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A NEW MATHEMATICAL PROPOSAL FOR GENERATION OF SHAPE INVARIANT POTENTIALS AND OPTICAL MEDIUM USING SUPERSYMMETRIC QUANTUM MECHANICS

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Abstract—A new mathematical method and proposal for generation of shape invariant potentials using supersymmetric quantum mechanics is introduced. For this purpose the potential term in the Schrödinger equation is expressed in terms of the super potential. The obtained equation transformed into well known ordinary second order differential equation. Using standard technique, the Nikiforov-Uvarov (NU) method the superpotential in the Schrödinger equation is expressed in terms of the parameters appeared in the NU-approach concluding to a nonlinear differential equation. By solving the obtained equation and using relation between superpotential and potential the shape invariant potentials are obtained. The proposed method is general and straightforward for introducing of the shape invariant potentials.

1. INTRODUCTION

Symmetries and invariance properties are among the most characteristic features of any physical system. They usually give a deeper insight into the physical nature of the systems, but also play an essential role from mathematical point of view. Symmetries typically lead to characteristic patterns in the energy spectrum of the system. These features are shared by the quantum mechanical problems. Technically these are relatively simple systems, and accordingly they include a number of exactly solvable examples. One of the most widely known symmetries of quantum mechanical problems is based on supersymmetry. In principle, this symmetry is related to bosonic and fermionic degrees of freedom. Particle physicists believe that supersymmetry exists close to the electroweak energy scale in nature. According to particle physics, supersymmetry naturally solves hierarchy problem, the large ratio between the electroweak scale and the other higher physical scales as the Planck or the Grand Unified Theory (GUT). Moreover, it explains the unification of the three fundamental coupling constants responsible for the weak, strong and electromagnetic interactions. Since the event of supersymmetry in theoretical particle physics [1] and its implications [2] in supersymmetric quantum mechanics, we are dealing with supersymmetric wave equations, including time-independent superpotentials. In despite of supersymmetry is a remarkable and exciting idea, but at the same time, its implementation is technically pretty complicated. One reason that physicists explored supersymmetry is because it offers an extension to the more familiar symmetries of quantum field theory. Actually, supersymmetric quantum mechanics (SUSYQM) introduced for the first time by Witten [2] as a simplest supersymmetric model of the quantum field theory. It leads to field theories and string theories with essential properties [3–12]. The improved behavior and the natural solution of the hierarchy problem are just a few of the nice features of supersymmetric theories. In addition, supersymmetry also predicts degenerate super-partner states corresponding to every physical particle state of the theory. This model manages a family of exactly soluble potentials [13] and is widely used to obtain their exact solutions in quantum mechanics [14, 15]. In principle, supersymmetric quantum mechanics is a formalism used for determination of energy eigenvalues and eigenfunctions of quantum mechanical problems [13, 14]. In this framework the various methods have been used in their solutions. Recently, an alternative method known as the Nikiforov-Uvarov (NU) method [15] has been introduced for solving the related equation in the both relativistic and non-relativistic quantum mechanics [16–24].

Also, there are different numerical methods for investigation of inhomogeneous media reported in [25–32]. In these papers different approaches were presented and studied in detail. The contents of this paper are organized as follows. In the next section we briefly review the mathematical NU-method based on reducing the second order differential equation to the hyper-geometric type with an appropriate coordinate transformation. In Section 3, a summary of the supersymmetry in quantum mechanics is presented. According to this formalism, we express potential term in the Schrdinger equation in terms of the super potential in Section 4. Next, this equation is transformed into well known ordinary second order differential equation. Finally the paper ends with a short conclusion.

2. THE NIKIFOROV-UVAROV METHOD

The NU-method provide us exact solutions for certain kind of the second order linear differential equations by reducing it to a generalized equation of hyper-geometric type [6]. This approach also provides exact solutions in terms of special orthogonal functions, as well as corresponding eigenvalues. In non relativistic quantum mechanics, the Schrödinger equation can be solved in this framework, for some real or complex potential in one dimension. Here, we use this method for obtaining some known shape invariant potentials, their wave functions and eigenvalues. By introducing an appropriate coordinate transformation, $s = s(x)$ one can rewrite the Schrödinger equation in the following form

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second order, and $\tilde{\tau}(s)$ is a polynomial at most first degree [15]. To find a particular solution, we use the following separation of variables

$$\psi(s) = \phi(s)y_n(s).$$

This factorization reduces Eq. (1) to a hyper-geometric type equation of the following form

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0$$

where $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$. It should mention that $\tau(s)$ must be negative and $\pi(s)$ is defined as

$$\pi(s) = \left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right) \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + \kappa\sigma(s)}, \tag{2}$$

where κ is a parameter. Determination of κ is the essential point in the calculation of $\pi(s)$. It is simply defined by setting the discriminate of the square root to zero [15]. Therefore, one gets a general quadratic equation for κ. The values of κ can be used for calculation of energy eigenvalues using the following equation.

$$\lambda = \kappa + \pi'(s) = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \quad n = 0,\, 1,\, 2,\, \ldots \tag{3}$$

Polynomial solutions $y_n(s)$ are given by Rodriguez relation which is presented as follows

$$y_n(s) = \frac{B_n}{\rho(s)}\left(\frac{d}{ds}\right)^n \left[\sigma^n(s)\rho(s)\right], \quad n = 0,\, 1,\, 2,\, \ldots$$

where B_n and $\rho(s)$ are a normalization constant and the weight function respectively in which the later satisfies the following condition

$$(\sigma\rho)' = \tau\rho. \tag{4}$$

On the other hand, the function $\phi(s)$ satisfies the condition

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}.$$

3. SUPERSYMMETRIC QUANTUM MECHANICS

Let us first summarize the ordinary supersymmetry [2,3] and set the notation. Supersymmetric quantum mechanics (SUSYQM) allows us to write a pair of supersymmetric potentials as follows ($\hbar = m = 1$) [3]

$$V_{\pm}(x) = W^2(x) \pm W'(x), \tag{5}$$

where $W(x)$ is a function that generally referred to as the superpotential in SUSYQM literature, as well as "prime" denotes differentiation with respect to x. The superpotential in terms of the ground state wave function $\psi_0(x)$ is given by

$$W(x) = -\frac{d}{dx}\ln\psi_0(x) \tag{6}$$

or equivalently

$$\psi_0(x) = Ne^{\left(-\int^{x} W(y)dy\right)}.$$

Now, we can define a pair of Hamiltonians describing a supersymmetric system as follows

$$H_{\pm} = -\frac{d^2}{dx^2} + V_{\pm}(x). \tag{7}$$

In terms of the bosonic operators $A^{\pm}$ which is defined as

$$A^{\pm} = \pm \frac{d}{dx} + W(x) \tag{8}$$

this Hamiltonian can also be rewritten as

$$H_{\pm} \equiv A^{\mp} A^{\pm} = -\frac{d^2}{dx^2} + W^2(x) \pm W'(x). \tag{9}$$

It must be noted that the energy eigenvalues of pair Hamiltonian $H_{\pm}(x)$ are equal unless for the ground state $\psi_o^{(0)}$. Actually, energy of the ground state of $H_{-}(x)$ is zero in the case of unbroken supersymmetry, $E_0^{(0)} = 0$ [13].

More specifically, from Eq. (9) we have

$$H_1 = -\frac{d^2}{dx^2} + V_1(x) = A_1^{+} A_1^{-} + E_1^{(0)} \tag{10}$$

which implies

$$W_1^2(x) - W_1'(x) = \left(V_1(x) - E_1^{(0)}\right). \tag{11}$$

This equation is the well-known Riccati equation. The supersymmetric partner Hamiltonian H_2 can be similarly written as

$$H_2 = -\frac{d^2}{dx^2} + V_2(x) = A_2^{-} A_2^{+} + E_2^{(0)}$$

and corresponding equation is

$$W_2^2(x) + W_2'(x) = V_2(x) - E_2^{(0)}.$$

By iteration method one can write supersymmetric partner Hamiltonian H_n in general form

$$H_n = -\frac{d^2}{dx^2} + V_n(x) = A_n^{-} A_n^{+} + E_n^{(0)}$$

and corresponding Riccati equation as

$$W_n^2(x) \pm W_n'(x) = V_n(x) - E_n^{(0)},$$

with

$$A_n^{\pm} = \pm \frac{d}{dx} + W_n(x), \quad n = 1, 2, 3, \ldots$$

The eigen-functions of the partner Hamiltonians satisfy

$$\psi_n^{(1)} = \frac{A^- \psi_{n+1}^{(0)}}{\sqrt{E_n^{(0)}}}, \quad \psi_{n+1}^{(1)} = \frac{A^+ \psi_n^{(0)}}{\sqrt{E_n^{(0)}}}$$

in the unbroken supersymmetry with the following eigenvalues

$$E_{n+1}^{(0)} = E_n^{(1)}, \quad E_0^{(0)} = 0, \quad n = 0,\, 1,\, 2,\, \ldots$$

This procedure is known as the hierarchy of Hamiltonians.

4. THE MODEL

The Schrödinger equation for supersymmetric partner potentials $V_\pm(x)$ can be written as follows

$$\psi''(x) + V_\pm(x)\psi(x) = E_\pm \psi(x)$$

or equivalently

$$\psi''(x) + \Big[E_\pm - W^2(x) \mp W'(x)\Big]\,\psi(x) = 0.$$

It can be easily shown that this equation with an appropriate coordinate transformation $s = f(W(x))$ takes the form

$$\psi''(s) + \left[\frac{W''f' + W'^2 f''}{W'^2 f'^2}\right]\psi'(s) + \left[\frac{E_\pm - W^2 \mp W'}{W'^2 f'^2}\right]\psi(s) = 0.$$

By comparing this equation with Eq. (1) we get

$$\tilde{\tau}(s) = \left[\frac{W''f' + W'^2 f''}{W'^2 f'^2}\right]\sigma(s) \tag{12}$$

and

$$\tilde{\sigma}(s) = \left[\frac{E_\pm - W^2 \mp W'}{W'^2 f'^2}\right]\sigma^2(s). \tag{13}$$

If we suppose a simple form to coordinate transformation $f(W(x)) = W(x)$, then (12) and (13) reduce to

$$W''(x) - \frac{\widetilde{\tau}(s)}{\sigma(s)} W'^2(x) = 0 \tag{14}$$

and

$$\tilde{\sigma}(s) = \left[\frac{E_{\pm} - W^2 \mp W'}{W'^2}\right]\sigma^2(s) \tag{15}$$

respectively. Now, as an illustration, we choose

$$\tilde{\tau}(s) = 0 \quad \text{and} \quad \sigma(s) = \omega/2, \tag{16}$$

Eqs. (14) and (15) take respectively the forms

$$\frac{\omega}{2}W''(x) = 0 \tag{17}$$

and

$$\tilde{\sigma}(s) = -s^2 + E_{\pm} \mp \omega/2. \tag{18}$$

The function $W(x) = \frac{1}{2}\omega x - b$ satisfies Eq. (17) clearly. The corresponding partner potential can be obtained from (5) as follows

$$V_{\pm}(x) = \frac{1}{4}\omega^2(x - 2b/\omega)^2 - \omega/2. \tag{19}$$

This potential is known as "*Shifted Oscillator*" potential. Substituting Eqs. (16) and (18) in relation (2), leads to

$$\pi(s) = \pm\sqrt{s^2 - (E_{\pm} \mp \omega/2) + \kappa\omega/2}.$$

This function can be written in two possible values

$$\pi(s) = \pm s \quad \text{for} \quad \kappa = \frac{2E_{\pm}}{\omega} \mp 1.$$

Imposing $\tau'(s) < 0$ implies

$$\pi(s) = -s \quad \text{for} \quad \kappa = \frac{2E_{\pm}}{\omega} + 1.$$

One can easily show that substituting these result into Eq. (3) led to energy eigenvalues

$$E_n^{(1)} = n\omega, \quad n = 0, 1, 2, \ldots$$

Similarly, the weight function $\rho(x)$ is simply found out from (4) as

$$\rho = e^{-2s^2/\omega}.$$

By substituting into the Rodriguez relation one gets

$$y_n(s) = B_n e^{2s^2/\omega}\left(\frac{d}{ds}\right)^n e^{-2s^2/\omega}$$

where B_n stands for the normalization constant. The other part of the wave function is simply found out to be $\phi = e^{-s^2/\omega}$. Finally, by combining two parts, one obtains

$$\psi(s) = e^{-s^2/\omega} H_n\left(\sqrt{\frac{\omega}{2}}s\right)$$

or equivalently

$$\psi(y) = e^{-\frac{1}{2}y^2} H_n(y), \quad y = \sqrt{\frac{\omega}{2}}\left(x - \frac{2b}{\omega}\right),$$

where $H_n(y)$ is Hermit polynomial.

In general, by choosing appropriate functions for $s = f(W(x))$, $\tilde{\tau}(s)$ and $\sigma(s)$, we made a table for all known shape invariant potentials as follows.

Table 1. Shape invariant potential obtained by the proposed new mathematical method.

N	$V(x)$ *Potential − Name*	$W(x)$	$\tilde{\tau}(x)$	$\sigma(x)$	$E_n^{(1)}$	$\psi_n(y)$	*Variables*
1	$\frac{1}{4}\omega^2(x-2b/\omega)^2 - \omega/2$ *Shifted − oscillator*	$\frac{1}{2}\omega x - b$	0	$\omega/2$	$n\omega$	$e^{-\frac{1}{2}y^2} H_n(y)$	$y = \sqrt{\omega/2}$ $(x - \frac{2b}{\omega})$
2	$a^2 + b^2 e^{-2\alpha x} - 2b(a+\alpha/2)e^{-\alpha x}$ *Morse*	$a - be^{-\alpha x}$	1	$x - a$	$a^2 - (a-n\alpha)^2$	$y^{\beta-n} e^{-y/2} L_n^{2(\beta-n)}(y)$	$y = \frac{2b}{\alpha} e^{-\alpha x}$ $\beta = a/\alpha$
3	$-\frac{e^2}{x} + \frac{\ell(\ell+1)}{x^2} + \frac{e^4}{4(\ell+1)^2}$ *Coulomb*	$\frac{e^2}{2(\ell+1)} - \frac{\ell+1}{x}$	$2x - \frac{e^2}{\ell+1}$	$(x - \frac{e^2/2}{\ell+1})^2$	$\frac{e^4}{4}[(\ell+1)^{-2}$ $-(n+\ell+1)^{-2}]$	$y^{\ell+1} e^{-y/2} L_n^{2\ell+1}(y)$	$y = \frac{e^2 x}{n+\ell+1}$
4	$a^2 + \frac{b^2}{a^2} + a(a-\alpha)\,\mathrm{cosech}^2\,\alpha x -$ $2b\coth\alpha x$ *Eckart*	$-a\coth\alpha x + \frac{b}{a}$ $b > a^2$	$2(s - \frac{b}{a})$	$(s - \frac{b}{a})^2 - a^2$	$-(a+n\alpha)^2 - a^2$ $+\frac{b^2}{a^2} - \frac{b^2}{(a+n\alpha)^2}$	$(y-1)^{-(\beta+n-\gamma)/2}$ $(y+1)^{-(\beta+n+\gamma)/2}$ $P_n^{(-\alpha-n+\gamma,-\beta-n-\gamma)}(y)$	$y = \coth\alpha x$ $\beta = a/\alpha$ $\gamma = \frac{b}{\alpha^2(\beta+n)}$
5	$a(a-\alpha)\cos ec^2\alpha x + 2b\cot\alpha x$ $-a^2 + b^2/a^2$ *Rosen − Morse (trigonometric)*	$-a\cot\alpha x - \frac{b}{a}$ $0 \le \alpha x \le \pi$	$2(s + \frac{b}{a})$	$(s + \frac{b}{a})^2 + a^2$	$(a+n\alpha)^2 - a^2$ $+\frac{b^2}{a^2} - \frac{b^2}{(a+n\alpha)^2}$	$(y^2-1)^{-(\beta+n)/2} e^{a\alpha x}$ $P_n^{(-\alpha-n+i\gamma,-\beta-n-i\gamma)}(y)$	$y = i\cot\alpha x$ $\beta = a/\alpha$ $\gamma = \frac{b}{\alpha^2(\beta+n)}$
6	$-a(a+\alpha)\sec h^2\alpha x + 2b\tanh\alpha x$ $+a^2 + b^2/a^2$ *Rosen − Morse (hyperbolic)*	$a\tanh\alpha x + \frac{b}{a}$ $a^2 > b$	$-2(s - \frac{b}{a})$	$-(s - \frac{b}{a})^2 + a^2$	$-(a-n\alpha)^2 + a^2$ $+\frac{b^2}{a^2} - \frac{b^2}{(a-n\alpha)^2}$	$(1-y)^{(\beta-n+\gamma)/2}$ $(1+y)^{(\beta-n-\gamma)/2}$ $P_n^{(\beta-n+\gamma,\,\beta-n-\gamma)}(y)$	$y = \tanh\alpha x$ $\beta = a/\alpha$ $\gamma = \frac{b}{\alpha^2(\beta-n)}$

5. CONCLUSION

Supersymmetric quantum mechanical shape invariant potentials using a new mathematical method have been proposed. In the proposed method a nonlinear differential equation of the superpotential in terms of the parameters of the NU-method is obtained and considering exact and well known solution for these parameters the superpotential and finally the shape invariant potentials were obtained.

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