + H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi\rho)\Bigg]\\ &+\Bigg[a_n H_n^{(2)\prime}(\chi\rho) + b_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\Bigg]\Bigg\}\end{aligned} \tag{39b}$$

$$\widetilde{H}_{zn} = -\frac{J_\rho}{4}\chi\Big[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n(\chi\rho) + b_n H_n^{(2)}(\chi\rho)\Big] \tag{39c}$$

$$\begin{aligned}\widetilde{E}_{\phi n} = -\frac{jkZ_0 J_\rho}{4}\Bigg\{&\Bigg[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi\rho) + b_n H_n^{(2)\prime}(\chi\rho)\Bigg]\\ &+\frac{h^2}{k^2}\frac{n}{\chi\rho}\Big[H_n^{(2)\prime}(\chi\rho_0)J_n(\chi\rho) + a_n H_n^{(2)}(\chi\rho)\Big]\Bigg\}\end{aligned} \tag{39d}$$

Enforcement of the boundary conditions (36) yields the equations

$$\begin{aligned}&j\chi\Big[H_n^{(2)\prime}(\chi\rho_0)J_n(\chi a) + a_n H_n^{(2)}(\chi a)\Big]\\ &\quad = k\zeta_1\Bigg\{\Bigg[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a) + H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi a)\Bigg]\\ &\qquad\quad + \Bigg[a_n H_n^{(2)\prime}(\chi a) + b_n\frac{n}{\chi a}H_n^{(2)}(\chi a)\Bigg]\Bigg\}\end{aligned}$$

$$\begin{aligned}&k^2\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi a) + b_n H_n^{(2)\prime}(\chi a)\right]\\ &\qquad + h^2\left[H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a) + a_n\frac{n}{\chi a}H_n^{(2)}(\chi a)\right]\\ &\quad = jk\chi\zeta_2\Big[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n(\chi a) + b_n H_n^{(2)}(\chi a)\Big]\end{aligned}$$

In the above equations, ζ_1 and ζ_2 are the normalized impedances related by $\zeta_1 = \frac{Z_1}{Z_0}$ and $\zeta_2 = \frac{Z_2}{Z_0}$ with an intrinsic impedance Z_0 in free space, respectively. These equations are solved for a_n and b_n resulting

$$a_n(\chi) = \frac{k}{\Delta_t(\chi)}\left\{\Big[k\Psi_n(\chi) - j\chi\zeta_2\Big]A(\chi a) + \zeta_1\frac{n}{\chi a}B(\chi a)\right\} \tag{40a}$$

$$b_n(\chi) = \frac{1}{\Delta_t(\chi)}\left\{\Big[j\chi - k\zeta_1\Psi_n(\chi)\Big]B(\chi a) - h^2\frac{n}{\chi a}A(\chi a)\right\} \tag{40b}$$

$$\Delta_t(\chi) = k\left\{\left[j\chi - k\zeta_1\Psi_n(\chi)\right]\left[k\Psi_n(\chi) - j\chi\zeta_2\right] + h^2\zeta_1\left(\frac{n}{\chi a}\right)^2\right\} \quad (40c)$$

$$\begin{aligned} A(\chi a) &= \frac{k\zeta_1}{H_n^{(2)}(\chi a)}\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a) + H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi a)\right] \\ &\quad - \frac{j\chi}{H_n^{(2)}(\chi a)}H_n^{(2)\prime}(\chi\rho_0)J_n(\chi a) \qquad (40d) \end{aligned}$$

$$\begin{aligned} B(\chi a) &= \frac{1}{H_n^{(2)}(\chi a)}\left\{jk\chi\zeta_2\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n(\chi a)\right. \\ &\quad \left. - k^2\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi a) - h^2H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a)\right\} \quad (40e) \end{aligned}$$

$$\Psi_n(\chi a) = \frac{H_n^{(2)\prime}(\chi a)}{H_n^{(2)}(\chi a)} \quad (40f)$$

(II) ϕ-Directed Dipole Field

When the dipole is oriented in ϕ-direction, two components A_ρ and A_ϕ are needed to express the electromagnetic field. The Fourier transforms of the these components are written as

$$\widetilde{A}_{\rho n} = \widetilde{A}^0_{\rho n} + \widetilde{A}^1_{\rho n}, \qquad \widetilde{A}_{\phi n} = \widetilde{A}^0_{\phi n} + \widetilde{A}^1_{\phi n} \quad (41a)$$

These are given by

$$\widetilde{A}^0_{\rho n} = -\frac{\mu J_\phi}{4}\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi\rho) + H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho)\right] \quad (41b)$$

$$\widetilde{A}^0_{\phi n} = \frac{\mu J_\phi}{4j}\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho) + H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi\rho)\right] \quad (41c)$$

$$\left.\begin{aligned} \widetilde{A}^1_{\rho n} &= -\frac{\mu J_\phi}{4}\left[b_nH_n^{(2)\prime}(\chi\rho) + a_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right], \\ \widetilde{A}^1_{\phi n} &= \frac{\mu J_\phi}{8j}\left[a_nH_n^{(2)\prime}(\chi\rho) + b_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right], \end{aligned}\right. \quad (41d)$$

The Fourier transform of the tangential components of the electromagnetic field is derived from (40) [52, 53].

$$\widetilde{E}_{zn} = -\frac{Z_0J_\phi}{4}\frac{h\chi}{k}\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n(\chi\rho) + b_nH_n^{(2)}(\chi\rho)\right] \quad (42a)$$

$$\begin{aligned} \widetilde{E}_{\phi n} &= -\frac{Z_0J_\phi}{4k}\left\{k^2\left[H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi\rho) + a_nH_n^{(2)\prime}(\chi\rho)\right]\right. \\ &\quad \left. + h^2\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho) + b_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right]\right\} \end{aligned}$$

$$\widetilde{H}_{\phi n} = \frac{jJ_\phi}{4}h\left\{\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi\rho)+H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho)\right]\right.$$
$$\left.+\left[b_nH_n^{(2)\prime}(\chi\rho)+a_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right]\right\} \tag{42c}$$

(42b)

$$\widetilde{H}_{zn} = \frac{jJ_\phi}{4}\chi\Big[H_n^{(2)\prime}(\chi\rho_0)J_n(\chi\rho)+a_nH_n^{(2)}(\chi\rho)\Big] \tag{42d}$$

From the boundary conditions (36) we obtain the following relations

$$k^2\left[H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi a)+a_nH_n^{(2)\prime}(\chi a)\right]$$
$$+h^2\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a)+b_n\frac{n}{\chi a}H_n^{(2)}(\chi a)\right]$$
$$= j\zeta_2k\chi\Big[H_n^{(2)\prime}(\chi\rho_0)J_n(\chi a)+a_nH_n^{(2)}(\chi a)\Big] \tag{43a}$$

$$j\chi\Big[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n(\chi a)+b_nH_n^{(2)}(\chi a)\Big]$$
$$= k\zeta_1\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi a)+H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a)\right.$$
$$\left.+b_nH_n^{(2)\prime}(\chi a)+a_n\frac{n}{\chi a}H_n^{(2)}(\chi a)\right] \tag{43b}$$

From these relations, the expansion coefficients a_n and b_n are determined as

$$a_n(\chi) = \frac{1}{\Delta_t(\chi)}\left\{\Big[j\chi-k\zeta_1\Psi_n(\chi a)\Big]A(\chi a)-h^2\frac{n}{\chi a}B(\chi a)\right\} \tag{44a}$$

$$b_n(\chi) = \frac{k}{\Delta_t(\chi)}\left\{\Big[k\Psi_n(\chi a)-j\zeta_2\chi\Big]B(\chi a)+\zeta_1\frac{n}{\chi a}A(\chi)\right\} \tag{44b}$$

$$A(\chi a) = \frac{1}{H_n^{(2)}(\chi a)}\left\{-k^2H_n^{(2)\prime}(\chi\rho_0)J_n'(\chi a)\right.$$
$$\left.-h^2\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a)+j\chi k\zeta_2H_n^{(2)\prime}(\chi\rho_0)J_n(\chi a)\right\} \tag{44c}$$

$$B(\chi) = \frac{1}{H_n^{(2)}(\chi a)}\left\{-j\chi\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n(\chi a)\right.$$
$$\left.+k\zeta_1\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi a)+H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi a}J_n(\chi a)\right]\right\} \tag{44d}$$

(III) z-Directed Dipole Field

When the dipole is oriented in z-direction, the primary field is derived from only A_z-component. But in the presence of the impedance cylinder we need one more component. We use here z-component of the vector potential F_z of the electric type. These are written as

$$\widetilde{A}_{zn} = \frac{\mu J_z}{4j}\Big[H_n^{(2)}(\chi\rho_0)J_n(\chi\rho) + a_n H_n^{(2)}(\chi\rho)\Big],\ \widetilde{F}_{zn} = \frac{\epsilon_0 Z_0 J_z}{4j} b_n H_n^{(2)}(\chi\rho) \tag{45a}$$

The Fourier transform of the tangential components of the electromagnetic field is given by [52, 53]

$$\widetilde{E}_{zn} = -\frac{Z_0 J_z}{4}\frac{\chi^2}{k}\Big[H_n^{(2)}(\chi\rho_0)J_n(\chi\rho) + a_n H_n^{(2)}(\chi\rho)\Big] \tag{45b}$$

$$\widetilde{E}_{\phi n} = -\frac{Z_0 J_z}{4}\left\{\frac{nh}{k\rho}\Big[H_n^{(2)}(\chi\rho_0)J_n(\chi\rho) + a_n H_n^{(2)}(\chi\rho)\Big] + j\chi b_n H_n^{(2)\prime}(\chi\rho)\right\} \tag{45c}$$

$$\widetilde{H}_{\phi n} = \frac{jJ_z}{4}\left\{\chi\left[H_n^{(2)}(\chi\rho_0)J_n'(\chi\rho) + a_n H_n^{(2)\prime}(\chi\rho)\right] + j\frac{nh}{k\rho} b_n H_n^{(2)}(\chi\rho)\right\} \tag{45d}$$

$$\widetilde{H}_{zn} = -\frac{J_z}{4}\frac{\chi^2}{k} b_n H_n^{(2)}(\chi\rho) \tag{45e}$$

It is noted that H_z component is not produced when the cylinder is perfect conductor. Hence the contribution proportional to the coefficient b_n is due to the surface impedance. From the boundary conditions (36), the equations for the expansion coefficients a_n and b_n are obtained by

$$\begin{aligned}&-\chi\Big[H_n^{(2)}(\chi\rho_0)J_n(\chi a) + a_n H_n^{(2)}(\chi a)\Big]\\ &\quad = j\zeta_1\left\{k\Big[H_n^{(2)}(\chi\rho_0)J_n'(\chi a) + a_n H_n^{(2)\prime}(\chi a)\Big] + j\frac{nh}{\chi a} b_n H_n^{(2)}(\chi a)\right\}\end{aligned} \tag{46a}$$

$$\begin{aligned}&-\left\{\frac{nh}{\chi a}\Big[H_n^{(2)}(\chi\rho_0)J_n(\chi a) + a_n H_n^{(2)}(\chi a)\Big] + jb_n k H_n^{(2)\prime}(\chi a)\right\}\\ &\quad = \zeta_2\chi b_n H_n^{(2)}(\chi a)\end{aligned} \tag{46b}$$

$$a_n(\chi) = \frac{1}{\Delta_z(\chi)}\left\{\left[\chi\zeta_2 + jk\Psi_n(\chi a)\right]A(\chi) + \zeta_1\frac{nh}{\chi a}B(\chi)\right\} \tag{47a}$$

$$b_n(\chi) = \frac{1}{\Delta_z(\chi)}\left\{\Big[\chi + jk\zeta_1\Psi_n(\chi a)\Big]B(\chi) - \frac{nh}{\chi a}A(\chi)\right\} \tag{47b}$$

$$\Delta_z(\chi a) = \Big[\chi + jk\zeta_1\Psi_n(\chi a)\Big]\left[\chi\zeta_2 + jk\Psi_n(\chi a)\right] + \zeta_1 h^2\left(\frac{n}{\chi a}\right)^2 \tag{47c}$$

$$A(\chi a) = \frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)}(\chi a)}\Big[-j\zeta_1 k J_n'(\chi a) - \chi J_n(\chi a)\Big] \tag{47d}$$

$$B(\chi a) = -\frac{nh}{\chi a}\frac{H_n^{(2)}(\chi\rho_0)}{H_n^{(2)}(\chi a)} J_n(\chi a) \tag{47e}$$

4.2. Far Scattered Field

Once the unknown coefficients $a_n(\chi)$ and $b_n(\chi)$ are determined, the scattered field in the far region may be obtained by applying the steepest descent method [54]. We write only the results.

(I) ρ-Directed Dipole Field

$$\begin{aligned} A_\rho &= \frac{\mu J_\rho}{8\pi j}\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0\right)\right] \\ &\quad\times\int_{-\infty}^{\infty}\left[a_n H_n^{(2)\prime}(\chi\rho) + b_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right]\exp\Big[-jh(z-z_0)\Big]dh \\ &\simeq \frac{\mu J_\rho}{4\pi}\frac{\exp(-jkR)}{R}\sum_{n=-\infty}^{\infty}a_n(k\sin\theta)\exp\left[jn(\phi-\phi_0)+j\frac{n\pi}{2}\right] \end{aligned} \tag{48a}$$

$$\begin{aligned} A_\phi &= \frac{\mu J_\rho}{8\pi}\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0\right)\right] \\ &\quad\times\int_{-\infty}^{\infty}\left[b_n H_n^{(2)\prime}(\chi\rho) + a_n\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho)\right]\exp\Big[-jh(z-z_0)\Big]dh \\ &\simeq j\frac{\mu J_\rho}{4\pi}\frac{\exp(-jkR)}{R}\sum_{n=-\infty}^{\infty}b_n(k\sin\theta)\exp\left[jn(\phi-\phi_0)+j\frac{n\pi}{2}\right] \end{aligned} \tag{48b}$$

where we assume that $kR\sin\theta \gg 1$. $a_n(ka\sin\theta)$ and $b_n(ka\sin\theta)$ mean that we set $\chi a = ka\sin\theta$ and $ha = ka\cos\theta$ in the coefficients given by (33a) and (33b). In the far region, electric field components are obtained from $E_\theta \simeq -j\omega A_\theta$ and $E_\phi \simeq -j\omega A_\phi$.

(II) ϕ-Directed Dipole Field

$$A_\rho \simeq -j\frac{\mu J_\phi}{4\pi}\frac{\exp(-jkR)}{R}\sum_{n=-\infty}^{\infty}b_n(k\sin\theta)\exp\left[jn\left(\phi-\phi_0\right)+j\frac{n\pi}{2}\right] \tag{49a}$$

$$A_\phi \simeq \frac{\mu J_\phi}{4\pi}\frac{\exp(-jkR)}{R}\sum_{n=-\infty}^{\infty}a_n(k\sin\theta)\exp\left[jn\left(\phi-\phi_0\right)+j\frac{n\pi}{2}\right] \tag{49b}$$

(III) z-Directed Dipole Field

$$A_z = \frac{\mu J_z}{8\pi j}\sum_{n=-\infty}^{\infty}\exp\left[jn\left(\phi-\phi_0\right)\right]\int_{-\infty}^{\infty} a_n H_n^{(2)}(\chi\rho)\exp\Big[-jh(z-z_0)\Big]dh$$

$$\simeq \frac{\mu J_z}{4\pi}\frac{\exp(-jkR)}{R}\sum_{n=-\infty}^{\infty} a_n(k\sin\theta)\exp\left[jn\left(\phi-\phi_0\right)+j\frac{n\pi}{2}\right] \quad (50a)$$

$$F_z \simeq Y_0\frac{\mu J_\rho}{4\pi}\frac{\exp(-jkR)}{R}\sum_{n=-\infty}^{\infty} b_n(k\sin\theta)\exp\left[jn(\phi-\phi_0)+j\frac{n\pi}{2}\right] \quad (50b)$$

Electromagnetic field is obtained by the relation

$$E_\theta \simeq -j\omega A_\theta = -j\omega\sin\theta A_z, \quad H_\theta \simeq -j\omega F_\theta = -j\omega\sin\theta F_z = -Z_0 E_\phi \quad (50c)$$

4.3. Surface Field

Surface field can be used as the secondary sources which produce the scattered fields.

(I) ρ-Directed Dipole

$$\widetilde{E}_{zn} = \frac{Z_0 J_\rho}{4j}\frac{h\chi}{k}\frac{k^2\zeta_1}{\Delta_t(\chi a)} \times\left\{-\Big[k\Psi_n(\chi a)-j\chi\zeta_2\Big]H_n^{(2)\prime}(\chi\rho_0)+k\frac{n}{\chi a}\cdot\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\right\}W_n(\chi a) \quad (51a)$$

$$\widetilde{H}_{zn} = -\frac{J_\rho}{4}\frac{k\chi}{\Delta_t(\chi a)} \times\left\{k\Big[j\chi-k\zeta_1\Psi_n(\chi a)\Big]\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)+h^2k\zeta_1\frac{n}{\chi a}H_n^{(2)\prime}(\chi\rho_0)\right\}W_n(\chi a) \quad (51b)$$

(II) ϕ-Directed Dipole

$$\widetilde{E}_{zn} = -\frac{Z_0 J_\phi}{4}\frac{kh\chi\zeta_1}{\Delta_t(\chi a)} \times\left\{-\Big[k\Psi_n(\chi a)-j\chi\zeta_2\Big]\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)+k\frac{n}{\chi a}H_n^{(2)\prime}(\chi\rho_0)\right\}W_n(\chi a) \quad (52a)$$

$$\widetilde{H}_{zn} = \frac{jJ_\phi}{4}\frac{k\chi}{\Delta_t(\chi a)}$$

$$\times\left\{k\left[j\chi - k\zeta_1\Psi_n(\chi a)\right]H_n^{(2)\prime}(\chi\rho_0) + h^2\zeta_1\frac{n}{\chi a}\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)\right\}W_n(\chi a) \tag{52b}$$

(III) z-Directed Dipole

$$\widetilde{E}_{zn} = -\frac{jZ_0J_z}{4}\frac{\chi^2\zeta_1}{\Delta_z(\chi a)}\left[\chi\zeta_2 + jk\Psi_n(\chi a)\right]H_n^{(2)}(\chi\rho_0)W_n(\chi a) \tag{53a}$$

$$\widetilde{H}_{zn} = \frac{jJ_z}{4}\frac{\chi^2\zeta_1}{\Delta_z(\chi a)}\frac{nh}{\chi a}H_n^{(2)}(\chi\rho_0)W_n(\chi a) \tag{53b}$$

In the above equations (50) $\sim$ (52),

$$W_n = \frac{H_n^{(2)\prime}(\chi a)}{H_n^{(2)}(\chi a)}J_n(\chi a) - J_n'(\chi a) = -\frac{j2}{\pi\chi a}\cdot\frac{1}{H_n^{(2)}(\chi a)} \tag{54}$$

5. FINITE IMPEDANCE CIRCULAR CYLINDER

The determination of the far-zone radiation from an arbitrarily oriented dipole near a finite impedance cylinder is done on the same lines as for PEC cylinder. From a theoretical point of view, the problem contains considerable mathematical complications so some method of approximation is utilized. In order to carry out a more accurate approximate solution to this problem the current distribution on the finite cylinder will be taken identical to the distribution on the infinite cylinder. In case of impedance cylinder, beside magnetic vector potential, electric vector potential is also incorporated.

5.1. General Calculations

We consider the general calculations or procedure which will be employed to calculate the arbitrary oriented dipole field scattered by finite impedance cylinder. In the far zone, the electric field can be written as [52].

$$E_\theta = -j\omega A_\theta - j\omega Z_0 F_\phi \tag{55a}$$

$$E_\phi = -j\omega A_\phi + j\omega Z_0 F_\theta \tag{55b}$$

where

$$A_u = \frac{\mu_0}{4\pi}\int_A \mathbf{J}\cdot\mathbf{i}_u\frac{e^{-jkR}}{R}dA$$

$$F_u = \frac{\epsilon_0}{4\pi}\int_A \mathbf{M}\cdot\mathbf{i}_u\frac{e^{-jkR}}{R}dA \qquad (u=\theta,\phi)$$

The far zone field of the unchanged portion of the current between the length of the cylinder $-l_1$ and l_2 is then computed and added to the field of the dipole to obtain an expression for the total far zone field of the configuration. In the above relations, we need the surface current densities on the cylinder, which are given by

$$J_\phi = -H_z(\rho = a), \qquad J_z = H_\phi(\rho = a)$$

$$M_\phi = E_z(\rho = a), \qquad M_z = -E_\phi(\rho = a)$$

which in turn can be obtained through

$$H_\phi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \int_{-\infty}^{\infty} \widetilde{H}_{\phi n} \exp[-jh(z - z_0)]dh \quad (56a)$$

$$E_\phi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \int_{-\infty}^{\infty} \widetilde{E}_{\phi n} \exp[-jh(z - z_0)]dh \quad (56b)$$

$$H_z = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \int_{-\infty}^{\infty} \widetilde{H}_{zn} \exp[-jh(z - z_0)]dh \quad (56c)$$

$$E_z = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)] \int_{-\infty}^{\infty} \widetilde{E}_{zn} \exp[-jh(z - z_0)]dh \quad (56d)$$

E and H fields are related through the boundary conditions (36). Which can be obtained by using the results of previous section.

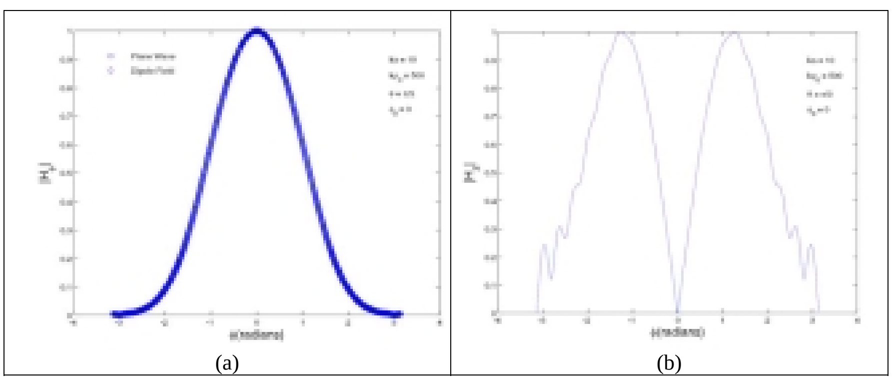

Figure 3. Surface field of a PEC cylinder for ρ-directed dipole.

6. RESULTS AND DISCUSSIONS

Surface fields in the case of PEC infinite cylinder are shown in Figs. 3–5, while far zone scattered field patters in the same case are given in Figs. 6–8. Fig. 3(a) shows surface field plots due to incident plane wave and ρ-directed dipole field with $ka = 10$, $k\rho_0 = 500$, $\theta = \frac{\pi}{3}$ and $\phi_0 = 0$. The patterns match those reported by [55] which indicates the validity of this method. Having validation of the method the surface fields due to all other orientations i.e, ϕ- and z-directed dipole are plotted which are shown in Figs. 4–5. Similarly in Fig. 6(a), far zone scattered field due to ρ-directed dipole are plotted which is similar to a well known far zone scattered field plot due to plane wave incident on PEC cylinder

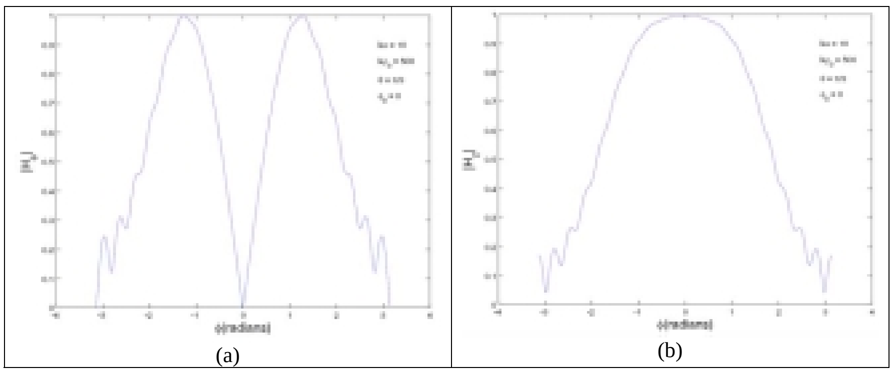

Figure 4. Surface field of a PEC cylinder for φ-directed dipole.

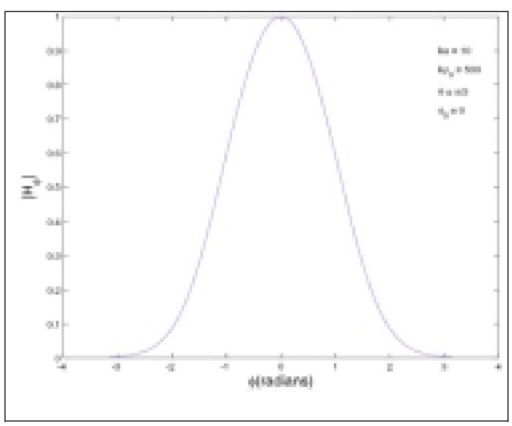

Figure 5. Surface field of a PEC cylinder for z-directed dipole.

which is again in good agreement with the literature [55]. Therefore the far zone scattered field due to all other orientations i.e., ϕ-, and z-directed dipole are plotted which are shown in Figs. 7–8. Next is the scattering of dipole field from finite PEC cylinder. For the case of the dipole contiguous to the cylinder, $E_{\theta_{total}}$and $E_{\phi_{total}}$ are plotted against θ for $kl_1 = 5.236$, $kl_2 = 10.472$ and $ka = 0.6317$. Our's and [50]'s radiation patterns agree sufficiently to warrant comparison in Fig. 9. In case of dipole field scattered from infinite impedance cylinder, surface fields are shown in Figs. 10–12, while far zone scattered field patters are given in Figs. 13–15. These patterns match with surface field plots and far scattered field plots of dipole field scattered from infinite PEC cylinder, which match those reported by Bowman, Seniour and Uslenghi [55] which indicates the validity of this method. Next is the case of finite impedance cylinder. For the case of the dipole contiguous

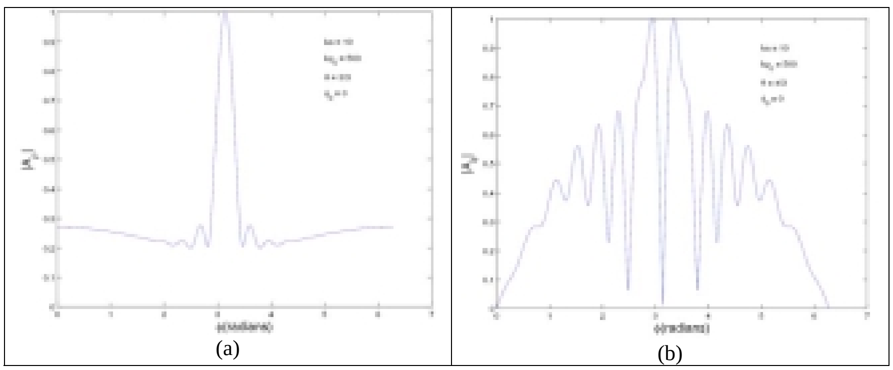

Figure 6. Far scattered field of a PEC cylinder for ρ-directed dipole.

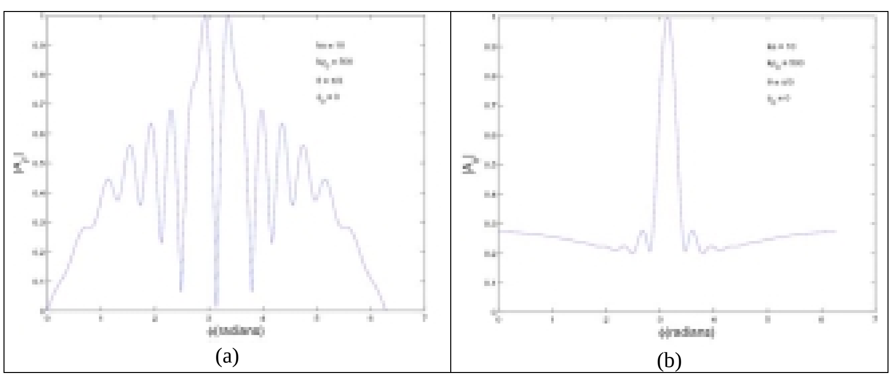

Figure 7. Far scattered field of a PEC cylinder for φ-directed dipole.

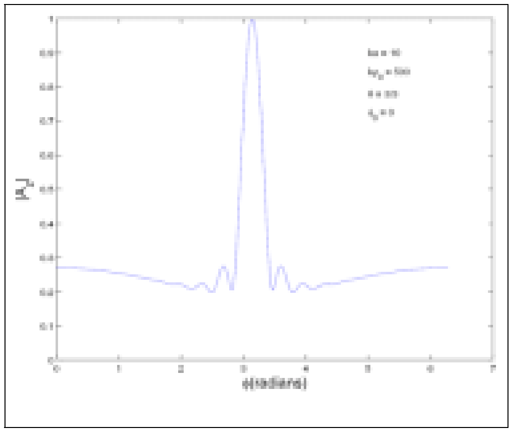

Figure 8. Far scattered field of a PEC cylinder for z-directed dipole.

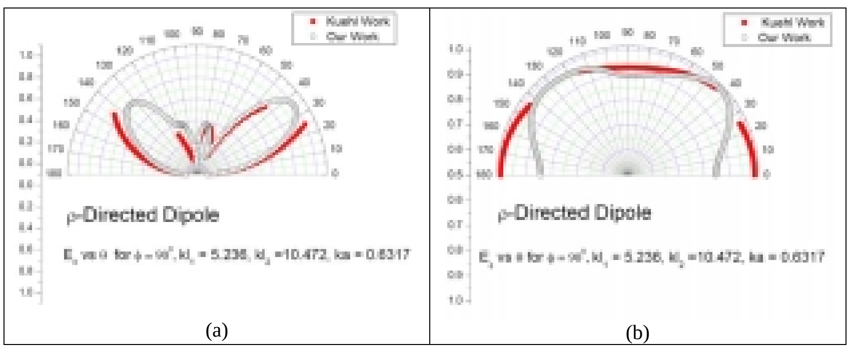

Figure 9. Comparison between Kuehl work and our work.

to the cylinder, $E_{\theta_{total}}$ and $E_{\phi_{total}}$ are plotted against θ for $kl_1 = 5.236$, $kl_2 = 10.472$ and $ka = 0.6317$, $ka = 1$. Our results are sufficiently agreed to the case when the cylinder is of PEC and the results of which are validated through [50]. Figs. 16–18. The exciting dipole is not necessarily situated in the plane equidistant from the two ends of the cylinder. Therefore the radiation pattern is not necessarily symmetrical about the midplane of the system [50].

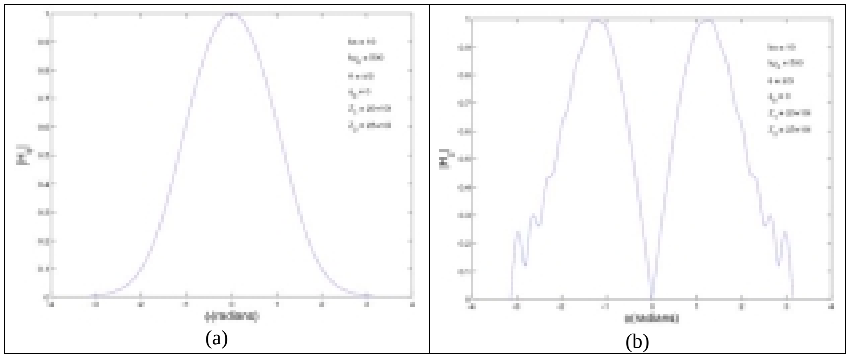

Figure 10. Surface field of an impedance cylinder for ρ-directed dipole.

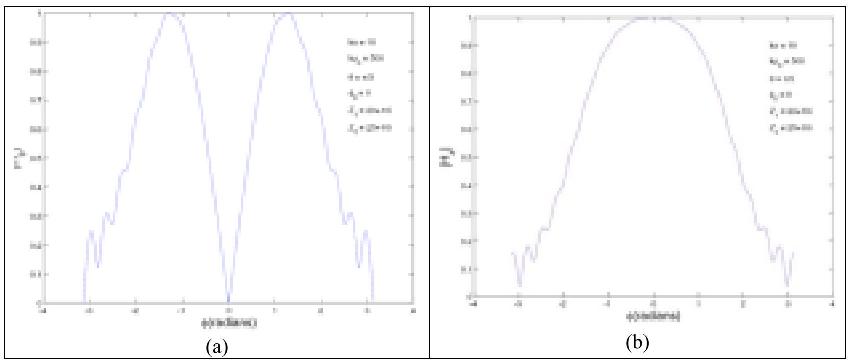

Figure 11. Surface field of an impedance cylinder for φ-directed dipole.

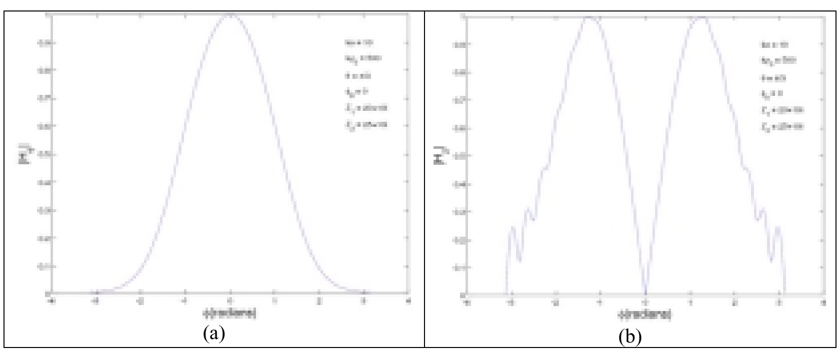

Figure 12. Surface field of an impedance cylinder for z-directed dipole.

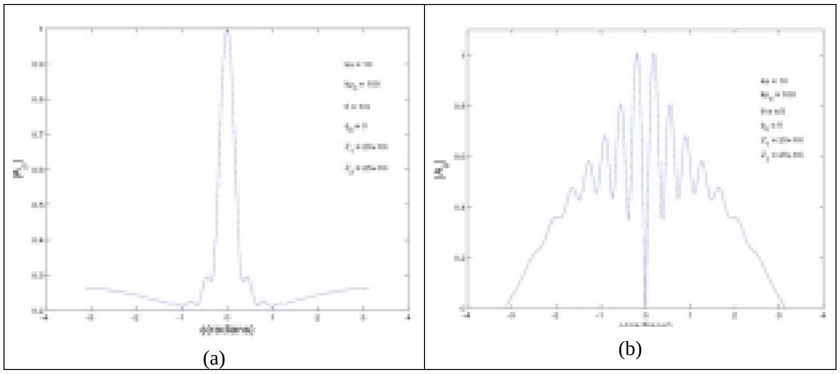

Figure 13. Far scattered field of an impedance cylinder for ρ-directed dipole.

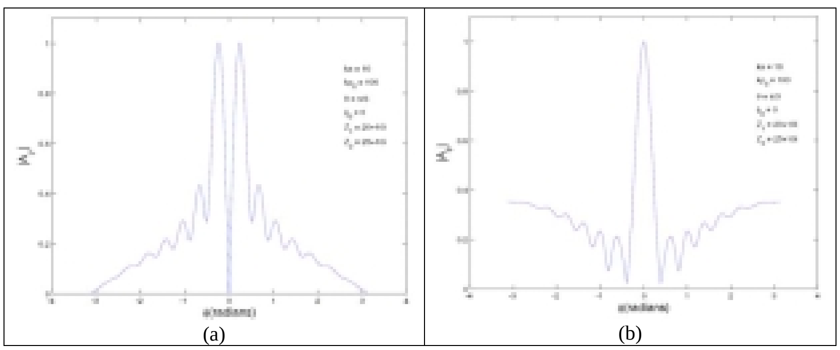

Figure 14. Far scattered field of an impedance cylinder for φ-directed dipole.

APPENDIX A. DIPOLE FIELD IN THE CIRCULAR CYLINDRICAL COORDINATES

The dyadic Green's functions for various configurations are collected by Tai [56], but we present here a different derivation so that it is easily applied to the present problem.

A small arbitrarily oriented dipole current is placed at $\mathbf{r}_0 = (\rho_0, \phi_0, z_0)$ and the problem is to derive the field at any point $\mathbf{r} = (\rho, \phi, z)$ produced by the dipole current. The field is given as the solution of the Maxwell equations

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \qquad \nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J}\delta(\mathbf{r} - \mathbf{r}_0) \tag{A1a}$$

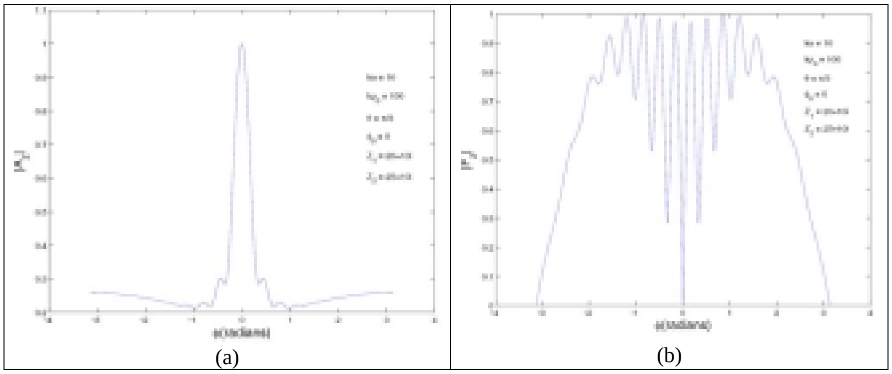

Figure 15. Far scattered field of an impedance cylinder for z-directed dipole.

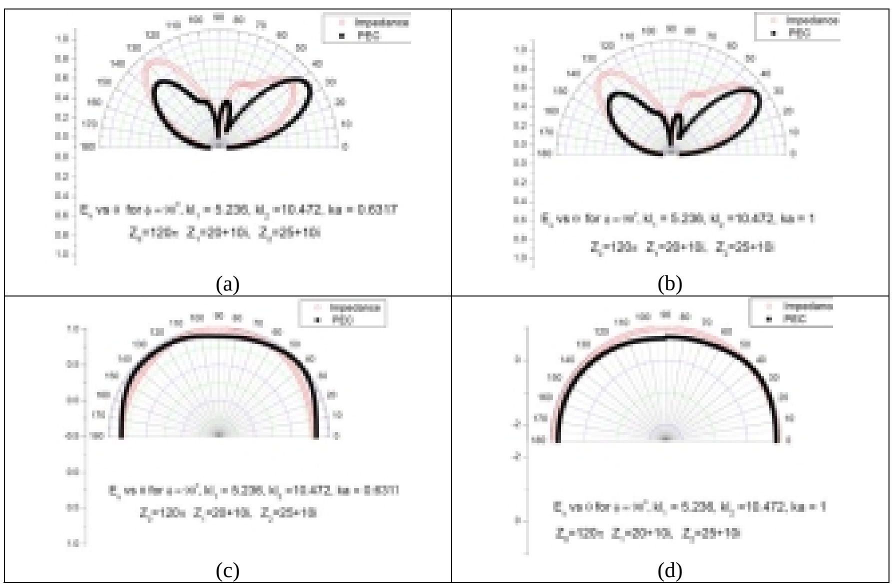

Figure 16. Comparison between PEC and impedance case (ρ-directed dipole).

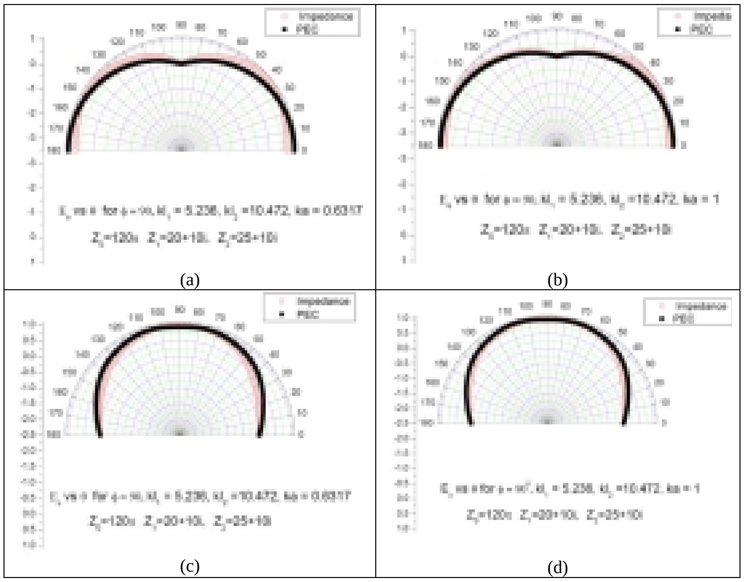

Figure 17. Comparison between PEC and impedance case (φ-directed dipole).

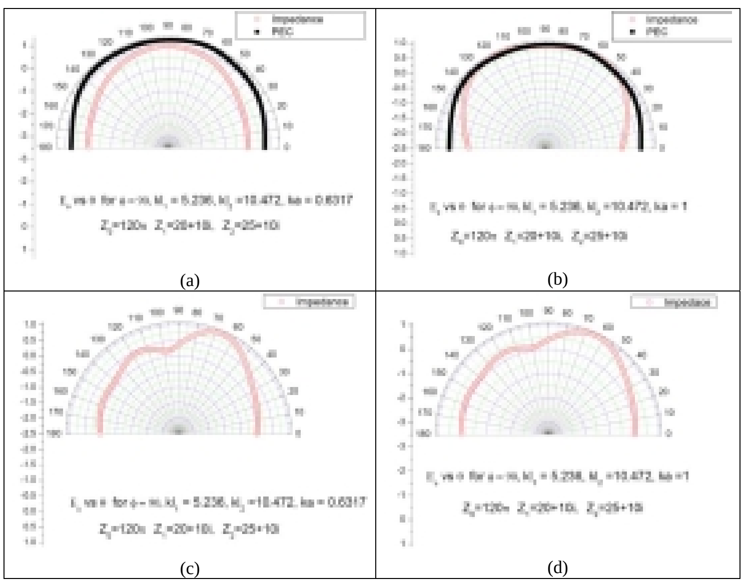

Figure 18. Comparison between PEC and impedance case (z-directed dipole).

where $\mathbf{J}$ is the current density and $\delta(\mathbf{r}-\mathbf{r}_0)$ is the Dirac's delta function. Symbols $\mathbf{r}$ and $\mathbf{r}_0$ denote the vectors of observation and source points, respectively. Parameters ϵ and μ are the permittivity and permeability of the medium of the region of interest. In present discussion, it is assumed that ϵ and μ are constants. We introduce the vector potential $\mathbf{A}$ of magnetic type, which has the relationship with the electromagnetic field [52, 53]

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -j\omega \left[\mathbf{A} + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{A}\right] \tag{A1b}$$

Substituting (A1b) into (A1a) yields the equations which the vector potential $\mathbf{A}$ must satisfy

$$\nabla\times\nabla\times\mathbf{A} - \nabla\nabla\cdot\mathbf{A} - k^2\mathbf{A} = \mu\mathbf{u}J_u\delta(\mathbf{r}-\mathbf{r}_0) \tag{A1c}$$

where $k=\omega\sqrt{\mu\epsilon}$ is the wave number, $\mathbf{u}$ is the unit vector representing the direction of the dipole current density and J_u is the component of the direction. Representing $\nabla = \nabla_t + \mathbf{i}_z\frac{\partial}{\partial z}$, $\mathbf{A} = \mathbf{A}_t + \mathbf{i}_z A_z$, $\mathbf{i}_z$ is the unit vector directed to positive z direction and subscript "t" denotes the transverse components. Above equation takes the following form

$$\nabla_t\times\nabla_t\times\mathbf{A}_t - \nabla_t\nabla_t\cdot\mathbf{A}_t - \frac{\partial^2\mathbf{A}_t}{\partial z^2} - k^2\mathbf{A}_t - \left[\nabla_t^2 A_z + \frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right]\mathbf{i}_z = \mu\mathbf{J}\delta(\mathbf{r}-\mathbf{r}_0) \tag{A1d}$$

where ∇_t^2 is the two dimensional Laplacian in cylindrical coordinates. Each component of the vector potential is expanded in terms of Fourier transform with respect to z parameter, which is defined by

$$A_\ell(\rho,\phi,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{A}_\ell(\rho,\phi,h)\exp\Big[-jh(z-z_0)\Big]dh \tag{A2a}$$

$$\widetilde{A}_\ell(\rho,\phi,h) = \int_{-\infty}^{\infty}A_\ell(\rho,\phi,h)\exp\Big[jh(z-z_0)\Big]dz, \quad \ell=\rho,\phi,z \tag{A2b}$$

where tilde means the transformed function. Any vector field, which is finite, uniform, and continuous and which vanishes at infinity, may also be expressed as the sum of a gradient of a scalar and curl of a zero divergence vector [57]. The solution of the transverse component of magnetic vector potential, used in (A1d), may be written for $\mathbf{r}\neq\mathbf{r}_0$ as

$$\widetilde{\mathbf{A}}_t = \nabla_t\Phi + \nabla_t\times(\mathbf{i}_z\Psi) \tag{A3}$$

where Φ and Ψ are the scalar functions satisfying two dimensional wave equation $\nabla_t^2 F + \chi^2 F = 0$ with $F=\Phi$ or $F=\Psi$ and $\chi^2 = k^2 - h^2$. The solution of the z-component of (A1d) is given by $\widetilde{A}_z = \Phi$ for $\mathbf{r} \neq \mathbf{r}_0$. The variation with respect to variable ϕ is expanded in terms of Fourier series and the expansion of the Dirac's delta function is given by

$$\delta(\phi - \phi_0) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\Big[jn(\phi - \phi_0)\Big] \tag{A4}$$

Let the coefficients of Fourier series of the transformed function $\widetilde{A}_\ell$ be denoted by $\widetilde{A}_{\ell n}$. Thus the solution for each component of the vector potential is obtained in the form

$$A_\ell(\rho, \phi, z) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\Big[jn(\phi - \phi_0)\Big] \int_{-\infty}^{\infty} \widetilde{A}_{\ell n}(\rho, h) \exp\Big[-jh(z - z_0)\Big] dh, \quad \ell = \rho, \phi, z \tag{A5a}$$

The coefficients $\widetilde{A}_{\ell n}(\rho, h)$ of Fourier series of transformed functions $\widetilde{A}_\ell(\rho, \phi, h)$ must satisfy the following inhomogeneous differential equations

$$\begin{aligned} &\left\{ \frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho}(\rho \widetilde{A}_{\rho n}) + \left(\chi^2 - \frac{n^2}{\rho^2}\right) \widetilde{A}_{\rho n} - j\frac{2n}{\rho^2} \widetilde{A}_{\phi n} \right\} \mathbf{i}_\rho \\ &+ \left\{ \frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho}(\rho \widetilde{A}_{\phi n}) + \left(\chi^2 - \frac{n^2}{\rho^2}\right) \widetilde{A}_{\phi n} + j\frac{2n}{\rho^2} \widetilde{A}_{\rho n} \right\} \mathbf{i}_\phi \\ &+ \left\{ \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d}{d\rho} \widetilde{A}_{zn}\right) + \left(\chi^2 - \frac{n^2}{\rho^2}\right) \widetilde{A}_{zn} \right\} \mathbf{i}_z = -\mu J_u \mathbf{u} \frac{\delta(\rho - \rho_0)}{2\pi\rho} \end{aligned} \tag{A5b}$$

In next subsection, radiated field due to electric dipole has been determined while specifying a particular direction of electric dipole. That is three situation has been considered, ρ-directed dipole, ϕ-directed dipole and z-directed dipole.

A.1. ρ-Directed Dipole Field

Let $\mathbf{u} = \mathbf{i}_\rho$ in the right hand side of (A5b), the equations for $\widetilde{A}_{\rho n}$ and $\widetilde{A}_{\phi n}$ become

$$\begin{aligned} \frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho}(\rho \widetilde{A}_{\rho n}) + \left(\chi^2 - \frac{n^2}{\rho^2}\right) \widetilde{A}_{\rho n} - j\frac{2n}{\rho^2} \widetilde{A}_{\phi n} &= -\mu J_\rho \frac{\delta(\rho - \rho_0)}{2\pi\rho} \\ \frac{d}{d\rho} \frac{1}{\rho} \frac{d}{d\rho}(\rho \widetilde{A}_{\phi n}) + \left(\chi^2 - \frac{n^2}{\rho^2}\right) \widetilde{A}_{\phi n} + j\frac{2n}{\rho^2} \widetilde{A}_{\rho n} &= 0 \end{aligned} \tag{A6a}$$

It is obvious from (A5b), $\widetilde{A}_{zn}$ satisfies homogeneous wave equation and is independent from $\widetilde{A}_{\rho n}$ and $\widetilde{A}_{\phi n}$. Set of equations (A6a) are coupled differential equations for $\widetilde{A}_{\rho n}$ and $\widetilde{A}_{\phi n}$. The solution of (A6a) can be constructed from that of corresponding homogeneous equation which is given in (A3), by applying the source conditions

$$\widetilde{A}_{\rho n}\Big|_{\rho_0-0}^{\rho_0+0} = 0,\ \widetilde{A}_{\phi n}\Big|_{\rho_0-0}^{\rho_0+0} = 0;\ \frac{d}{d\rho}\widetilde{A}_{\rho n}\Big|_{\rho_0-0}^{\rho_0+0} = -\frac{\mu J_\rho}{2\pi\rho_0},\ \frac{d}{d\rho}\widetilde{A}_{\phi n}\Big|_{\rho_0-0}^{\rho_0+0} = 0 \tag{A6b}$$

The scalar functions Φ and Ψ of $\widetilde{\mathbf{A}}_t$ given by (A3) may be written as

$$\begin{aligned}
\Phi(\rho,\phi,h) &= \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]\Phi_1(\rho) \\
\Psi(\rho,\phi,h) &= \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]\Psi_1(\rho)
\end{aligned}$$

The scalar functions Φ_1 and Ψ_1 of $\widetilde{\mathbf{A}}_{tn}$ given by (A3) are by $J_n(\chi\rho)$ for $\rho < \rho_0$ and $H_n^{(2)}(\chi\rho)$ for $\rho > \rho_0$, respectively. $\widetilde{\mathbf{A}}_t$ may be obtained using (A3), that is

$$\begin{aligned}
\widetilde{\mathbf{A}}_t(\rho,\phi,h) &= \nabla_t\Phi(\rho,\phi,h) + \nabla_t \times (\mathbf{i}_z\Psi(\rho,\phi,h)) \\
\widetilde{A}_\rho(\rho,\phi,h) &= \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]\left[\frac{\partial}{\partial\rho}\Phi_1(\rho) + \frac{jn}{\rho}\Psi_1(\rho)\right] \\
&= \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]\tilde{A}_{\rho n}(\rho) \\
\widetilde{A}_\phi(\rho,\phi,h) &= \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]\left[\frac{jn}{\rho}\Phi_1(\rho) - \frac{\partial}{\partial\rho}\Psi_1(\rho)\right] \\
&= \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]\tilde{A}_{\phi n}(\rho)
\end{aligned}$$

where derivative with respect to ϕ has been employed to simplify the expression.

$$\widetilde{A}_{\rho n} = \begin{pmatrix} P\left[\chi J_n'(\chi\rho) + \alpha_1\dfrac{jn}{\rho}J_n(\chi\rho)\right], & \rho < \rho_0 \\ Q\left[\chi H_n^{(2)\prime}(\chi\rho) + \alpha_1\dfrac{jn}{\rho}H_n^{(2)}(\chi\rho)\right], & \rho > \rho_0 \end{pmatrix}$$

$$\widetilde{A}_{\phi n} = \begin{pmatrix} P\left[-\chi\alpha_1 J_n'(\chi\rho) + \dfrac{jn}{\rho} J_n(\chi\rho)\right], & \rho < \rho_0 \\ Q\left[\chi\alpha_1 H_n^{(2)\prime}(\chi\rho) + \dfrac{jn}{\rho} H_n^{(2)}(\chi\rho)\right], & \rho > \rho_0 \end{pmatrix}$$

As $\widetilde{A}_{tn}$ is continuous at $\rho = \rho_0$, therefore by considering this fact, we can set $\widetilde{A}_{tn}$ in the form

$$\widetilde{A}_{\rho n} = \begin{pmatrix} \left[A_n H_n^{(2)\prime}(\chi\rho_0) + B_n \dfrac{n}{\chi\rho_0} H_n^{(2)}(\chi\rho_0)\right]\left[J_n'(\chi\rho) + \alpha\dfrac{n}{\chi\rho} J_n(\chi\rho)\right] \\ \rho < \rho_0 \\ \left[A_n J_n'(\chi\rho_0) + B_n \dfrac{n}{\chi\rho_0} J_n(\chi\rho_0)\right]\left[H_n^{(2)\prime}(\chi\rho) + \alpha\dfrac{n}{\chi\rho} H_n^{(2)}(\chi\rho)\right] \\ \rho > \rho_0 \end{pmatrix}$$

$$\widetilde{A}_{\phi n} = j\begin{pmatrix} \left[A_n H_n^{(2)\prime}(\chi\rho_0) + B_n \dfrac{n}{\chi\rho_0} H_n^{(2)}(\chi\rho_0)\right]\left[\alpha J_n'(\chi\rho) + \dfrac{n}{\chi\rho} J_n(\chi\rho)\right] \\ \rho < \rho_0 \\ \left[A_n J_n'(\chi\rho_0) + B_n \dfrac{n}{\chi\rho_0} J_n(\chi\rho_0)\right]\left[\alpha H_n^{(2)\prime}(\chi\rho) + \dfrac{n}{\chi\rho} H_n^{(2)}(\chi\rho)\right] \\ \rho > \rho_0 \end{pmatrix} \tag{A7a}$$

where $\alpha = j\alpha_1$. Coefficients A_n, B_n and α may be determined from the conditions given in (A6b). From the first two conditions of (A6b), we have $A_n\alpha = B_n$ for $\widetilde{A}_{\rho n}$ and $B_n\alpha = A_n$ for $\widetilde{A}_{\phi n}$. Hence we have the relation $\alpha = \pm 1$. It follows that $A_n = B_n$ for $\alpha = 1$, and $A_n = -B_n$ for $\alpha = -1$. That means that the solution of (A7a) must contain two terms in order to satisfy these conditions given in (A6a) the solution is given by a linear combination of these functions. Thus $\widetilde{A}_{\rho n}$ and $\widetilde{A}_{\phi n}$ are written as

$$\widetilde{A}_{\rho n} = C_n \begin{pmatrix} H_{n-1}^{(2)}(\chi\rho_0) J_{n-1}(\chi\rho) \\ J_{n-1}(\chi\rho_0) H_{n-1}^{(2)}(\chi\rho) \end{pmatrix} + D_n \begin{pmatrix} H_{n+1}^{(2)}(\chi\rho_0) J_{n+1}(\chi\rho) \\ J_{n+1}(\chi\rho_0) H_{n+1}^{(2)}(\chi\rho) \end{pmatrix} \quad \begin{matrix} \rho < \rho_0 \\ \rho > \rho_0 \end{matrix}$$

$$\widetilde{A}_{\phi n} = jC_n \begin{pmatrix} H_{n-1}^{(2)}(\chi\rho_0) J_{n-1}(\chi\rho) \\ J_{n-1}(\chi\rho_0) H_{n-1}^{(2)}(\chi\rho) \end{pmatrix} - jD_n \begin{pmatrix} H_{n+1}^{(2)}(\chi\rho_0) J_{n+1}(\chi\rho) \\ J_{n+1}(\chi\rho_0) H_{n+1}^{(2)}(\chi\rho) \end{pmatrix}$$

$$\rho < \rho_0$$
$$\rho > \rho_0 \tag{A7b}$$

From the last condition of (A6b) we have $C_n = D_n$ and from the third condition of (A6b) we obtain $C_n = \frac{\mu J_\rho}{j8}$. Thus we finally obtain

$$\widetilde{A}_{\rho n} = \frac{\mu J_\rho}{8j} \begin{pmatrix} H^{(2)}_{n-1}(\chi\rho_0) J_{n-1}(\chi\rho) + H^{(2)}_{n+1}(\chi\rho_0) J_{n+1}(\chi\rho) \\ J_{n-1}(\chi\rho_0) H^{(2)}_{n-1}(\chi\rho) + J_{n+1}(\chi\rho_0) H^{(2)}_{n+1}(\chi\rho) \end{pmatrix} \begin{matrix} \rho < \rho_0 \\ \rho > \rho_0 \end{matrix}$$

$$\widetilde{A}_{\phi n} = \frac{\mu J_\rho}{8} \begin{pmatrix} H^{(2)}_{n-1}(\chi\rho_0) J_{n-1}(\chi\rho) - H^{(2)}_{n+1}(\chi\rho_0) J_{n+1}(\chi\rho) \\ J_{n-1}(\chi\rho_0) H^{(2)}_{n-1}(\chi\rho) - J_{n+1}(\chi\rho_0) H^{(2)}_{n+1}(\chi\rho) \end{pmatrix} \begin{matrix} \rho < \rho_0 \\ \rho > \rho_0 \end{matrix} \tag{A7c}$$

where we have used the Wronski's relation for the Bessel functions

$$J_n(z)\frac{d}{dz}H^{(2)}_n(z) - H^{(2)}_n(z)\frac{d}{dz}J_n(z) = -\frac{2j}{\pi z} \tag{A7d}$$

to simplify the solution. Consider the two relations

$$\begin{aligned} H^{(2)}_{n-1}(\chi\rho_0)J_{n-1}(\chi\rho) &= \left[H^{(2)\prime}_n(\chi\rho_0) + \frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0)\right]\left[J'_n(\chi\rho) + \frac{n}{\chi\rho}J_n(\chi\rho)\right] \\ &= H^{(2)\prime}_n(\chi\rho_0)J'_n(\chi\rho) + \frac{n}{\chi\rho}J_n(\chi\rho)H^{(2)\prime}_n(\chi\rho_0) \\ &\quad + \frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0)J'_n(\chi\rho) + \frac{n}{\chi\rho_0}\frac{n}{\chi\rho}H^{(2)}_n(\chi\rho_0)J_n(\chi\rho) \\ H^{(2)}_{n+1}(\chi\rho_0)J_{n+1}(\chi\rho) &= \left[\frac{n}{\chi\rho}J_n(\chi\rho) - J'_n(\chi\rho)\right]\left[\frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0) - H^{(2)\prime}_n(\chi\rho_0)\right] \\ &= \frac{n}{\chi\rho_0}\frac{n}{\chi\rho}H^{(2)}_n(\chi\rho_0)J_n(\chi\rho) - \frac{n}{\chi\rho}J_n(\chi\rho)H^{(2)\prime}_n(\chi\rho_0) \\ &\quad - \frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0)J'_n(\chi\rho) + H^{(2)\prime}_n(\chi\rho_0)J'_n(\chi\rho) \end{aligned}$$

Adding, we get

$$\begin{aligned} &H^{(2)}_{n-1}(\chi\rho_0)J_{n-1}(\chi\rho) + H^{(2)}_{n+1}(\chi\rho_0)J_{n+1}(\chi\rho) \\ &= 2H^{(2)\prime}_n(\chi\rho_0)J'_n(\chi\rho) + 2\frac{n}{\chi\rho_0}\frac{n}{\chi\rho}H^{(2)}_n(\chi\rho_0)J_n(\chi\rho) \end{aligned}$$

Therefore, a part of (A7c) may be written in a more appropriate form as

$$\widetilde{A}_{\rho n} = \frac{\mu J_\rho}{4j}\left[J'_n(\chi\rho)H^{(2)\prime}_n(\chi\rho_0) + \frac{n}{\chi\rho}J_n(\chi\rho)\frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0)\right], \quad \rho < \rho_0 \tag{A7e}$$

Similarly

$$\widetilde{A}_{\phi n} = \frac{\mu J_\rho}{4}\left[\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0)J_n'(\chi\rho) + H_n^{(2)\prime}(\chi\rho_0)\frac{n}{\chi\rho}J_n(\chi\rho)\right], \quad \rho < \rho_0 \tag{A7f}$$

Using (A7e) and (A7f), in combination with (A1b), the expression of the electromagnetic field is given by

$$\begin{aligned}
E_\rho &= -\frac{\omega\mu J_\rho}{8\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\Big[jn(\phi-\phi_0)\Big]\exp\Big[-jh(z-z_0)\Big]dh \\
&\quad\times\begin{pmatrix} \frac{h^2}{k^2}J_n'(\chi\rho)H_n^{(2)\prime}(\chi\rho_0) + \frac{n}{\chi\rho}J_n(\chi\rho)\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0) \\ \frac{h^2}{k^2}J_n'(\chi\rho_0)H_n^{(2)\prime}(\chi\rho) + \frac{n}{\chi\rho_0}J_n(\chi\rho_0)\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho) \end{pmatrix} &\text{(A8a)} \\
E_\phi &= \frac{\omega\mu J_\rho}{8\pi j}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\Big[jn(\phi-\phi_0)\Big]\exp\Big[-jh(z-z_0)\Big]dh \\
&\quad\times\begin{pmatrix} J_n'(\chi\rho)\frac{n}{\chi\rho_0}H_n^{(2)}(\chi\rho_0) + \frac{h^2}{k^2}\frac{n}{\chi\rho}J_n(\chi\rho)H_n^{(2)\prime}(\chi\rho_0) \\ \frac{n}{\chi\rho_0}J_n(\chi\rho_0)H_n^{(2)\prime}(\chi\rho) + \frac{h^2}{k^2}J_n'(\chi\rho_0)\frac{n}{\chi\rho}H_n^{(2)}(\chi\rho) \end{pmatrix} &\text{(A8b)} \\
E_z &= \frac{\omega\mu J_\rho}{8\pi j}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\Big[jn(\phi-\phi_0)\Big]\frac{h\chi}{k^2}\begin{pmatrix} J_n(\chi\rho)H_n^{(2)\prime}(\chi\rho_0) \\ J_n'(\chi\rho_0)H_n^{(2)}(\chi\rho) \end{pmatrix} \\
&\quad\times\exp\Big[-jh(z-z_0)\Big]dh &\text{(A8c)}
\end{aligned}$$

Above results agree with the expression given by Tai [56, p. 96]. The purpose to derive the field expressions in (A8) is just to verify the expressions derived for magnetic vector potentials.

By substituting (A7c) into (A5a) and after simple mathematical manipulations, the vector potential for ρ-directed dipole is obtained as

$$\begin{aligned}
A_\rho &= \frac{\mu J_\rho}{8\pi j}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)] \\
&\times\int_{-\infty}^{\infty}\begin{Bmatrix} H_{n-1}^{(2)}(\chi\rho_0)J_{n-1}(\chi\rho) + H_{n+1}^{(2)}(\chi\rho_0)J_{n+1}(\chi\rho) \\ J_{n-1}(\chi\rho_0)H_{n-1}^{(2)}(\chi\rho) + J_{n+1}(\chi\rho_0)H_{n+1}^{(2)}(\chi\rho) \end{Bmatrix}\exp[-jh(z-z_0)]dh \\
&= \frac{\mu J_\rho}{8\pi j}\sum_{n=-\infty}^{\infty}\exp[jn(\phi-\phi_0)]\int_{-\infty}^{\infty}\exp[-jh(z-z_0)]
\end{aligned}$$

$$\times \begin{Bmatrix} H_n^{(2)}(\chi\rho_0) J_n(\chi\rho_0) \exp[j(\phi-\phi_0)] + H_n^{(2)}(\chi\rho_0) J_n(\chi\rho_0) \exp[-j(\phi-\phi_0)] \\ J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \exp[j(\phi-\phi_0)] + J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \exp[-j(\phi-\phi_0)] \end{Bmatrix} dh$$

$$= \frac{\mu J_\rho}{8\pi j} \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]$$

$$\times \int_{-\infty}^{\infty} \begin{Bmatrix} J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \\ H_n^{(2)}(\chi\rho_0) J_n(\chi\rho) \end{Bmatrix} \cos(\phi-\phi_0) \exp[-jh(z-z_0)] dh, \quad \begin{matrix} \rho > \rho_0 \\ \rho < \rho_0 \end{matrix} \tag{A9a}$$

$$A_\phi = -\frac{\mu J_\rho}{8\pi j} \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]$$

$$\times \int_{-\infty}^{\infty} \begin{Bmatrix} J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \\ H_n^{(2)}(\chi\rho_0) J_n(\chi\rho) \end{Bmatrix} \sin(\phi-\phi_0) \exp[-jh(z-z_0)] dh, \quad \begin{matrix} \rho > \rho_0 \\ \rho < \rho_0 \end{matrix} \tag{A9b}$$

It may be noted that (A7e), (A9a), and (A9b) may be used when appropriate.

Equations (A9a) and (A9b) may be written as

$$A_\rho = \frac{\mu J_\rho}{8\pi j} \sum_{n=-\infty}^{\infty} \exp[jn(\phi-\phi_0)]$$

$$\times \int_{-\infty}^{\infty} \begin{Bmatrix} J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \\ H_n^{(2)}(\chi\rho_0) J_n(\chi\rho) \end{Bmatrix} \cos(\phi-\phi_0) \exp[-jh(z-z_0)] dh, \quad \begin{matrix} \rho > \rho_0 \\ \rho < \rho_0 \end{matrix}$$

$$= \frac{\mu J_\rho}{8\pi j} \int_{-\infty}^{\infty} H_0^{(2)} \left(\chi \sqrt{\rho^2 - 2\rho\rho_0 \cos(\phi-\phi_0) + \rho_0^2} \right) \times \cos(\phi-\phi_0) \exp[-jh(z-z_0)] dh$$

$$A_\phi = -\frac{\mu J_\rho}{8\pi j} \int_{-\infty}^{\infty} H_0^{(2)} \left(\chi \sqrt{\rho^2 - 2\rho\rho_0 \cos(\phi-\phi_0) + \rho_0^2} \right) \times \sin(\phi-\phi_0) \exp[-jh(z-z_0)] dh$$

Where we have used the addition theorem for the Hankel function given by

$$H_0^{(2)} \left(\chi \sqrt{\rho^2 - 2\rho\rho_0 \cos(\phi-\phi_0) + \rho_0^2} \right)$$

$$= \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \exp[jn(\phi-\phi_0)], & \rho > \rho_0 \\ \displaystyle\sum_{n=-\infty}^{\infty} H_n^{(2)}(\chi\rho_0) J_n(\chi\rho) \exp[jn(\phi-\phi_0)], & \rho_0 > \rho \end{cases} \tag{A10a}$$

Using the formula [58, p. 731]

$$\frac{1}{8\pi j}\int_{-\infty}^{\infty} dh \exp[-jh(z-z_0)] H_0^{(2)}(\chi r) = \frac{\exp(-jkR)}{4\pi R}, \quad r^2+(z-z_0)^2 = R^2, \quad \chi^2+h^2=k^2 \tag{A10b}$$

A_ρ and A_ϕ can be written as

$$A_\rho = \mu J_\rho \frac{\exp(-jkR)}{4\pi R}\cos(\phi-\phi_0), \quad A_\phi = -\mu J_\rho \frac{\exp(-jkR)}{4\pi R}\sin(\phi-\phi_0) \tag{A10c}$$

where

$$R = \sqrt{\rho^2 - 2\rho\rho_0\cos(\phi-\phi_0) + \rho_0^2 + (z-z_0)^2}$$

Thus we can see that we need two components of the vector potential to represent the ρ−directed dipole field.

A.2. ϕ-Directed Dipole Field

Let $\mathbf{u} = \mathbf{i}_\phi$ in the right hand side of (A5b), the equations for $\widetilde{A}_{\rho n}$ and $\widetilde{A}_{\phi n}$ become

$$\begin{aligned} \frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\widetilde{A}_{\rho n}) + \left(\chi^2 - \frac{n^2}{\rho^2}\right)\widetilde{A}_{\rho n} - j\frac{2n}{\rho^2}\widetilde{A}_{\phi n} &= 0 \\ \frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}(\rho\widetilde{A}_{\phi n}) + \left(\chi^2 - \frac{n^2}{\rho^2}\right)\widetilde{A}_{\phi n} + j\frac{2n}{\rho^2}\widetilde{A}_{\rho n} &= -\mu J_\phi \frac{\delta(\rho-\rho_0)}{2\pi\rho} \end{aligned} \tag{A11a}$$

These are coupled differential equations for $\widetilde{A}_{\rho n}$ and $\widetilde{A}_{\phi n}$ and the source condition in this case is given by

$$\widetilde{A}_{\rho n}\Big|_{\rho_0-0}^{\rho_0+0} = 0,\ \widetilde{A}_{\phi n}\Big|_{\rho_0-0}^{\rho_0+0} = 0;\ \frac{d}{d\rho}\widetilde{A}_{\rho n}\Big|_{\rho_0-0}^{\rho_0+0} = 0,\ \frac{d}{d\rho}\widetilde{A}_{\phi n}\Big|_{\rho_0-0}^{\rho_0+0} = -\frac{\mu J_\rho}{2\pi\rho} \tag{A11b}$$

We can also assume the solution (A7a) for ϕ directed dipole field and same equations to (A7b) are satisfied. From the continuity of $\frac{d\widetilde{A}_{\rho n}}{d\rho}$

we obtain $D_n = -C_n$ and from the discontinuity of $\frac{d\widetilde{A}_{\phi n}}{d\rho}$, we obtain $C_n = -\frac{\mu J_\phi}{8}$. Then the result becomes

$$\widetilde{A}_{\rho n} = -\frac{\mu J_\rho}{8}\begin{pmatrix} H^{(2)}_{n-1}(\chi\rho_0)J_{n-1}(\chi\rho) - H^{(2)}_{n+1}(\chi\rho_0)J_{n+1}(\chi\rho) \\ J_{n-1}(\chi\rho_0)H^{(2)}_{n-1}(\chi\rho) - J_{n+1}(\chi\rho_0)H^{(2)}_{n+1}(\chi\rho)\end{pmatrix}, \quad \begin{matrix}\rho < \rho_0 \\ \rho > \rho_0\end{matrix}$$

$$\widetilde{A}_{\phi n} = \frac{\mu J_\phi}{8j}\begin{pmatrix} H^{(2)}_{n-1}(\chi\rho_0)J_{n-1}(\chi\rho) + H^{(2)}_{n+1}(\chi\rho_0)J_{n+1}(\chi\rho) \\ J_{n-1}(\chi\rho_0)H^{(2)}_{n-1}(\chi\rho) + J_{n+1}(\chi\rho_0)H^{(2)}_{n+1}(\chi\rho)\end{pmatrix} \quad \begin{matrix}\rho < \rho_0 \\ \rho > \rho_0\end{matrix} \tag{A11c}$$

Using recursion relations for Bessel functions, we can obtain

$$\widetilde{A}_{\rho n} = -\frac{\mu J_\phi}{4}\left[\frac{n}{\chi\rho}J_n(\chi\rho)H^{(2)\prime}_n(\chi\rho_0) + J'_n(\chi\rho)\frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0)\right], \ \rho < \rho_0$$

$$\widetilde{A}_{\phi n} = \frac{\mu J_\phi}{4j}\left[J'_n(\chi\rho)H^{(2)\prime}_n(\chi\rho_0) + \frac{n}{\chi\rho}J_n(\chi\rho)\frac{n}{\chi\rho_0}H^{(2)\prime}_n(\chi\rho_0)\right], \ \rho < \rho_0$$

Using (A11c) the electromagnetic field is given by

$$\begin{aligned} E_\rho &= \frac{j\omega\mu J_\phi}{8\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\Big[jn(\phi-\phi_0)\Big]\exp\Big[-jh(z-z_0)\Big]dh \\ &\quad\times\begin{pmatrix}\frac{h^2}{k^2}J'_n(\chi\rho)\frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0) + \frac{n}{\chi\rho}J_n(\chi\rho)H^{(2)\prime}_n(\chi\rho_0) \\ \frac{h^2}{k^2}\frac{n}{\chi\rho_0}J_n(\chi\rho_0)H^{(2)\prime}_n(\chi\rho) + J'_n(\chi\rho_0)\frac{n}{\chi\rho}H^{(2)}_n(\chi\rho)\end{pmatrix} \end{aligned} \tag{A12a}$$

$$\begin{aligned} E_\phi &= -\frac{\omega\mu J_\phi}{8\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\Big[jn(\phi-\phi_0)\Big]\exp\Big[-jh(z-z_0)\Big]dh \\ &\quad\times\begin{pmatrix}J'_n(\chi\rho)H^{(2)\prime}_n(\chi\rho_0) + \frac{h^2}{k^2}\frac{n}{\chi\rho}J_n(\chi\rho)\frac{n}{\chi\rho_0}H^{(2)}_n(\chi\rho_0) \\ J'_n(\chi\rho_0)H^{(2)\prime}_n(\chi\rho) + \frac{h^2}{k^2}\frac{n}{\chi\rho_0}J_n(\chi\rho_0)\frac{n}{\chi\rho}H^{(2)}_n(\chi\rho)\end{pmatrix} \end{aligned} \tag{A12b}$$

$$\begin{aligned} E_z &= -\frac{\omega\mu J_\phi}{8\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\exp\Big[jn(\phi-\phi_0)\Big]\frac{h\chi}{k^2}\frac{n}{\chi\rho_0} \\ &\quad\times\begin{pmatrix}J_n(\chi\rho)H^{(2)}_n(\chi\rho_0) \\ J_n(\chi\rho_0)H^{(2)}_n(\chi\rho)\end{pmatrix}\exp\Big[-jh(z-z_0)\Big]dh \end{aligned} \tag{A12c}$$

Considering again (A11c), the vector potential is obtained by

$$A_\rho = \frac{\mu J_\phi}{8\pi j} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)]$$

$$\times \int_{-\infty}^{\infty} \left\{ \begin{array}{c} J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \\ H_n^{(2)}(\chi\rho_0) J_n(\chi\rho) \end{array} \right\} \sin(\phi - \phi_0) \exp[-jh(z - z_0)] dh, \quad \begin{array}{l} \rho > \rho_0 \\ \rho < \rho_0 \end{array} \tag{A13a}$$

$$A_\phi = \frac{\mu J_\phi}{8\pi j} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)]$$

$$\times \int_{-\infty}^{\infty} \left\{ \begin{array}{c} J_n(\chi\rho_0) H_n^{(2)}(\chi\rho) \\ H_n^{(2)}(\chi\rho_0) J_n(\chi\rho) \end{array} \right\} \cos(\phi - \phi_0) \exp[-jh(z - z_0)] dh, \quad \begin{array}{l} \rho > \rho_0 \\ \rho < \rho_0 \end{array} \tag{A13b}$$

The integration can be performed exactly and the result is given by

$$A_\rho = \mu J_\phi \, \frac{\exp(-jkR)}{4\pi R} \sin(\phi - \phi_0), \qquad A_\phi = \mu J_\phi \frac{\exp(-jkR)}{4\pi R} \cos(\phi - \phi_0) \tag{A13c}$$

A.3. z-Directed Dipole Field

In this case only the single component A_z can represent the field. Let $\mathbf{u} = \mathbf{i}_z$ in the right hand side of (A5b), the equation for $\widetilde{A}_{zn}$ becomes

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d\widetilde{A}_{zn}}{d\rho} + \left(\chi^2 - \frac{n^2}{\rho^2} \right) \widetilde{A}_{zn} = -\mu J_z \frac{\delta(\rho - \rho_0)}{2\pi\rho} \tag{A14a}$$

The source condition is given by

$$\widetilde{A}_{zn} \Big|_{\rho_0 - 0}^{\rho_0 + 0} = 0, \qquad \frac{d}{d\rho} \widetilde{A}_{zn} \Big|_{\rho_0 - 0}^{\rho_0 + 0} = -\frac{\mu J_z}{2\pi\rho_0} \tag{A14b}$$

The solution is readily derived and the result is given by

$$\widetilde{A}_{zn} = \frac{\mu J_z}{4j} \left(\begin{array}{c} J_n(\chi\rho) H_n^{(2)}(\chi\rho_0) \\ H_n^{(2)}(\chi\rho) J_n(\chi\rho_0) \end{array} \right), \quad \begin{array}{l} \rho < \rho_0 \\ \rho > \rho_0 \end{array} \tag{A14c}$$

Thus the vector potential A_z is obtained as

$$A_z = \frac{\mu J_z}{8\pi j} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)]$$

$$\times \int_{-\infty}^{\infty} \left\{ \begin{array}{c} J_n(\chi\rho) H_n^{(2)}(\chi\rho_0) \\ H_n^{(2)}(\chi\rho) J_n(\chi\rho_0) \end{array} \right\} \exp[-jh(z-z_0)] dh, \quad \begin{array}{l} \rho < \rho_0 \\ \rho > \rho_0 \end{array}$$

$$= \mu J_z \frac{\exp(-jkR)}{4\pi R} \tag{A14d}$$

The electric field produced by current element $J_z \delta(\mathbf{r} - \mathbf{r}_0)$ is obtained by

$$E_\rho = \frac{j\omega\mu J_z}{8\pi} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)]$$

$$\times \int_{-\infty}^{\infty} \frac{h\chi}{k^2} \left(\begin{array}{c} J_n'(\chi\rho) H_n^{(2)}(\chi\rho_0) \\ H_n^{(2)\prime}(\chi\rho) J_n(\chi\rho_0) \end{array} \right) \exp[-jh(z-z_0)] dh \tag{A15a}$$

$$E_\phi = -\frac{\omega\mu J_z}{8\pi} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)]$$

$$\times \int_{-\infty}^{\infty} \frac{h}{k^2} \frac{n}{\rho} \left(\begin{array}{c} J_n(\chi\rho) H_n^{(2)}(\chi\rho_0) \\ H_n^{(2)}(\chi\rho) J_n(\chi\rho_0) \end{array} \right) \exp[-jh(z-z_0)] dh \tag{A15b}$$

$$E_z = -\frac{\mu J_z}{8\pi} \sum_{n=-\infty}^{\infty} \exp[jn(\phi - \phi_0)]$$

$$\times \int_{-\infty}^{\infty} \frac{h^2}{k^2} \left(\begin{array}{c} J_n(\chi\rho) H_n^{(2)}(\chi\rho_0) \\ H_n^{(2)}(\chi\rho) J_n(\chi\rho_0) \end{array} \right) \exp[-jh(z-z_0)] dh \tag{A15c}$$

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